

ERGODICITY AND STABILITY OF NONSTATIONARY QUEUEING SYSTEMS

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D. B. ANDREEV, M. A. ELESIN, E. A. KRYLOV, A. V. KUZNETSOV, AND A. I. ZEIFMAN

ABSTRACT. We study stability and ergodicity of a special class of nonhomogeneous birth-death processes and consider applications of estimates of queue-length process for $M_t/M_t/S$ and $M_t/M_t/S/S$ queues.

1. INTRODUCTION

Nonstationary (nonhomogeneous in time) Markov chains have been studied over the last 20 years; see, for example, [11, 14, 17, 3]. Nonstationary Markov chains appear, in particular, when describing queueing processes. Most of the research on those chains is devoted to various problems of their approximations (see [4, 19, 20] and references therein).

Important are the problems on the rate of convergence to a limit regime and on stability of nonstationary queueing systems. The first study on these problems was initiated by B. V. Gnedenko; see [8, 9, 10] and [15, 16].

The main tool of the present paper is the method used for the first time in [8] and developed in [25, 26, 27, 28, 29, 1, 12]. The method is based on estimates using the log-norm of a linear operator function and on special transforms of the reduced operator function.

We apply this method to a class of Markov queueing systems with a special form of nonstationarity, which, however, is quite general for many applications.

2. MAIN NOTATION

In what follows we consider a nonhomogeneous Markov process $X(t)$, $t \geq 0$ (a queue-length process or a total number of customers arrived to a system), that is, a birth-death process with the phase space $E = \{0, 1, \dots, N\}$, $N \leq \infty$. It is assumed that the birth intensities $\lambda_n(t)$, $t \geq 0$, and death intensities $\mu_n(t)$, $t \geq 0$, $n \in E$, depend on time. Namely

$$(1) \quad \mathbb{P}(X(t+h) = j / X(t) = i) = \begin{cases} \lambda_i(t) \cdot h + o(h) & \text{if } j = i + 1, \\ \mu_i(t) \cdot h + o(h) & \text{if } j = i - 1, \\ 1 - (\lambda_i(t) + \mu_i(t)) \cdot h + o(h) & \text{if } j = i, \\ o(h) & \text{if } |i - j| > 1. \end{cases}$$

Here $h > 0$ and all symbols $o(h)$ are such that $o(h) = o(t, h) \rightarrow 0$ as $h \rightarrow 0$, $t \geq 0$, uniformly with respect to $i \in E$. Let

$$(2) \quad p_{ij}(s, t) = \mathbb{P}(X(t) = j \mid X(s) = i), \quad i, j \in E, 0 \leq s \leq t,$$

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and $p_i(t) = \mathbb{P}(X(t) = i)$, $i \in E$, $t \geq 0$, be transition probabilities and probabilities of states, respectively. Denote by $\mathbf{p}(t) = (p_0(t), \dots, p_N(t))^T$, $t \geq 0$, and by $A(t) = (a_{ij}(t))$, $t \geq 0$, the vector column of probabilities of states, and the matrix of intensities related to (1). Note that

$$(3) \quad a_{ij}(t) = \begin{cases} \lambda_{i-1}(t) & \text{if } j = i - 1, \\ \mu_{i+1}(t) & \text{if } j = i + 1, \\ -(\lambda_i(t) + \mu_i(t)) & \text{if } j = i, \\ 0 & \text{if } |i - j| > 1. \end{cases}$$

Then the behavior of the process is described by the forward Kolmogorov system

$$(4) \quad \frac{d\mathbf{p}}{dt} = A(t)\mathbf{p}, \quad \mathbf{p} = \mathbf{p}(t), \quad t \geq 0.$$

The Cauchy operator $U(t, s)$, $0 \leq s \leq t$, of equation (4) is determined by the matrix

$$U^T(t, s) = P(s, t) := (p_{ij}(s, t))_{i,j=0}^N.$$

The norms of vectors and matrices used below are, as a rule, the ℓ_1 -norms denoted by $\|\cdot\|$, that is, $\|\mathbf{x}\| = \sum_{i \in E} |x_i|$ and

$$\|B\| = \sup_{j \in E} \sum_{i \in E} |b_{ij}|$$

for $\mathbf{x} = (x_0, \dots, x_N)^T$ and $B = (b_{ij})_{i,j=0}^N$, respectively.

