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TWO-BOUNDARY PROBLEMS FOR A RANDOM WALK WITH NEGATIVE GEOMETRIC JUMPS

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Abstract. Two-boundary problems for a random walk with negative geometric jumps are considered, and the corresponding results for a usual semicontinuous random walk are generalized for them. The following results are obtained: the probability distribution of ruin is found and expressed in terms of the lower and upper boundaries; formulas are given for the joint distribution of the infimum, supremum, and the walk itself at an arbitrary time instance; the transient probabilities and ergodic distribution are evaluated for the process describing the evolution of the random walk with two boundaries.

Two-boundary problems for random walks and stochastic processes have several applications in the queue theory, storage and inventory theories, reliability theory, and in many other fields.

Two-boundary problems have been studied for semicontinuous random walks and for semicontinuous stochastic processes. Several methods are known for solving those problems, namely combinatorial [1], resolvent [2]–[6], factorization [7], and renewal [8] methods.

In this paper we solve two-boundary problems for random walks with negative geometric jumps. This model is a generalization of a usual model of semicontinuous random walks.

1. Main notation

Let $\alpha \in \{0, 1, \dots\}$ be a nonnegative integer-valued random variable,

$$\mathsf{E}[\theta^\alpha] = \sum_{i=0}^{\infty} a_i \theta^i, \qquad a_i = \mathsf{P}[\alpha = i], \qquad \mathsf{P}[\alpha > 1] > 0, \qquad |\theta| \le 1.$$

Consider the random variable

$$\xi = \alpha - \beta, \qquad \xi \in \{0, \pm 1, \dots\} = \mathbf{Z},$$

where $\beta \in \{1, 2, \dots\}$ is a positive integer-valued random variable distributed geometrically with parameter $b \in [0, 1)$, that is,

$$\mathsf{P}[\beta = n] = (1-b)b^{n-1}, \qquad n > 0.$$

It is clear that

$$\mathsf{E}[\theta^\xi] = \mathsf{E}[\theta^\alpha] \frac{(1-b)/\theta}{1 - b/\theta} \overset{\text{def}}{=} \sum_{i=-\infty}^{\infty} p_i \theta^i, \qquad |\theta| = 1.$$

An easy evaluation yields the distribution of the random variable ξ:

$$p_i = \mathsf{P}[\xi = i] = (1-b) b^{-i-1}\, \mathsf{E}[b^\alpha; \alpha > i], \qquad i \in \mathbf{Z}. \tag{1}$$

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Some of the properties of the distribution p_i, $i \in \mathbf{Z}$, are listed in the following result.

Lemma 1. *For all* $i \in \mathbf{Z}$,

$$bp_i - p_{i-1} = \begin{cases} 0, & i < 0, \\ -(1-b)a_i, & i \geq 0, \end{cases} \tag{2}$$

$$b\,\mathsf{P}[\xi \geq i] - \mathsf{P}[\xi \geq i-1] = -(1-b)\,\mathsf{P}[\alpha \geq i],$$

$$b\,\mathsf{P}[\xi \leq i] - \mathsf{P}[\xi \leq i-1] = -(1-b)\,\mathsf{P}[\alpha \leq i],$$

where $\mathsf{P}[\alpha = m] = 0$ *for* $m < 0$.

These equalities follow from the explicit expression (1) for the distribution of the random variable ξ.

Let $\{\xi, \xi_i'\}$, $i > 0$, be a sequence of independent identically distributed random variables and let ξ_n, $n \geq 0$, be a random walk such that

$$\xi_0 = 0, \qquad \xi_n = \xi_1' + \cdots + \xi_n', \quad n > 0.$$

The moment generating function of increments of the random walk ξ_n, $n \geq 0$, satisfies

$$\sum_{n=0}^{\infty} t^n\,\mathsf{E}\left[\theta^{\xi_n}\right] = \frac{1}{1 - t\,\mathsf{E}[\theta^{\xi}]} = \left\{1 - t\,\mathsf{E}[\theta^{\alpha}]\frac{(1-b)/\theta}{1-b/\theta}\right\}^{-1}, \qquad |\theta| = 1,\ t \in [0,1).$$

The equation

$$1 - \frac{b}{\theta} - (1-b)\frac{t}{\theta}\,\mathsf{E}[\theta^{\alpha}] = 0$$

has a unique root $\theta = c(t)$ in the domain $|\theta| < 1$. Moreover the root is positive and $b < c(t) < 1$. This result is well known for semicontinuous random walks [9]; the proof for the case under consideration is similar.

