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AN ESTIMATE OF THE RATE OF CONVERGENCE TO THE NORMAL LAW

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Abstract. The paper contains an estimate of the rate of convergence to the normal law for nonidentically distributed random variables.

A generalization of a V. M. Zolotarev result for the case of nonidentically distributed random variables is obtained in [1]. This paper continues the line of research initiated in [1] on applications of pseudomoments to the rate of convergence to the normal law. However below we use other pseudomoments different from those used in [1].

Let $\xi_1, \xi_2, \dots, \xi_n, \dots$ be a sequence of independent random variables such that $\mathsf{M}\,\xi_i = 0$ and $\operatorname{Var}\xi_i = \sigma_i^2$. Let $F_i(x)$ and $f_i(t)$ be the distribution function and characteristic function of the random variable ξ_i, $i \ge 1$, respectively. Denote by $\Phi_n(x)$ the distribution function of the random variable $(\xi_1 + \xi_2 + \dots + \xi_n)/B_n$ and put $B_n^2 = \sigma_1^2 + \dots + \sigma_n^2$, $\overline{\sigma}_i = \min(1; \sigma_i)$, and $\overline{B}_n^2 = \overline{\sigma}_1^2 + \dots + \overline{\sigma}_n^2$. Further let $\Phi(x)$ be the standard normal distribution function, $\rho_n = \sup_x |\Phi_n(x) - \Phi(x)|$, and

$$\lambda_{ni}^{(1)} = \int_{|x| \le B_n} \max\left(1, |x|^3\right) \left| d\left(F_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right)\right|,$$

$$\lambda_{ni}^{(2)} = \int_{|x| \ge B_n} x^2 \left| d\left(F_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right)\right|,$$

$$\Lambda_n^{(1)} = \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \lambda_{ni}^{(1)}, \qquad \Lambda_n^{(2)} = \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \lambda_{ni}^{(2)}, \qquad b_n = \min(\overline{\sigma}_1, \dots, \overline{\sigma}_n).$$

Theorem. *For all $n \ge 1$,*

$$\rho_n \le C \left(\frac{1}{\overline{B}_n} \Lambda_n^{(1)} + \Lambda_n^{(2)}\right) b_n^{-3}, \tag{1}$$

where C is a universal constant.

Proof. Let n be fixed and $c \in (0; 1)$ be a certain constant; the exact value of c will be chosen below. Assume that $\Lambda_n^{(2)} \le c$ and note that estimate (1) is obvious otherwise.

For $n = 1$ we have

$$\rho_1 = \sup_x |F_1(x\sigma) - \Phi(x)| = \sup_x |F_1(x) - \Phi(x/\sigma_1)| = \sup_x \left| \int_{-\infty}^{+\infty} d(F_1(x) - \Phi(x/\sigma_1)) \right|$$
$$\le \int_{-\infty}^{+\infty} |d(F_1(x) - \Phi(x/\sigma_1))| \le \lambda_{n1}^{(1)} + \frac{1}{\sigma_1^2} \lambda_{n1}^{(2)},$$

and inequality (1) follows. Now let $n \ge 2$.

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Put $\phi_n(t) = \prod_{i=1}^n f_i(t/B_n)$. We apply the following inequality:

$$\rho_n \le \min\left\{\max\left(\int_0^X \left|\phi_n(t) - e^{-t^2/2}\right| \frac{dt}{t}, \frac{6\sqrt{2}}{X}\right); X > 0\right\} \tag{2}$$

(see [2, p. 377]) with $X = TB_n$, $T = c^{5/2}\Lambda_n^{-1}$, and $\Lambda_n = \max\bigl(\Lambda_n^{(1)}; \Lambda_n^{(2)}\bigr)$. Let $T_1 = \sqrt{c}$, $T_2 = \min(T_1, T)$. Using the inequalities

$$\left|\phi_n(t) - e^{-t^2/2}\right| \le \sum_{i=1}^n \left|\omega_i\left(\frac{t}{B_n}\right)\right| |\psi_{ni}(t)|,$$

where $\omega_i(t) = f_i(t) - \exp\bigl\{-t^2\sigma_i^2/2\bigr\}$, $\psi_{ni}(t) = \prod_{k=1}^{i-1}\exp\bigl\{-t^2\sigma_k^2/2B_n^2\bigr\}\prod_{k=i+1}^n f_k(t/B_n)$, and

$$|\omega_i(t)| \le \min\left(\frac{|t|^3}{6}\lambda_{ni}^{(1)} + 2t^2\lambda_{ni}^{(2)}; \lambda_{ni}^{(1)} + \frac{1}{B_n^2}\lambda_{ni}^{(2)}\right),$$

we get

$$\begin{aligned}|f_k(t)| &\le \left|\exp\left\{-t^2\sigma_k^2/2\right\} + \omega_k(t)\right| \le \exp\left\{-t^2\sigma_k^2/2\right\} + \frac{|t^3|}{6}\lambda_{nk}^{(1)} + 2t^2\lambda_{nk}^{(2)} \\ &\le \exp\left\{-t^2\overline{\sigma}_k^2/2\right\} + t^2\left(\frac{c^{5/2}}{6} + 2c\right) \le \exp\left\{-c_1t^2\overline{\sigma}_k^2\right\}\end{aligned}$$