The set of all stochastic vectors is denoted by Ω :

$$\Omega = \{\mathbf{x} = (x_0, \dots, x_N)^T : \mathbf{x} \geq 0, \|\mathbf{x}\| = 1\}.$$

3. STABILITY

Since

$$(5) \quad p_0(t) = 1 - \sum_{i \geq 1} p_i(t), \quad t \geq 0,$$

we obtain from (4) the system of differential equations:

$$(6) \quad \frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \quad t \geq 0,$$

where

$$(7) \quad \begin{aligned} B(t) &= \{a_{ij}(t) - a_{i0}(t), i, j = 1, \dots, N\}, \\ \mathbf{z}(t) &= (p_1(t), \dots, p_N(t))^T, \\ \mathbf{f}(t) &= (a_{10}(t), \dots, a_{N0}(t))^T, \quad t \geq 0. \end{aligned}$$

The solution of equation (6) is given by

$$(8) \quad \mathbf{z}(t) = V(t, s) \cdot \mathbf{z}(s) + \int_s^t V(t, \tau) \cdot \mathbf{f}(\tau) d\tau, \quad 0 \leq s \leq t,$$

where $V(t, s)$ is the Cauchy operator of system (6).

When study the stability and ergodicity, we often apply the inequality

$$(9) \quad \|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\| \leq \|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\| \leq 2 \|\mathbf{z}^{(1)} - \mathbf{z}^{(2)}\|, \quad t \geq 0,$$

where

$$\mathbf{z}^{(i)} = \mathbf{z}^{(i)}(t) \quad \text{and} \quad \mathbf{p}^{(i)} = \mathbf{p}^{(i)}(t), \quad t \geq 0, \quad i = 1, 2,$$

is an arbitrary pair of solutions of systems (6) and (4), respectively.

The following transform:

$$(10) \quad D = \begin{pmatrix} d_0 & d_0 & d_0 & \dots \\ 0 & d_1 & d_1 & \dots \\ 0 & 0 & d_2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where $d_i > 0$, $i = 0, 1, \dots, N-1$, is introduced in [27, 28] in order to reduce the matrix $B(t)$ of system (6) to a more convenient form. Note that the transform is determined by an $N \times N$ upper triangle matrix. We have

$$D^{-1} = \begin{pmatrix} d_0^{-1} & -d_1^{-1} & 0 & \dots \\ 0 & d_1^{-1} & -d_2^{-1} & 0 & \dots \\ \dots & 0 & \dots & d_2^{-1} & \dots & \dots \\ & \dots & \dots & \dots & \dots & \dots \\ & & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Applying this transform to the matrix $B(t)$ in (7) that corresponds to a nonhomogeneous birth-death process we get the matrix $DB(t)D^{-1}$:

$$(11) \quad DB(t)D^{-1} = \begin{pmatrix} -(\lambda_0(t) + \mu_1(t)) & d_0 \cdot d_1^{-1} \cdot \mu_1 & 0 & \dots \\ d_1 \cdot d_0^{-1} \cdot \lambda_1 & -(\lambda_1(t) + \mu_2(t)) & d_1 \cdot d_2^{-1} \cdot \mu_2 & 0 & \dots \\ 0 & d_2 \cdot d_1^{-1} \cdot \lambda_2(t) & \dots & \dots & \dots \\ \vdots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Put $d_0 = 1$. For the case of a finite phase space we also put $d_N = 0$. Let $\delta_{i+1} = d_{i+1}/d_i$, $i = 0, \dots, N-1$. Then the log-norm is given by

$$(12) \quad \gamma(B(t))_{1D} = \gamma(DB(t)D^{-1})_1 = -\inf_k \alpha_k(t) = -\underline{\alpha}(t), \quad t \geq 0$$

(see, for example, [5, 29]), where

$$(13) \quad \alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \delta_{k+1} \cdot \lambda_{k+1}(t) - \delta_k^{-1} \cdot \mu_k(t), \quad t \geq 0, \quad k = 0, \dots, N-1.$$