Definition 1. A sequence $R_k(t)$, $k > 0$, is called the resolvent of the random walk ξ_n, $n \geq 0$, if

$$R(\theta, t) = \sum_{k=1}^{\infty} \theta^k R_k(t) = \left[\frac{b}{\theta} + (1-b)\frac{t}{\theta}\,\mathsf{E}[\theta^{\alpha}] - 1\right]^{-1}, \qquad |\theta| < c(t). \tag{3}$$

Since the right-hand side of (3) is analytic for $|\theta| < c(t)$, it can be expanded in a power series. In what follows we need the first term of the sequence $R_k(t)$, $k > 0$:

$$R_1(t) = (b + (1-b)t\,\mathsf{P}[\alpha = 0])^{-1}.$$

Definition 2. A sequence R_k, $k > 0$, is called the potential of the random walk ξ_n, $n \geq 0$, if

$$R(\theta) = \sum_{k=1}^{\infty} \theta^k R_k = \left[\frac{b}{\theta} + (1-b)\frac{1}{\theta}\,\mathsf{E}[\theta^{\alpha}] - 1\right]^{-1}, \qquad |\theta| < c(1) \leq 1. \tag{4}$$

The following two equalities are used later when solving two-boundary problems.

Lemma 2. *For all* $k > 0$,

$$R_k(t) + t(1-b)b^{-k-1}\sum_{i=1}^{k} R_i(t)b^i\,\mathsf{E}[b^{\alpha}; \alpha \leq k-i] = b^{-k}, \tag{5}$$

$$S_k(t) + t(1-b)b^{-k-1}\sum_{i=1}^{k} S_i(t)b^i\,\mathsf{E}[b^{\alpha}; \alpha \leq k-i] = \frac{1}{1-b}[b^{-k} - 1], \tag{6}$$

where

$$S_k(t) = \sum_{i=1}^{k} R_i(t).$$

Proof. Denote the left-hand side of (5) by A_k, $k > 0$. Then

$$\sum_{k=1}^{\infty} \theta^k A_k = R(\theta, t) + t(1-b)\frac{1}{b}R(\theta, t)\frac{\mathsf{E}[\theta^\alpha]}{1-\theta/b}, \qquad |\theta| < b.$$

It follows from the definition of the resolvent (3) that

$$R(\theta, t)\left[\frac{b}{\theta} + (1-b)\frac{t}{b}\,\mathsf{E}[\theta^\alpha] - 1\right] = 1, \qquad |\theta| < c(t),$$

whence

$$\sum_{k=1}^{\infty} \theta^k A_k = \frac{\theta}{b}\left\{1 - \frac{\theta}{b}\right\}^{-1}, \qquad |\theta| < b.$$

Comparing the coefficients for θ^k, $k > 0$, on both sides of the latter equality, we get $A_k = b^{-k}$, $k > 0$. Equality (6) is proved similarly. □

Now we turn to solving the two-boundary problems.

2. Ruin problem

Fix an integer $N > 1$ and put $\mathsf{P}[\xi_0 = k] = 1$, $k \in \{1, \dots, N-1\}$. Let

$$\tau_k = \inf\{n > 0 \colon \xi_n \notin \{1, \dots, N-1\}\}$$

be the first moment when the random walk ξ_n, $n \geq 0$, exits the set $\{1, \dots, N-1\}$. We also introduce the first moments when the random walk ξ_n, $n \geq 0$, exits the set $\{1, \dots, N-1\}$ through the upper and lower boundary, respectively:

$$\tau_k^+ = \inf\bigl\{n > 0 \colon \xi_n \notin \{1, \dots, N-1\}, \xi_{\tau_k} \geq N\bigr\},$$
$$\tau_k^- = \inf\bigl\{n > 0 \colon \xi_n \notin \{1, \dots, N-1\}, \xi_{\tau_k} \leq 0\bigr\}.$$

Theorem 1. *Let*

$$\mathsf{E}\left[t^{\tau_k^-}\right] = \mathsf{E}[t^{\tau_k}; \xi_{\tau_k} \leq 0], \qquad \mathsf{E}\left[t^{\tau_k^+}\right] = \mathsf{E}[t^{\tau_k}; \xi_{\tau_k} \geq N]$$

be the moment generating functions of the "ruin" times τ_k^- and τ_k^+, $k \in \{1, \dots, N-1\}$. Then

$$\mathsf{E}\left[t^{\tau_k^-}\right] = \frac{b^N}{1-b}\frac{R_{N-k}(t)}{\hat{R}_N(b,t)},$$

$$\mathsf{E}\left[t^{\tau_k^+}\right] = 1 - \frac{R_{N-k}(t)}{\hat{R}_N(b,t)}\left[\frac{b^N}{1-b} + (1-b)(1-t)\hat{S}_N(b,t)\right] + (1-b)(1-t)S_{N-k}(t), \tag{7}$$

where $R_k(t)$, $k > 0$, is the resolvent (3) of the random walk ξ_n, $n \geq 0$,

$$\hat{R}_N(b,t) = \sum_{i=N}^{\infty} b^i R_i(t), \qquad S_k(t) = \sum_{i=1}^{k} R_i(t), \qquad \hat{S}_N(b,t) = \sum_{i=N}^{\infty} b^i S_i(t).$$

