

A STRONG LAW OF LARGE NUMBERS FOR GENERALIZED ALMOST SURE CENTRAL LIMIT THEOREMS

UDC 519.21

RITA GIULIANO ANTONINI AND LUCA PRATELLI

ABSTRACT. We prove a Strong Law of Large Numbers in which the variables are assumed to be asymptotically negligible and a generalized Almost Sure Central Limit Theorem is given. As an application we obtain a result about the so-called *intersective ASCLT*.

1. INTRODUCTION

In this paper we study the asymptotic behaviour of the “weighted” random variables

$$(1.1) \quad \frac{1}{\phi(n)} \sum_{k=1}^n (\phi(k) - \phi(k-1)) X_k,$$

where $(X_k)_{k \geq 1}$ is a sequence of mean square integrable random variables and ϕ is a function increasing to ∞ . Results concerning random variables such as (1.1) are known as *Almost Sure Central Limit Theorems* (ASCLT) (see Berkes–Csáki [3] for an exhaustive list of references). In Section 3 we state and prove our main result; it may be listed among the most comprehensive ones about the ASCLT, and we get it as an application of a generalized Strong Law of Large Numbers, where covariances of the random variables involved are assumed to be *asymptotically negligible* (see Definition 2.2). We point out that our result takes into account also *almost orthogonal* random sequences studied by M. Weber. As an application of the main result, in Section 4 we prove a new version of the so-called *intersective law*, more general than the one recently obtained by Giuliano Antonini and Weber [8].

2. PRELIMINARY RESULTS

In what follows we denote by $(X_n)_{n \geq 1}$ a sequence of L^2 -random variables and by

$$\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$$

a decreasing function with $\int_1^\infty \psi(t) dt < \infty$. We start with two definitions.

Definition 2.1. A sequence $(X_n)_n$ is *mean square controlled by a random variable* Z if there exists a number C such that, for each n , we have

$$\mathbb{E}[X_n^2] \leq C \int_{\{|Z| \leq n\}} Z^2 d\mathbb{P}.$$

If $\sup_n \mathbb{E}[X_n^2]$ is finite, $(X_n)_n$ is clearly mean square controlled by 1.

2000 *Mathematics Subject Classification.* Primary 60F15; Secondary 60F05.

Definition 2.2. A sequence $(X_n)_n$ has ψ -negligible covariance if, for every pair of integers p and q , with $p < q$, we have

$$|\text{Cov}(X_p, X_q)| \leq \psi(\log(q - p + 1)).$$

Notice that the condition of ψ -negligible covariance yields

$$\lim_{q-p \rightarrow \infty} \text{Cov}(X_p, X_q) = 0.$$

The following result turns out to be a very useful strong law of large numbers.

Lemma 2.3. Let $(X_n)_n$ be a sequence of positive random variables such that $\sup_n \mathbb{E}[X_n]$ is finite; assume moreover that $(X_n)_n$ has ψ -negligible covariance and is mean square controlled by an integrable random variable. Then the random variables

$$Y_n = \frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}[X_k])$$

converge to 0 in L^2 and almost surely.

Proof. Let $a \geq 1$ be fixed and put $a_n = \lfloor a^n \rfloor$. We note that $(a_n)_n$ is an ultimately increasing sequence of integers. Moreover, since X_n is positive, for any integer m with $a_n < m < a_{n+1}$ we have

$$\frac{1}{m} \left(a_n Y_{a_n} - \sum_{k=a_n+1}^m \mathbb{E}[X_k] \right) \leq Y_m \leq \frac{1}{m} \left(a_{n+1} Y_{a_{n+1}} + \sum_{k=m+1}^{a_{n+1}} \mathbb{E}[X_k] \right).$$

If $b_m = (a_{n+1} - m)/m$ and $c_m = (m - a_n)/m$, from the above relation we get

$$\begin{aligned} |Y_m| &\leq \left(|Y_{a_{n+1}}| \frac{a_{n+1}}{m} + b_m \sup_k \mathbb{E}[X_k] \right) \vee \left(|Y_{a_n}| \frac{a_n}{m} + c_m \sup_k \mathbb{E}[X_k] \right) \\ &\leq \left(|Y_{a_{n+1}}| a + 2(a-1) \sup_k \mathbb{E}[X_k] \right) \vee \left(|Y_{a_n}| + 2(a-1) \sup_k \mathbb{E}[X_k] \right) \\ &\leq |Y_{a_{n+1}}| a + |Y_{a_n}| + 2(a-1) \sup_k \mathbb{E}[X_k], \end{aligned}$$

since $a_{n+1} \leq am$ and $b_m \vee c_m \leq 2(a-1)$ for $m = a_n + 1 + \dots + a_{n+1}$. Hence it will be enough to prove the statement for the subsequence $(Y_{a_n})_n$; to this end we show that

$$\sum_{n \geq 1} \mathbb{E}[Y_{a_n}^2] < \infty.$$

First, note that

$$\sum_{n \geq 1} |Z| a_n^{-2} \sum_{k=1}^{a_n} I_{\{|Z| \leq k\}} = \sum_{k \geq 1} |Z| I_{\{|Z| \leq k\}} \sum_{a_n \geq k} a_n^{-2} \leq \frac{4a^2 |Z|}{a^2 - 1} \sum_{k \geq 1} I_{\{|Z| \leq k\}} k^{-2} \leq 8 + \frac{4}{a-1}.$$