Now we use the change $\mathbf{z}(t) = D^{-1}\mathbf{y}(t)$, $t \geq 0$, and transform system (6) to a system with the matrix $DB(t)D^{-1}$, $t \geq 0$. As in [26, 27] we put

$$(14) \quad \|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|.$$

Then we get

$$(15) \quad \|\mathbf{z}\|_{1D} \geq \frac{g}{2} \|\mathbf{z}\|$$

by (10), where $g = \inf_k d_k$. If the weights δ_k , $k = 0, \dots, N-1$, in (13) are such that

$$\int_0^{+\infty} \underline{\alpha}(t) dt = +\infty,$$

then the birth-death process is weakly ergodic and

$$(16) \quad \left\| \mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t) \right\| \leq \frac{4}{g} \exp \left\{ -\int_s^t \underline{\alpha}(u) du \right\} \left\| \mathbf{z}^{(1)}(s) - \mathbf{z}^{(2)}(s) \right\|_{1D}, \quad 0 \leq s \leq t.$$

Here

$$(17) \quad \left\| \mathbf{z}^{(1)}(s) - \mathbf{z}^{(2)}(s) \right\|_{1D} \leq \sum_{i \geq 1} q_i \left| p_i^{(1)}(s) - p_i^{(2)}(s) \right|, \quad t \geq 0,$$

according to notation (14), where $q_i = \sum_{m=0}^{i-1} d_m$, $i = 1, \dots, N$.

Estimate (16) holds for ergodic birth-death processes. This is seen by putting

$$\mathbf{p}^{(2)}(t) = \boldsymbol{\pi}, \quad t \geq 0.$$

We consider a class of nonhomogeneous birth-death processes satisfying the condition

$$(18) \quad \lambda_n(t) = \lambda_n \cdot a(t), \quad \mu_n(t) = \mu_n \cdot b(t), \quad t \geq 0, \quad n \in E,$$

where

$$(19) \quad \mu_0 = 0, \quad \lambda_n > 0, \quad n = 0, \dots, N-1, \quad \mu_n > 0, \quad n = 1, \dots, N-1.$$

We assume that the following limits exist:

$$(20) \quad \lim_{n \rightarrow \infty} \lambda_n = \lambda > 0, \quad \lim_{n \rightarrow \infty} \mu_n = \mu > 0$$

if $N = \infty$. For the case of $N < \infty$ we put

$$(21) \quad \lambda = \lambda_{N-1}, \quad \mu = \mu_N.$$

We also assume that the functions $a(t) \geq 0$, $t \geq 0$, and $b(t) \geq 0$, $t \geq 0$, are locally integrable on $[0, \infty)$ and possess the ‘‘uniform’’ mean values

$$(22) \quad \tilde{a} = \lim_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t a(\tau) d\tau, \quad \tilde{b} = \lim_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t b(\tau) d\tau.$$

Condition (22) holds, in particular, if the functions $a(t)$ and $b(t)$ are periodic (in general, with different periods) or asymptotically periodic.

It is easy to show that condition (22) implies that the functions $a(t)$ and $b(t)$ are integral bounded on $[0, \infty)$, that is,

$$(23) \quad \sup_{t \geq 0} \int_t^{t+1} a(\tau) d\tau = \mathbf{a} < \infty, \quad \sup_{t \geq 0} \int_t^{t+1} b(\tau) d\tau = \mathbf{b} < \infty.$$

Note that the existence of ordinary means

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t a(\tau) d\tau, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t b(\tau) d\tau$$

does not imply the integral boundedness of $a(t)$ and $b(t)$.

In the current paper we study the stability of birth-death processes with respect to the perturbations of a more general form as compared to those in [25, 26, 27, 30, 12].

We assume that the perturbed Markov chain $\bar{X}(t)$, $t \geq 0$, is a process with bounded jumps, that is, its matrix of intensities

$$\bar{A}(t) = (\bar{a}_{ij}(t))_{i,j=0}^N$$

is such that $\bar{a}_{ij}(t) = 0$ for all $t \geq 0$, $|i - j| > R$, and some positive integer R . Moreover we assume that

$$\bar{a}_{ij}(t) = \bar{a}_{ij} \bar{a}(t)$$

for $i > j$, while

$$\bar{a}_{ij}(t) = \bar{a}_{ij} \bar{b}(t)$$

for $i < j$, and

$$(24) \quad L = \sup_{i,j, i \neq j} (a_{ij}, \bar{a}_{ij}) < \infty,$$

$$(25) \quad \sup_{t \geq 0} \int_t^{t+1} (|\hat{a}(\tau)| + |\hat{b}(\tau)|) d\tau \leq \varepsilon,$$

where $\hat{a}(t) = a(t) - \bar{a}(t)$ and $\hat{b}(t) = b(t) - \bar{b}(t)$.