Corollary 1. *Let*

$$\mathsf{P}_k^- = \mathsf{P}[\xi_{\tau_k} \leq 0] \quad \textit{and} \quad \mathsf{P}_k^+ = \mathsf{P}[\xi_{\tau_k} \geq N]$$

be the "ruin" probabilities corresponding to the lower and upper boundaries, respectively. Denote by $\mathsf{E}[\tau_k]$ *the mean "ruin" time. Then*

$$\mathsf{P}_k^- = \frac{b^N}{1-b}\frac{R_{N-k}}{\hat{R}_N(b)}, \qquad \mathsf{P}_k^+ = 1 - \frac{b^N}{1-b}\frac{R_{N-k}}{\hat{R}_N(b)} = 1 - \mathsf{P}_k^-,$$

$$\mathsf{E}[\tau_k] = b^N \frac{R_{N-k}}{\hat{R}_N(b)} S_{N-1} + R_{N-k} - (1-b)S_{N-k},$$

where R_k*,* $k > 0$*, is the potential of the random walk* ξ_n*,* $n \ge 0$*,*

$$\hat{R}_N(b) = \sum_{i=N}^{\infty} b^i R_i, \qquad S_k = \sum_{i=1}^{k} R_i.$$

Proof of Theorem 1. Let $\mathsf{P}[\xi_0 = N-k] = 1$, $k \in \{1,\dots,N-1\}$. By the full probability formula, the functions

$$\varphi_k(t) = \mathsf{E}\left[t^{\tau_{N-k}^-}\right], \qquad k \in \{1,\dots,N-1\},\ t \in [0,1),$$

satisfy the following system of equations:

$$\varphi_k(t) = t\,\mathsf{P}[\xi \le k-N] + t\sum_{i=1}^{N-1} \varphi_i(t) p_{k-i}, \qquad k \in \{1,\dots,N-1\}. \tag{8}$$

System (8) is regular, that is,

$$t\sum_{i=1}^{N-1} p_{k-i} < 1 - tb^{N-2}\,\mathsf{E}[b^\alpha].$$

Therefore it has a unique solution (see [10]). We are going to find this solution in an explicit form. When solving this problem we follow a method that was useful for solving other two-boundary problems.

The functions $b\varphi_{k+1}(t)$ are such that

$$b\varphi_{k+1}(t) = tb\,\mathsf{P}[\xi \le k+1-N] + tb\sum_{i=1}^{N-1}\varphi_i(t)p_{k+1-i}, \qquad k \in \{0,\dots,N-2\}.$$

Subtracting this equation from (8) and applying (2) we get

$$b\varphi_{k+1}(t) - \varphi_k(t) = -t(1-b)\sum_{i=1}^{k+1}\varphi_i(t)a_{k+1-i}, \qquad k \in \{1,\dots,N-2\}. \tag{9}$$

Consider equation (9) for all $k \ge 1$:

$$b\varphi_{k+1}(t) - \varphi_k(t) = -t(1-b)\sum_{i=1}^{k+1}\varphi_i(t)a_{k+1-i}, \qquad k \ge 1, \tag{10}$$

and solve it by using the moment generating function method. Put

$$\Phi(\theta,t) = \sum_{k=1}^{\infty}\theta^k\varphi_k(t), \qquad |\theta| < c(t).$$

Rewriting equation (10) for the corresponding moment generating functions we obtain

$$\Phi(\theta,t)\left(\frac{b}{\theta} + (1-b)\frac{t}{\theta}\,\mathsf{E}[\theta^\alpha] - 1\right) = \varphi_1(t)\frac{1}{R_1(t)}, \qquad |\theta| < c(t).$$

By the definition of the resolvent (3) we get from the latter equation that

$$\varphi_k(t) = \varphi_1(t)\frac{R_k(t)}{R_1(t)}, \qquad k \geq 1. \tag{11}$$

The function $\varphi_k(t)$ in (11) satisfies equation (10) for all $k \geq 1$, thus it is a solution of equation (9) for $k \in \{1, \dots, N-2\}$. A solution of initial equation (8) belongs to the set of solutions of equation (9) for which the function $\varphi_1(t)$ is a free parameter. It remains to find a function $\varphi_1^*(t)$ such that $\varphi_k(t)$ defined by (11) satisfies equation (8). Substituting the function $\varphi_k(t)$ from (11) into the initial equation (8) and using Lemma 2 we get

$$\varphi_1^*(t) = \frac{b^N}{1-b}\frac{R_1(t)}{\hat{R}_N(b,t)}, \qquad \hat{R}_N(b,t) = \sum_{i=N}^{\infty} R_i(t) b^i.$$

Thus

$$\varphi_k(t) = \mathsf{E}\left[t^{\tau_{N-k}^-}\right] = \frac{b^N}{1-b}\frac{R_k(t)}{\hat{R}_N(b,t)}, \qquad k \in \{1, \dots, N-1\},$$

and the first equality of Theorem 1 is proved.