for $|t| \le T_2$, while

$$|\psi_{ni}(t)| \le e \cdot \exp\left\{-t^2\frac{\overline{B}_n^2}{B_n^2}c_1\right\}$$

for $|t| \le T_2B_n$, where $c_1 = \bigl(1 - e^{-c/2}\bigr)/c - 13c/6$ and c is chosen such that $c_1 > 0$. Then (see inequality (12) in [1])

$$\left|\phi_n(t) - e^{-t^2/2}\right| \le e\left(\frac{|t|^3}{6}\Lambda_n^{(1)} + 2t^2\Lambda_n^{(2)}\right)\exp\left\{-t^2\frac{\overline{B}_n^2}{B_n^2}c_1\right\}. \tag{3}$$

We represent the integral in (2) as follows:

$$\begin{aligned}I &= \int_0^{TB_n}\left|\phi_n(t) - e^{-t^2/2}\right|\frac{dt}{t} \\ &= \int_0^{T_2B_n}\left|\phi_n(t) - e^{-t^2/2}\right|\frac{dt}{t} + \int_{T_2B_n}^{TB_n}\left|\phi_n(t) - e^{-t^2/2}\right|\frac{dt}{t} = I_1 + I_2.\end{aligned} \tag{4}$$

Inequality (3) implies

$$I_1 = \int_0^{T_2B_n}\left|\phi_n(t) - e^{-t^2/2}\right|\frac{dt}{t} \le C^{(1)}\left(\frac{1}{\overline{B}_n}\Lambda_n^{(1)} + \Lambda_n^{(2)}\right). \tag{5}$$

The integral I_2 is nonzero if $T_2 = T_1$, in which case $\Lambda_n^{(1)} \le c^2$. Thus

$$\begin{aligned}\left|f_i\left(\frac{t}{B_n}\right)\right| &\le \exp\left\{-t^2\sigma_i^2/(2B_n^2)\right\} + \left|\omega_i\left(\frac{t}{B_n}\right)\right| \\ &\le \exp\left\{\left(e^{-c/2} - 1\right)\overline{\sigma}_i^2 + \lambda_{ni}^{(1)} + \frac{1}{B_n^2}\lambda_{ni}^{(2)}\right\},\end{aligned}$$

$$\prod_{\substack{k=1\\k\ne i,j}}^n\left(\exp\left\{-t^2\sigma_1^2/(2B_n^2)\right\} + \left|\omega_k\left(\frac{t}{B_n}\right)\right|\right) \le e^{2+c}\exp\left\{-\left(\frac{1-e^{-c/2}}{c} - c\right)c\overline{B}_n^2\right\}$$

for $T_2B_n \le |t| \le TB_n$. Moreover by inequality (13) in [1],

$$\begin{aligned}\left|\phi_n(t) - e^{-t^2/2}\right| &\le \sum_{i=1}^n \left|\omega_i\left(\frac{t}{B_n}\right)\right| \sum_{\substack{j=1\\ j\ne i}}^n e^{-t^2\sigma_j^2/(2B_n^2)} \prod_{\substack{k=1\\ k\ne i,j}}^n \left(e^{-t^2\sigma_k^2/(2B_n^2)} + \left|\omega_k\left(\frac{t}{B_n}\right)\right|\right)\\ &\quad + \prod_{k=1}^n \left|\omega_k\left(\frac{t}{B_n}\right)\right|\\ &\le \sum_{i=1}^n \left|\omega_i\left(\frac{t}{B_n}\right)\right| e^{-t^2b_n^2/(2B_n^2)}(n-1)e^{2+c}\exp\left\{-c_3c\overline{B}_n^2\right\}\\ &\quad + \prod_{k=1}^n \left(\lambda_{nk}^{(1)} + \frac{1}{B_n^2}\lambda_{nk}^{(2)}\right),\end{aligned}$$

where $c_3 = \left(1 - e^{-c/2}\right)/c - c > 0$.

Thus

$$\begin{aligned}I_2 &= \int_{T_2B_n}^{TB_n} \left|\phi_n(t) - e^{-t^2/2}\right| \frac{dt}{t}\\ &\le (n-1)e^{2+c}\exp\left\{-c_3c\overline{B}_n^2\right\} \sum_{i=1}^n \left(\lambda_{ni}^{(1)} + \frac{1}{B_n^2}\lambda_{ni}^{(2)}\right) \int_{T_2B_n}^{TB_n} e^{-t^2b_n^2/(2B_n^2)}\frac{dt}{t}\\ &\quad + \prod_{k=1}^n \left(\lambda_{nk}^{(1)} + \frac{1}{B_n^2}\lambda_{nk}^{(2)}\right) \int_{T_2B_n}^{TB_n} \frac{dt}{t}\\ &\le C^{(3)}\left(\frac{\Lambda_n^{(1)}}{\overline{B}_n} + \Lambda_n^{(2)}\right) b_n^{-3}\end{aligned} \tag{6}$$

and (1) follows from (2) and (4)–(6). □

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