Put

$$(26) \quad f_k = \frac{\Delta \lambda \mu_{k+1} - \lambda_{k-1} \mu + \sqrt{(\Delta \lambda \mu_{k+1} - \lambda_{k-1} \mu)^2 + 4 \Delta \lambda \mu \cdot (\lambda_{k-1} \mu_{k+1} - \lambda_k \mu_k)}}{2 \Delta \lambda \mu},$$

$$k = 0, \dots, N-1,$$

where $\Delta > 0$ and λ and μ are defined by (20) and (21), respectively.

In what follows we use the notation

$$\tilde{E} = \begin{cases} \{0, 1, \dots, N-1\} & \text{if } N < \infty, \\ E & \text{if } N = \infty. \end{cases}$$

Put

$$(27) \quad c_0 = \min \left(1, \inf_k f_k, \inf_k \frac{\mu_{k+1}}{\mu} \right).$$

It is proved in [12] that for every $0 < c \leq c_0$ there exists a positive sequence $\{\delta_k\}$ such that

$$(28) \quad \underline{\alpha}(t) \geq l(t) = c \cdot (\mu b(t) - \Delta \lambda a(t)), \quad t \geq 0,$$

for some $\Delta > 1$ if $\inf_k f_k > 0$. Moreover $\liminf \delta_k > 1$ for $N = \infty$ and

$$(29) \quad 0 < m_1 \leq \frac{\mu_k}{\mu_{k+1} - c\mu} \leq \delta_k \leq \frac{\lambda_{k-1} + c\Delta\lambda}{\lambda_k} \leq m_2 < \infty$$

under the latter condition.

Conditions (22) and (28) imply that there exists a constant K such that

$$(30) \quad \exp \left(- \int_s^t l(u) du \right) \leq K \exp(-l(t-s))$$

for all $0 \leq s \leq t$, where

$$l = c \left(\mu \tilde{b} - \Delta \lambda \tilde{a} \right) - \varepsilon.$$

Let

$$(31) \quad \mu \tilde{b} - \Delta \lambda \tilde{a} > 0, \quad \mathbf{p}(0) = \bar{\mathbf{p}}(0),$$

and the ‘‘perturbed’’ Markov chain $\tilde{X}(t)$, $t \geq 0$, has bounded jumps and satisfies conditions (24) and (25).

We rewrite condition (6) as follows:

$$(32) \quad \frac{d\mathbf{z}}{dt} = \bar{B}(t)\mathbf{z} + \bar{\mathbf{f}}(t) + (B(t) - \bar{B}(t))\mathbf{z} + \mathbf{f}(t) - \bar{\mathbf{f}}(t)$$

$$= \bar{B}(t)\mathbf{z} + \bar{\mathbf{f}}(t) + \hat{B}(t)\mathbf{z} + \hat{\mathbf{f}}(t),$$

where $\hat{B}(t) = B(t) - \bar{B}(t)$ and $\hat{\mathbf{f}}(t) = \mathbf{f}(t) - \bar{\mathbf{f}}(t)$, $t \geq 0$. Then

$$\mathbf{z}(t) = \bar{V}(t, 0) \cdot \mathbf{z}(0) + \int_0^t \bar{V}(t, \tau) \cdot \bar{\mathbf{f}}(\tau) d\tau + \int_0^t \bar{V}(t, \tau) \cdot [\hat{B}(\tau)\mathbf{z}(\tau) + \hat{\mathbf{f}}(\tau)] d\tau$$

$$= \bar{\mathbf{z}}(t) + \int_0^t \bar{V}(t, \tau) \cdot [\hat{B}(\tau)\mathbf{z}(\tau) + \hat{\mathbf{f}}(\tau)] d\tau, \quad t \geq 0,$$

and we get

$$(33) \quad \begin{aligned} \|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_{1D} &\leq \int_0^t \|\bar{V}(t, \tau)\|_{1D} \cdot \|\hat{B}(\tau)\mathbf{z}(\tau) + \hat{\mathbf{f}}(\tau)\|_{1D} d\tau \\ &\leq \int_0^t \|\bar{V}(t, \tau)\|_{1D} \cdot \left\{ \|\hat{B}(\tau)\|_{1D} \cdot \|\mathbf{z}(\tau)\|_{1D} + \|\hat{\mathbf{f}}(\tau)\|_{1D} \right\} d\tau. \end{aligned}$$