Now we prove the second equality. Let $\mathsf{P}[\xi_0 = N-k] = 1$, $k \in \{1, \dots, N-1\}$. By the full probability formula, the functions

$$\psi_k(t) = \mathsf{E}\left[t^{\tau_{N-k}}; \xi_{\tau_{N-k}} \geq N\right], \qquad t \in [0,1),$$

satisfy the system of equations

$$\psi_k(t) = t\,\mathsf{P}[\xi \geq k] + t\sum_{i=1}^{N-1}\psi_i(t)p_{k-i}, \qquad k \in \{1, \dots, N-1\}. \tag{12}$$

The equation for $b\psi_{k+1}(t)$ is given by

$$b\psi_{k+1}(t) = tb\,\mathsf{P}[\xi \geq k+1] + tb\sum_{i=1}^{N-1}\psi_i(t)p_{k+1-i}, \qquad k \in \{0, \dots, N-2\}.$$

Subtracting this equation from (12) we obtain that for all $k \geq 1$,

$$b\psi_{k+1}(t) - \psi_k(t) = -t(1-b)\sum_{i=1}^{k+1}\psi_i(t)a_{k+1-i} - t(1-b)\,\mathsf{P}[\alpha \geq k+1], \qquad k \geq 1. \tag{13}$$

Put

$$\Psi(\theta, t) = \sum_{k=1}^{\infty}\theta^k\psi_k(t), \qquad |\theta| < c(t).$$

Rewriting equation (13) for the corresponding moment generating functions and using the definition of the resolvent (3) we get

$$\Psi(\theta,t) = \frac{\theta}{1-\theta} + \frac{\psi_1(t)-1}{R_1(t)}R(\theta,t) + (1-b)(1-t)\frac{\theta}{1-\theta}R(\theta,t), \qquad |\theta| < c(t).$$

Comparing the coefficients for θ^k, $k > 0$, on both sides of the latter equality we obtain

$$\psi_k(t) = 1 + \frac{\psi_1(t)-1}{R_1(t)}R_k(t) + (1-b)(1-t)S_{k-1}(t), \qquad k \geq 1. \tag{14}$$

To find a function $\psi_1^*(t)$ such that $\psi_k(t)$ defined by (14) satisfies the initial system of equations (12), we substitute the right-hand side of (14) into the system of equations (12) and apply Lemma 2. Then we obtain the following relation:

$$\frac{\psi_1^*(t)-1}{R_1(t)} = -\frac{1}{\hat{R}_N(b,t)}\left[\frac{b^N}{1-b} + (1-b)(1-t)\hat{S}_N(b,t)\right] + (1-b)(1-t),$$

where

$$\hat{R}_N(b,t)=\sum_{i=N}^{\infty} b^i R_i(t), \qquad \hat{S}_N(b,t)=\sum_{i=N}^{\infty} b^i S_i(t).$$

Therefore

$$\psi_k(t)=1-\frac{R_k(t)}{\hat{R}_N(b,t)}\left[\frac{b^N}{1-b}+(1-b)(1-t)\hat{S}_N(b,t)\right]+(1-b)(1-t)S_k(t),$$
$$k\in\{1,\dots,N-1\},$$

and the second equality of Theorem 1 is proved.

Further

$$\sum_{n=0}^{\infty} t^n\,\mathsf{P}[\tau_{N-k}>n]=\frac{1-\varphi_k(t)-\psi_k(t)}{1-t}=\frac{(1-b)\hat{S}_N(b,t)}{\hat{R}_N(b,t)}R_k(t)-(1-b)S_k(t).$$

Since

$$(1-b)\hat{S}_N(b,t)=\hat{R}_N(b,t)+b^N S_{N-1}(t),$$

we have

$$\sum_{n=0}^{\infty} t^n\,\mathsf{P}[\tau_{N-k}>n]=b^N\frac{R_k(t)}{\hat{R}_N(b,t)}S_{N-1}(t)+R_k(t)-(1-b)S_k(t),$$

whence equality (7) follows. Corollary 1 follows from Theorem 1 for $t=1$. □

3. THE JOINT DISTRIBUTION OF $\inf\{\xi_0,\dots,\xi_n\}$, $\sup\{\xi_0,\dots,\xi_n\}$, AND ξ_n, $n\geq 0$

Assume that $\mathsf{P}[\xi_0=0]=1$ and consider the random variables

$$\mu_n^+=\sup\{\xi_0,\dots,\xi_n\}, \qquad \mu_n^-=\inf\{\xi_0,\dots,\xi_n\}.$$

Let k and r be positive integers, $R_m(t)=0$, $m\leq 0$.

Theorem 2. *The joint distribution*

$$\mathsf{P}^n_{(-r,k)}[i]=\mathsf{P}[-r<\mu_n^-,\xi_n=i,\mu_n^+<k], \qquad -r<i<k,$$

of the random variables μ_n^-, ξ_n, *and* μ_n^+ *satisfies*

$$\sum_{n=0}^{\infty} t^n\,\mathsf{P}^n_{(-r,k)}[i]=b^N\frac{R_k(t)}{\hat{R}_N(b,t)}R_{i+r}(t)+bR_{i+1}(t)-R_i(t), \qquad N=k+r.$$

Proof of Theorem 2. Fix $N>1$ and assume that $\mathsf{P}[\xi_0=N-k]=1$, $k\in\{1,\dots,N-1\}$. Let

$$\tau_{N-k}=\inf\{n>0\colon \xi_n\notin\{1,\dots,N-1\}\}$$

be the first moment when the random walk ξ_n, $n\geq 0$, exits the set $\{1,\dots,N-1\}$. Denote by

$$Q_{k,d}(n)=\mathsf{P}[\xi_n=N-d,\tau_{N-k}>n], \qquad d\in\{1,\dots,N-1\},$$

the transient probability of the random walk ξ_n, $n>0$, on the interval $0\leq n<\tau_{N-k}$.