Since $(X_n)_n$ has ψ -negligible covariance and is mean square controlled by an integrable random variable Z , we get

$$\begin{aligned} \sum_{n \geq 1} \mathbb{E}[Y_{a_n}^2] &\leq \sum_{n \geq 1} a_n^{-2} \sum_{k=1}^{a_n} \mathbb{E}[Z^2 I_{\{|Z| \leq k\}}] + 2 \sum_{n \geq 1} a_n^{-2} \sum_{q=2}^{a_n} \sum_{p=1}^{q-1} \psi(\log(q-p+1)) \\ &\leq \mathbb{E} \left[|Z| \sum_{n \geq 1} |Z| a_n^{-2} \sum_{k=1}^{a_n} I_{\{|Z| \leq k\}} \right] + 2 \sum_{n \geq 1} a_n^{-2} \sum_{q=2}^{a_n} \sum_{h=2}^q \psi(\log(h)) \\ &\leq (8 + 4(a-1)^{-1}) \mathbb{E}[|Z|] + 2 \sum_{n \geq 1} a_n^{-1} \sum_{h=2}^{a_n} \psi(\log(h)). \end{aligned}$$

Moreover, note that

$$\sum_{n \geq 1} a_n^{-1} \sum_{h=2}^{a_n} \psi(\log(h)) \sim \sum_{n \geq 1} a^{-n} \int_2^{a_n} \psi(\log(x)) dx \sim \sum_{n \geq 1} a^{-n} \int_{\log 2}^{n \log a} \psi(u) e^u du.$$

Thus, it suffices to prove that $\sum_{n \geq 1} a^{-n} \int_{\log 2}^{n \log a} \psi(u) e^u du < \infty$. We get the statement thanks to the integrability of ψ on $[1, \infty[$ since

$$\begin{aligned} \sum_{n \geq 1} a^{-n} \int_{\log 2}^{n \log a} \psi(u) e^u du &= \int_{\log 2}^{\infty} \psi(u) e^u \left(\sum_{n \geq 1} a^{-n} I_{\{u \leq n \log a\}} \right) du \\ &= \int_{\log 2}^{\infty} \psi(u) e^u \left(\sum_{n \geq u/\log a} a^{-n} \right) du \\ &\leq \int_{\log 2}^{\infty} \psi(u) e^u a^{-u/\log a} \frac{a}{a-1} du = \frac{a}{a-1} \int_{\log 2}^{\infty} \psi(u) du. \end{aligned}$$

This completes the proof of the theorem. □

Remark 2.4. The above argument shows that the statement of Lemma 2.3 holds also under the condition

$$\sum_{q=2}^n \sum_{p=1}^{q-1} \text{Cov}(X_p, X_q) \leq \psi_1(\log(n)), \tag{2.1}$$

where ψ_1 is a decreasing function with $\int_1^\infty \psi_1(t)/t^3 dt < \infty$. This condition does not necessarily imply that $(X_n)_n$ has ψ -negligible covariance and is weaker than the condition of Peligrad and Shao [14], used in a recent paper by Dudziński [6]. Moreover, we could also weaken the assumption that the X_n are positive random variables by only requiring that they are bounded from below by a constant.

We get a stronger result than Lemma 2.3 when $(X_n - \mathbb{E}[X_n])_n$ is an *almost-orthogonal* sequence, namely when the quadratic form defined on $l^2(\mathbb{N})$ by

$$(2.1) \quad (x_n)_n \mapsto \sum_{h,k} \text{Cov}(X_h, X_k) x_h x_k$$

is bounded. The condition of almost-orthogonality plays an important role in the study of random series such as $\sum_n c_n (X_n - \mathbb{E}[X_n])$ (see Kac, Salem, and Zygmund [10]) since, in this setting, the Rademacher–Menshov Theorem is still valid and $\sum_n c_n (X_n - \mathbb{E}[X_n])$ is almost surely convergent for any sequence $(c_n)_n$ of real numbers with

$$\sum_n c_n^2 \log^2 n < \infty.$$

In particular, the almost-orthogonality of $(X_n - \mathbb{E}[X_n])_n$ assures that, for any $\beta > 3$, the random variable $(n/\log^\beta n)^{1/2} Y_n$ converges to 0 almost surely. A condition which assures almost-orthogonality has been introduced by Weber [16]; it states that

$$(2.2) \quad \sup_{k \geq 1} \sum_{h=1}^{\infty} |\text{Cov}(X_h, X_k)| < \infty$$

implies the almost-orthogonality of $(X_n)_n$. By assuming (2.3) we are able to prove the following result.