According to (30) and (23) we have

$$(34) \quad \begin{aligned} \|\mathbf{z}(t)\|_{1D} &\leq \|V(t, 0)\| \cdot \|\mathbf{z}(0)\|_{1D} + \int_0^t \|V(t, \tau)\|_{1D} \cdot \|\mathbf{f}(\tau)\|_{1D} d\tau \\ &\leq Ke^{-lt}\|\mathbf{z}(0)\|_{1D} + \lambda_0 e^{-lt} \int_0^t e^{l\tau} a(\tau) d\tau \\ &\leq Ke^{-lt}\|\mathbf{z}(0)\|_{1D} + \lambda_0 e^{-lt} \left(\sum_{n=0}^{[t]-1} \int_n^{n+1} e^{l\tau} a(\tau) d\tau + \int_{[t]}^t e^{l\tau} a(\tau) d\tau \right) \\ &\leq Ke^{-lt}\|\mathbf{z}(0)\|_{1D} + \lambda_0 a \left(1 + \frac{e^l}{e^l - 1} \right). \end{aligned}$$

Let $\hat{a}_{ij}(t) = a_{ij}(t) - \bar{a}_{ij}(t)$. Then

$$D\hat{B}D^{-1} = \begin{pmatrix} g_{11} & g_{12} & \dots \\ g_{21} & g_{22} & \dots \\ \dots & \dots & \dots \end{pmatrix},$$

where

$$\begin{aligned} g_{11} &= \hat{a}_{11} + \hat{a}_{21} + \dots - \hat{a}_{10} - \hat{a}_{20} - \dots, \\ g_{12} &= d_0 d_1^{-1} (\hat{a}_{12} + \hat{a}_{22} + \dots - \hat{a}_{11} - \hat{a}_{21} - \dots), \\ g_{21} &= d_1 d_0^{-1} (\hat{a}_{21} + \hat{a}_{31} + \dots - \hat{a}_{20} - \hat{a}_{30} - \dots), \\ g_{22} &= (\hat{a}_{22} + \hat{a}_{32} + \dots - \hat{a}_{21} - \hat{a}_{31} - \dots). \end{aligned}$$

Since the jumps are bounded, the entries of the above matrix are zero if the absolute value of the difference of indices is greater than R . Putting

$$(35) \quad \mathbf{m}_3 = (\max(\mathbf{m}_2, \mathbf{m}_1^{-1}))^R,$$

we get

$$(36) \quad \left\| \hat{B}(t) \right\|_{1D} = \left\| D\hat{B}(t)D^{-1} \right\|_1 \leq 2LR\mathbf{m}_3 \left(|\hat{a}(t)| + |\hat{b}(t)| \right),$$

$$(37) \quad \left\| \hat{\mathbf{f}}(t) \right\|_{1D} \leq LR\mathbf{m}_3 |\hat{a}(t)|,$$

and

$$(38) \quad \gamma(\bar{B}(t))_{1D} \leq \gamma(B(t))_{1D} + \left\| \hat{B}(t) \right\| \leq -l(t) + 2LR\mathbf{m}_3 \left(|\hat{a}(t)| + |\hat{b}(t)| \right).$$

The last inequality implies that

$$(39) \quad \begin{aligned} \|\bar{V}(t, \tau)\| &\leq K \exp \left(-l(t - \tau) + \int_\tau^t 2LR\mathbf{m}_3 (|\hat{a}(\tau)| + |\hat{b}(\tau)|) d\tau \right) \\ &\leq 2K \exp \{ -(l - 2LR\mathbf{m}_3 \varepsilon)(t - \tau) \} \end{aligned}$$

by (24) and (25). Thus

$$\begin{aligned}
 \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 &\leq \frac{4}{g} \|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\|_{1D} \\
 &\leq \frac{8}{g} KLRm_3 \int_0^t \exp\{-(l - 2LRm_3\varepsilon)(t - u)\} \left(|\hat{a}(u)| + |\hat{b}(u)| \right) \\
 &\quad \times \left(2K\|\mathbf{z}(0)\|_{1D} + 2\lambda_0\mathbf{a} \left(1 + \frac{e^l}{e^l - 1} \right) + 1 \right) du \\
 (40) \quad &\leq \frac{16}{g} \varepsilon KLRm_3 \left(2K\|\mathbf{z}(0)\|_{1D} + 2\lambda_0\mathbf{a} \left(1 + \frac{e^l}{e^l - 1} \right) + 1 \right) \\
 &\quad \times \left(1 + \frac{e^{(l-2LRm_3\varepsilon)}}{e^{(l-2LRm_3\varepsilon)} - 1} \right).
 \end{aligned}$$

Therefore the following result holds.