By the full probability formula, the moment generating functions

$$Q^t_{k,d}=\sum_{n=0}^{\infty} t^n Q_{k,d}(n), \qquad k,d\in\{1,\dots,N-1\},\ t\in[0,1),$$

satisfy the system of equations

$$Q^t_{k,d}=\delta_{k,d}+t\sum_{i=1}^{N-1} p_{k-i}Q^t_{i,d}, \qquad k\in\{1,\dots,N-1\}, \tag{15}$$

where $\delta_{k,r}$ is the Kronecker symbol. The corresponding equation for $bQ^t_{k+1,d}$ is

$$bQ^t_{k+1,d} = b\delta_{k+1,d} + tb\sum_{i=1}^{N-1} p_{k+1-i}Q^t_{i,d}, \qquad k \in \{0,\dots,N-2\}.$$

Subtracting this equation from (15) we obtain that for all $k > 0$,

$$bQ^t_{k+1,d} - Q^t_{k,d} = b\delta_{k+1,d} - \delta_{k,d} - t(1-b)\sum_{i=1}^{k+1} a_{k+1-i}Q^t_{i,d}. \tag{16}$$

Consider the moment generating function

$$Q^t_d(\theta) = \sum_{k=1}^{\infty}\theta^k Q^t_{k,d}, \qquad |\theta| < c(t).$$

Rewriting (16) for the corresponding moment generating functions we obtain

$$Q^t_d(\theta) = \left(\frac{Q^t_{1,d}}{R_1(t)} - b\delta_{1,d}\right)R(\theta,t) + b\theta^{d-1}R(\theta,t) - \theta^d R(\theta,t), \qquad |\theta| < c(t),$$

where $R(\theta,t)$ is the moment generating function of resolvent (3) for the random walk ξ_n, $n > 0$. Comparing the coefficients for θ^k, $k > 0$, and putting $R_m(t) = 0$, $m \le 0$, we get

$$Q^t_{k,d} = \left(\frac{Q^t_{1,d}}{R_1(t)} - b\delta_{1,d}\right)R_k(t) + bR_{k-d+1}(t) - R_{k-d}(t). \tag{17}$$

We look for a function $Q^t_{1,d} = \tilde{Q}^t_{1,d}$ for which solution (17) of the infinite system (16) is also a solution of the initial system. Substituting expression (17) into the initial system (15) we get by Lemma 2 that

$$\frac{\tilde{Q}^t_{1,d}}{R_1(t)} - b\delta_{1,d} = b^N\frac{R_{N-d}(t)}{\hat{R}_N(b,t)}.$$

Combining this expression and (17) we obtain

$$Q^t_{k,d} = b^N\frac{R_{N-d}(t)}{\hat{R}_N(b,t)}R_k(t) + bR_{k-d+1}(t) - R_{k-d}(t), \qquad k,d \in \{1,\dots,N-1\}. \tag{18}$$

Now put $\mathsf{P}[\xi_0 = 0] = 1$, and let k and r be positive integers, $N = k + r$. Let

$$\tau = \inf\{n > 0\colon \xi_n \notin \{-r+1,\dots,k-1\}\}$$

be the first moment when the random walk ξ_n, $n > 0$, exits the set

$$\{-r+1,\dots,k-1\}.$$

Denote by

$$\mathsf{P}^n_{(-r,k)}[i] = \mathsf{P}[\xi_n = i, \tau > n], \qquad -r < i < k,$$

the transient probability for the random walk ξ_n on the interval $0 \le n < \tau$. Since the random walk ξ_n, $n \ge 0$, is homogeneous, we find from (18) that

$$\sum_{n=0}^{\infty} t^n\,\mathsf{P}^n_{(-r,k)}[i] = b^N\frac{R_k(t)}{\hat{R}_N(b,t)}R_{i+r}(t) + bR_{i+1}(t) - R_i(t), \qquad -r < i < k.$$

Since

$$\mathsf{P}[\tau > n] = \mathsf{P}\left[-r < \mu^-_n, \mu^+_n < k\right],$$

we have

$$\mathsf{P}^n_{(-r,k)}[i] = \mathsf{P}\left[-r < \mu^-_n, \xi_n = i, \mu^+_n < k\right], \qquad -r < i < k,$$

whence Theorem 2 follows. □

4. A random walk ξ_n, $n \geq 0$, and two absorbing boundaries

Fix a positive integer N and consider a random sequence $X_n \in \{0, 1, \dots, N\}$, $n \geq 0$, whose one-step transient probabilities are given by

$$k \to \begin{cases} i, & \mathsf{P}[\xi = i - k],\ i \in \{1, \dots, N-1\}, \\ 0, & \mathsf{P}[\xi \leq -k], \\ N, & \mathsf{P}[\xi \geq N - k] \end{cases} \tag{19}$$

for all $k \in \{0, \dots, N\}$.