Theorem 1. *Assume that condition (31) holds and the “perturbed” Markov chain $\bar{X}(t)$, $t \geq 0$, has bounded jumps and satisfies conditions (24) and (25). Then estimate (40) holds.*

Now we consider the case of a finite phase space. The equation for the “perturbed” chain, similar to (6), can be written as follows:

$$(41) \quad \frac{d\bar{\mathbf{z}}}{dt} = B(t)\bar{\mathbf{z}} + \mathbf{f}(t) + (\bar{B}(t) - B(t))\bar{\mathbf{z}} + \bar{\mathbf{f}}(t) - \mathbf{f}(t) = B(t)\bar{\mathbf{z}} + \mathbf{f}(t) - \hat{B}(t)\bar{\mathbf{z}} - \hat{\mathbf{f}}(t).$$

Then

$$(42) \quad \|\bar{\mathbf{z}}(t)\|_1 \leq \|\bar{\mathbf{p}}(t)\|_1 \leq 1,$$

$$(43) \quad \left\| \hat{B}(t) \right\|_1 \leq 2L \left(|\hat{a}(t)| + |\hat{b}(t)| \right),$$

$$(44) \quad \left\| \hat{\mathbf{f}}(t) \right\|_1 \leq L|\hat{a}(t)|,$$

$$(45) \quad \|V(t, \tau)\|_1 \leq \|D\| \cdot \|D^{-1}\| \cdot \exp \left\{ - \int_s^t l(u) du \right\} \leq NG \cdot \frac{2}{g} \cdot K e^{-l(t-s)},$$

where $G = \max d_i$.

Thus for the ℓ_1 -norm we get

$$\begin{aligned}
 \|\mathbf{z}(t) - \bar{\mathbf{z}}(t)\| &\leq \int_0^t \|V(t, \tau)\| \cdot \left\| \hat{B}(\tau)\bar{\mathbf{z}}(\tau) + \hat{\mathbf{f}}(\tau) \right\| d\tau \\
 &\leq \int_0^t \|V(t, \tau)\| \cdot \left\{ \|\hat{B}(\tau)\| \cdot \|\bar{\mathbf{z}}(\tau)\| + \|\hat{\mathbf{f}}(\tau)\| \right\} d\tau \\
 (46) \quad &\leq \frac{6GNKL}{g} \cdot \int_0^t e^{-l(t-\tau)} \left(|\hat{a}(\tau)| + |\hat{b}(\tau)| \right) d\tau \\
 &\leq \frac{6GNKL}{g} \cdot \varepsilon \left(1 + \frac{e^l}{e^l - 1} \right).
 \end{aligned}$$

Therefore the following result holds.

Theorem 2. *Let all the assumptions of Theorem 1 hold and $N < \infty$. If the intensities of the “perturbed” Markov chain $\bar{X}(t)$, $t \geq 0$, satisfy conditions (24) and (25), then*

$$(47) \quad \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \leq \frac{12GNKL\varepsilon}{g} \left(1 + \frac{e^l}{e^l - 1} \right).$$

4. EXAMPLES

First we consider a queueing system close to $M_t/M_t/S$. The corresponding references for the system $M_t/M_t/S$ can be found in [13, 21, 22, 2]. We assume that the number of customers in the queue (the queue length) is a birth-death process $X(t)$ whose intensities are locally integrable on $[0, \infty)$:

$$(48) \quad \lambda_n(t) = a(t), \quad \mu_n(t) = b(t) \cdot \min(n, S).$$

The stability of these processes is studied in [29, 12]. Assume that the uniform means \tilde{a} and \tilde{b} exist. We also assume that the traffic intensity

$$\rho = \frac{\tilde{a}}{S\tilde{b}}$$

is such that $(1 - 1/S)^2 \leq \rho < 1$. Then

$$c_0 = \min\left(\frac{1}{S}, \frac{\Delta - 1}{\Delta}\right) = \frac{\Delta - 1}{\Delta} = 1 - \rho^{1/2},$$

$\delta_k = \rho^{-1/2} > 1$, and

$$(49) \quad l(t) = \left(1 - \rho^{1/2}\right) (Sb(t) - \rho^{-1}a(t)).$$

Note that estimate (49) is quite precise, since the true value of the convergence parameter for a stationary queueing system $M/M/S$ is equal to

$$\dot{\alpha} = \left(\sqrt{\tilde{a}} - \sqrt{S\tilde{b}}\right)^2$$

(see [6]).