The homogeneous (discrete time) Markov process X_n, $n \geq 0$, describes the evolution of the random walk ξ_n, $n \geq 0$, in the strip $0, \dots, N$ with two reflecting boundaries with absorption (the upper boundary is at the level N, while the lower one is at the level 0).

If the process X_n, $n \geq 0$, reaches the boundary 0 at a moment n_0, it stays there for a random time $\nu(0)$ until the first positive jump of the random walk ξ_{n-n_0}, $n \geq n_0$, occurs. Then the process is "reflected" and appears at a state $i \in \{1, \dots, N-1\}$ with probability $\mathsf{P}[\xi = i]$ (the probability of its appearing on the upper boundary is $\mathsf{P}[\xi \geq N]$). It is clear that $\nu(0)$ is a geometric random variable and

$$\mathsf{P}[\nu(0) = n] = \mathsf{P}[\xi > 0](\mathsf{P}[\xi \leq 0])^{n-1}, \qquad n > 0.$$

If the process X_n, $n \geq 0$, reaches the upper boundary N at a moment n_N, it stays there for a random time $\nu(N)$ until the first negative jump of the random walk ξ_{n-n_N}, $n \geq n_N$, occurs. Then the process is "reflected" and appears at a state $i \in \{1, \dots, N-1\}$ with probability $\mathsf{P}[\xi = i - N]$ (the probability of its appearing on the lower boundary is $\mathsf{P}[\xi \leq -N]$). It is clear that $\nu(N)$ is a geometric random variable and

$$\mathsf{P}[\nu(N) = n] = \mathsf{P}[\xi < 0](\mathsf{P}[\xi \geq 0])^{n-1}, \qquad n > 0.$$

Random walks with two reflecting boundaries are useful for queueing systems having a bounded number of waiting places as well as in storage and inventory theories, and in other applied probability fields.

4.1. Hitting the boundaries.

Let $\mathsf{P}[X_0 = k] = 1$, and let

$$\tau_k^0 = \inf\{n > 0 \colon X_n = 0\}, \qquad k \in \{1, \dots, N\},$$

be the first time when the process X_n, $n \geq 0$, reaches the lower boundary 0. Also let

$$\tau_k^N = \inf\{n > 0 \colon X_n = N\}, \qquad k \in \{0, \dots, N-1\},$$

be the first time when the process X_n, $n \geq 0$, reaches the upper boundary N.

Theorem 3. *The moment generating functions of the random variables τ_k^0 and τ_k^N and their means $\mathsf{E}[\tau_k^0]$ and $\mathsf{E}[\tau_k^N]$ satisfy*

$$\mathsf{E}\left[t^{\tau_k^0}\right] = \frac{b^N}{1-b} \frac{1 + (1-b)(1-t)S_{N-k}(t)}{b^N/(1-b) + (1-b)(1-t)\hat{S}_N(b,t)},$$

$$\mathsf{E}\left[\tau_k^0\right] = (1-b)\left[b^{-N}\hat{R}_N(b) + S_{N-1} - S_{N-k}\right],$$

$$\sum_{n=0}^{\infty} t^n\, \mathsf{P}\left[\tau_k^N > n\right] = (1-b)\left\{\frac{\hat{R}_{N+1}(b,t)}{\hat{R}_{N+1}(b,t) - b\hat{R}_N(b,t)} R_{N-k}(t) - S_{N-k}(t)\right\},$$

$$\mathsf{E}\left[\tau_k^N\right] = (1-b)\left\{\frac{\hat{R}_{N+1}(b)}{\hat{R}_{N+1}(b) - b\hat{R}_N(b)} R_{N-k} - S_{N-k}\right\},$$

where

$$\hat{R}_N(b,t) = \sum_{i=N}^{\infty} b^i R_i(t), \qquad \hat{R}_N(b) = \sum_{i=N}^{\infty} b^i R_i = \hat{R}_N(b,1),$$

$$\hat{S}_N(b,t) = \sum_{i=N}^{\infty} b^i S_i(t), \qquad S_k(t) = \sum_{i=1}^{k} R_i(t).$$

Proof of Theorem 3. Let $\mathsf{P}[X_0 = N-k] = 1$. Using the transient probabilities (19) of the process X_n, $n \geq 0$, we see that the moment generating function

$$\tilde{\varphi}_k(t) = \mathsf{E}\left[t^{\tau^0_{N-k}}\right], \qquad t \in (0,1),\ k \in \{0,\dots,N-1\},$$

satisfies the system of equations

$$\tilde{\varphi}_k(t) = t\,\mathsf{P}[\xi \leq k-N] + t\tilde{\varphi}_0(t)\,\mathsf{P}[\xi \geq k] + t\sum_{i=1}^{N-1} \tilde{\varphi}_i(t) p_{k-i}, \qquad k \in \{0,\dots,N-1\}, \tag{20}$$

by the full probability formula. System (20) is strongly regular, that is,

$$t\,\mathsf{P}[\xi \geq k] + t\sum_{i=1}^{N-1} \mathsf{P}[\xi = k-i] = t\,\mathsf{P}[\xi > k-N] < 1 - tb^{N-1}\,\mathsf{E}[b^{\alpha}].$$

Therefore it has a unique solution (see [10]).

Using the transient probabilities (19) of the process X_n, $n \geq 0$, we prove that the moment generating function

$$\tilde{\psi}_k(t) = \mathsf{E}\left[t^{\tau^N_{N-k}}\right], \qquad k \in \{1,\dots,N\},\ t \in [0,1),$$

satisfies the system of equations

$$\tilde{\psi}_k(t) = t\,\mathsf{P}[\xi \geq k] + t\,\mathsf{P}[\xi \leq k-N]\tilde{\psi}_N(t) + t\sum_{i=1}^{N} \tilde{\psi}_i(t) p_{k-i}, \qquad k \in \{1,\dots,N\}, \tag{21}$$

by the full probability formula. The latter system is strongly regular, since

$$t\,\mathsf{P}[\xi < k-N] + t\,\mathsf{P}[k-N \leq \xi \leq k-1] < t\,\mathsf{P}[\xi \leq N-1] < 1 - t\,\mathsf{E}\left[1 - b^{\alpha-N}; \alpha > N\right].$$

Thus it has a unique solution (see [10]).

Applying to systems (20) and (21) the method used in the preceding sections for solving corresponding systems of equations, we complete the proof of Theorem 3. ☐

4.2. Transient probabilities and ergodic distribution of the process X_n, $n \geq 0$.

Let $N > 0$ and let ν_t be a geometric random variable with parameter t: $\mathsf{P}[\nu_t = n] = (1-t)t^n$, $n \geq 0$, $t \in [0,1)$.

Theorem 4. *Let* $\mathsf{P}[X_0 = k] = 1$, $k \in \{0,\dots,N-1\}$. *Then*

$$\begin{aligned}\mathsf{P}[X_{\nu_t} = i] = {} & \frac{b^N}{1-b}\frac{R_{i+1}(t) - R_i(t)}{\hat{R}_{N+1}(b,t)}\left[1 + (1-b)(1-t)S_{N-k}(t)\right] \\ & + (1-t)\left[bR_{i-k+1}(t) - R_{i-k}(t)\right], \qquad i \in \{0,\dots,N-1\},\end{aligned}$$

$$\mathsf{P}[X_{\nu_t} = N] = \frac{\hat{R}_{N+1}(b,t) - b\hat{R}_N(b,t)}{(1-b)\hat{R}_{N+1}(b,t)}\left[1 + (1-b)(1-t)S_{N-k}(t)\right] - (1-t)R_{N-k}(t).$$

Moreover there exists the ergodic distribution

$$\Pi_i = \lim_{n\to\infty} \mathsf{P}[X_n = i], \qquad i \in \{0,\dots,N\},$$

of the process X_n, $n \geq 0$, such that

$$\Pi_i = \frac{b^{N+1}}{1-b}\frac{R_{i+1}-R_i}{\hat{R}_{N+1}(b)}, \quad i \in \{0,\dots,N-1\}, \qquad \Pi_N = \frac{\hat{R}_{N+1}(b) - b\hat{R}_N(b)}{(1-b)\hat{R}_{N+1}(b)}.$$

Proof of Theorem 4. Let $\mathsf{P}[X_0 = N-k] = 1$, $k \in \{0,\dots,N\}$. Denote by

$$\mathsf{P}_{kr}(n) = \mathsf{P}[X_n = N-r], \qquad r \in \{0,\dots,N\},$$

the transient probability of the process X_n, $n \geq 0$. By the full probability formula and (19), the functions

$$\mathsf{P}^t_{k,r} = \sum_{n=0}^{\infty} t^n\,\mathsf{P}_{kr}(n), \qquad t \in [0,1),$$

satisfy the system

$$\mathsf{P}^t_{k,r} = \delta_{k,r} + t\,\mathsf{P}^t_{N,r}\,\mathsf{P}[\xi < k-N] + t\,\mathsf{P}^t_{0,r}\,\mathsf{P}[\xi \geq k] + t\sum_{i=1}^{N}\mathsf{P}^t_{i,r}\,p_{k-i}, \tag{22}$$
$$k,r \in \{0,\dots,N\}.$$

This problem is the most complicated among similar problems considered in this paper. Nevertheless the method used for solving the preceding two-boundary problems works in this case, too.

The functions $b\,\mathsf{P}^t_{k+1,r}$, $k \in \{-1,\dots,N-1\}$, satisfy the following equations:

$$b\,\mathsf{P}^t_{k+1,r} = b\delta_{k+1,r} + tb\,\mathsf{P}^t_{N,r}\,\mathsf{P}[\xi < k+1-N] + tb\,\mathsf{P}^t_{0,r}\,\mathsf{P}[\xi \geq k+1] + tb\sum_{i=1}^{N}\mathsf{P}^t_{i,r}\,p_{k+1-i}.$$

Subtracting (22) from the latter equality we get that for $k \in \{0,\dots,N-1\}$,

$$b\,\mathsf{P}^t_{k+1,r} - \mathsf{P}^t_{k,r} = b\delta_{k+1,r} - \delta_{k,r} - t(1-b)\,\mathsf{P}^t_{0,r}\,\mathsf{P}[\alpha \geq k+1] - t(1-b)\sum_{i=1}^{k+1}\mathsf{P}^t_{i,r}\,a_{k+1-i}. \tag{23}$$