Now we assume that a queueing system is close to $M_t/M_t/S$ in the sense that customers may arrive to the system and be served in groups. The number of customers in every group does not exceed R and the perturbations of arrival and service intensities of customers satisfy conditions (24) and (25).

Then the following estimate of the “stability” holds:

$$(50) \quad \begin{aligned} \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 &\leq 16\varepsilon KLRm_3 \left(2K\|\mathbf{z}(0)\|_{1D} + 2a \left(1 + \frac{e^t}{e^t - 1}\right) + 1\right) \\ &\times \left(1 + \frac{e^{(l-2LRm_3\varepsilon)}}{e^{(l-2LRm_3\varepsilon)} - 1}\right), \end{aligned}$$

where $m_3 = \rho^{-R/2}$. If there are H customers in the system at the moment $t = 0$, then

$$\|\mathbf{z}(0)\|_{1D} \leq \sum_{k=0}^H \rho^{-k/2}.$$

It is worthwhile to mention that if the arrival and service intensities of customers are asymptotically periodic, then the corresponding process is “almost ergodic” in the sense that the limit

$$(51) \quad \tilde{p}_\infty = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{p}(\tau) d\tau$$

exists and is independent of the initial probability distribution of states.

Now we consider queueing systems close to $M_t/M_t/S/0$.

There is a number of papers devoted to the queueing system $M_t/M_t/S/0$ (the system with losses). Usually it is assumed that the intensities are constant (see references in [18]

and follow up papers [7, 24, 23]). The number of customers $X(t)$ for this queue is a birth-death process with a finite phase space $E = \{0, 1, \dots, S\}$ and with intensities

$$(52) \quad \lambda_n(t) = a(t), \quad \mu_n(t) = nb(t).$$

The first results and estimates for the general case are obtained in [27] (also see [29, 12]). Some of the results of [27] are obtained in a follow-up paper [20], where the non-Markovian case is also studied. We assume again that the uniform means \tilde{a} and \tilde{b} exist and moreover $\tilde{b} > 0$. Putting $\delta_k = 1$ for all k we prove that $\alpha_k(t) \geq b(t)$ for all k and

$$(53) \quad l(t) = b(t).$$

Let a queueing system be close to $M_t/M_t/S/0$ in the sense that the total number of customers in the system does not exceed S and the perturbations of the arrival and service intensities satisfy conditions (24) and (25).

Then Theorem 1 implies the following estimate of the “stability”:

$$(54) \quad \begin{aligned} & \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \\ & \leq 16\varepsilon KLS \left(2KS + 2a \left(1 + \frac{e^l}{e^l - 1} \right) + 1 \right) \left(1 + \frac{e^{(l-2LRm_3\varepsilon)}}{e^{(l-2LRm_3\varepsilon)} - 1} \right). \end{aligned}$$

Theorem 2 implies a sharper estimate, which is, naturally, of the same order:

$$(55) \quad \|\mathbf{p}(t) - \bar{\mathbf{p}}(t)\|_1 \leq 12\varepsilon KLS \left(1 + \frac{e^l}{e^l - 1} \right).$$

In the case of asymptotically periodic arrival and service intensities, the corresponding process is “almost ergodic” in the sense that the limit

$$(56) \quad \tilde{p}_\infty = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbf{p}(\tau) d\tau$$

exists and is independent of the initial probability distribution of customers. Note that the limit above can be used as the “pseudostationary distribution”.

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VOLOGDA STATE PEDAGOGICAL UNIVERSITY, VOLOGDA, RUSSIA

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VOLOGDA STATE PEDAGOGICAL UNIVERSITY, VOLOGDA, RUSSIA

VOLOGDA STATE PEDAGOGICAL UNIVERSITY, VOLOGDA, RUSSIA

VOLOGDA STATE PEDAGOGICAL UNIVERSITY, VOLOGDA, RUSSIA

Current address: Vologda Scientific Coordinate Centre of Central Economics and Mathematics Institute, Russian Academy of Sciences, Vologda, Russia

E-mail address: zai@uni-vologda.ac.ru

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