Put

$$\mathsf{P}_r(\theta,t) = \sum_{k=0}^{\infty}\theta^k\,\mathsf{P}^t_{k,r}, \qquad |\theta| < c(t).$$

Solving equations (23) by the moment generating function method we obtain that for all $k \geq 0$,

$$\mathsf{P}_r(\theta,t) = \frac{\mathsf{P}^t_{0,r}}{1-\theta}\,[1 + (1-b)(1-t)R(\theta,t)] + b(1-\delta_{r,0})\theta^{r-1}R(\theta,t) - \theta^r R(\theta,t),$$
$$|\theta| < c(t).$$

Comparing the coefficients for θ^k, $k \geq 0$, on both sides of this equality, we see that

$$\mathsf{P}^t_{k,r} = \mathsf{P}^t_{0,r}[1 + (1-b)(1-t)S_k(t)] + b(1-\delta_{r,0})R_{k+1-r}(t) - R_{k-r}(t), \tag{24}$$
$$R_m(t) = 0, \qquad m \leq 0.$$

Now we find a function $\mathsf{P}^*_{0,r}(t)$ such that $\mathsf{P}^t_{k,r}$ defined by (24) for $k \in \{0,\dots,N\}$ is a solution of the initial system (22). Substituting the right-hand side of (24) into (22) and applying Lemma 2 we prove that

$$\mathsf{P}^*_{00}(t) = \frac{\hat{R}_{N+1}(b,t) - b\hat{R}_N(b,t)}{(1-b)(1-t)\hat{R}_{N+1}(b,t)},$$

$$\mathsf{P}^*_{0r}(t) = \frac{b^{N+1}}{1-b}\frac{R_{N+1-r}(t) - R_{N-r}(t)}{(1-t)\hat{R}_{N+1}(b,t)}, \qquad r \in \{1,\dots,N\},$$

where

$$\hat{R}_N(b,t)=\sum_{i=N}^{\infty} b^i R_i(t).$$

Substituting $\mathsf{P}^*_{0r}(t)$ into (24) we find that for all $k,r\in\{0,\dots,N\}$,

$$\begin{aligned}\mathsf{P}^t_{k,r}&=\frac{b^{N+1}}{1-b}\frac{R_{N+1-r}(t)-R_{N-r}(t)}{(1-t)\hat{R}_{N+1}(b,t)}\,[1+(1-b)(1-t)S_k(t)]\\&\quad+bR_{k+1-r}(t)-R_{k-r}(t),\qquad r\in\{1,\dots,N\},\end{aligned}$$

$$\mathsf{P}^t_{k,0}=\frac{\hat{R}_{N+1}(b,t)-b\hat{R}_N(b,t)}{(1-t)(1-b)\hat{R}_{N+1}(b,t)}\,[1+(1-b)(1-t)S_k(t)]-R_k(t). \tag{25}$$

Since

$$\mathsf{P}[X_{\nu_t}=i]=(1-t)\,\mathsf{P}^t_{N-k,N-i},\qquad k,\,i\in\{0,\dots,N\},$$

the first two identities of Theorem 4 follow from (25).

The process X_n, $n\geq 0$, is a Markov chain with a finite set of states $\{0,\dots,N\}$. If $\mathsf{P}[\alpha>1]>0$, then there exists n_0 such that $\min_{k,r}\mathsf{P}_{k,r}(n_0)>0$. Then (see [11]) there exists the ergodic distribution

$$\Pi_i=\lim_{n\to\infty}\mathsf{P}[X_n=i],\qquad i\in\{0,\dots,N\}.$$

To get the ergodic distribution in an explicit form, one needs to evaluate the limits

$$\Pi_i=\lim_{t\to 1}(1-t)\,\mathsf{P}^t_{k,N-i},\qquad i\in\{0,\dots,N\}.$$

Multiplying equalities (25) by $(1-t)$ and passing to the limits as $t\to 1$, we get by changing $r\to N-i$ that

$$\Pi_i=\frac{b^{N+1}}{1-b}\frac{R_{i+1}-R_i}{\hat{R}_{N+1}(b)},\quad i\in\{0,\dots,N-1\},\qquad \Pi_N=\frac{\hat{R}_{N+1}(b)-b\hat{R}_N(b)}{(1-b)\hat{R}_{N+1}(b)}$$

where R_k, $k>0$, is potential (4) of the random walk ξ_n, $n\geq 0$. Theorem 4 is proved. □

Remark 1. Putting $b=0$ in our theorems and corollaries we obtain the corresponding results for a lower semicontinuous random walk ξ_n, $n\geq 0$, whose increments have the moment generating function

$$\sum_{n=0}^{\infty}t^n\,\mathsf{E}\left[\theta^{\xi_n}\right]=\left\{1-\frac{t}{\theta}\,\mathsf{E}[\theta^{\alpha}]\right\}^{-1},\qquad |\theta|=1,\ t\in(0,1).$$

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