Lemma 2.5. *Let $(X_n)_n$ be a sequence of random variables with ψ -negligible covariance and $\sup_n \text{Var}[X_n] < \infty$. If $\int_1^\infty \psi(u) \exp(u) du$ is finite, the sequence $(X_n - \mathbb{E}[X_n])_n$ is almost-orthogonal. In particular, for any $\beta > 3$, the random variable*

$$(2.3) \quad \frac{1}{\sqrt{n \log^\beta n}} \sum_{k=1}^n (X_k - \mathbb{E}[X_k])$$

converges to 0 almost surely and in L^2 .

Proof. The relation

$$\begin{aligned} \sup_k \sum_{h=1}^{\infty} |\text{Cov}(X_h, X_k)| &\leq \sup_k \text{Var}(X_k) + 2 \sum_{l=2}^{\infty} \psi(\log l) \\ &\leq \sup_k \text{Var}(X_k) + 2\psi(\log 2) + 2 \int_2^{\infty} \psi(\log(u)) du \\ &\leq \sup_k \text{Var}(X_k) + 4\psi(\log 2) + 2 \int_1^{\infty} \psi(u) e^u du \end{aligned}$$

yields immediately that the sequence $(X_n - \mathbb{E}[X_n])_n$ is almost-orthogonal and the random variable defined in (2.4) converges in L^2 . Thus, the lemma is proven by the above remarks on almost-orthogonality. \square

3. THE MAIN RESULT

With the notation of the previous section, let $(X_n)_n$ be a sequence of positive and L^2 -random variables. We prove the following result:

Theorem 3.1. *Let ϕ be an increasing positive function on \mathbb{R}^+ , with $\sup \phi = \infty$. Moreover, let ψ be a decreasing positive function on \mathbb{R}^+ , with $\int_0^\infty \psi(u) du < \infty$. For every pair p and q of integers, with $p < q$, suppose*

$$|\text{Cov}(X_p, X_q)| \leq \psi(\log(\phi(q) - \phi(p) + 1)).$$

Then

$$U_n = \frac{1}{\phi(n)} \sum_{k=1}^n (\phi(k) - \phi(k-1)) (X_k - \mathbb{E}[X_k])$$

converges to 0 in L^2 and almost surely. Moreover, if $\int_0^\infty \psi(u) \exp(u) du$ is finite, then for any $\beta > 3$, $U_n[\phi(n)/\log^\beta(\phi(n))]^{1/2}$ converges to 0 in L^2 and almost surely.

Proof. For every $t > 0$, put

$$V_t = \frac{1}{t} \sum_{k=1}^{\lfloor \phi^{-1}(t) \rfloor} (\phi(k) - \phi(k-1)) (X_k - \mathbb{E}[X_k]).$$

Since $U_n = V_{\phi(n)}$, it suffices to prove that, as t goes to ∞ , the random variable V_t converges to 0 in L^2 and almost surely. Since, for any real number $t \in]m, m+1]$, $m \in \mathbb{N}$, we have

$$\frac{1}{t} \left(mV_m - \sum_{k=\lfloor \phi^{-1}(m) \rfloor + 1}^{\lfloor \phi^{-1}(t) \rfloor} b_k \mathbb{E}[X_k] \right) \leq V_t \leq \frac{1}{t} \left((m+1)V_{m+1} - \sum_{k=\lfloor \phi^{-1}(t) \rfloor + 1}^{\lfloor \phi^{-1}(m+1) \rfloor} b_k \mathbb{E}[X_k] \right),$$

where $b_k = \phi(k) - \phi(k - 1)$, it will be enough to prove the theorem for $(V_m)_m$. To this end, for $k \geq 1$, put

$$\tilde{X}_k = \sum_{j=\lfloor \phi^{-1}(k-1) \rfloor + 1}^{\lfloor \phi^{-1}(k) \rfloor} (\phi(j) - \phi(j - 1))X_j$$

and $c_k = \lfloor \phi^{-1}(k) \rfloor$. (If $c_{k-1} + 1 > c_k$, we mean $\tilde{X}_k \equiv 0$.) Clearly \tilde{X}_k is a positive random variable that satisfies

$$\|\tilde{X}_k\|_{L^2} \leq \sum_{j=c_{k-1}+1}^{c_k} (\phi(j) - \phi(j - 1))\|X_j\|_{L^2} \leq \left(\sup_n E[X_n^2] \right)^{1/2}.$$

Moreover, for every pair p and q , with $p < q$, we have

$$\begin{aligned} \left| \text{Cov}(\tilde{X}_p, \tilde{X}_q) \right| &\leq \sum_{i=c_{p-1}+1}^{c_p} \sum_{j=c_{q-1}+1}^{c_q} b_i b_j \psi(\log(\phi(j) - \phi(i) + 1)) \\ &\leq \psi(\log(\phi(c_{q-1} + 1) - \phi(c_p) + 1)) \leq \psi(\log(q - \phi(c_p))) \\ &\leq \psi(\log(q - p)) \leq \tilde{\psi}(\log(q - p + 1)), \end{aligned}$$

where $\tilde{\psi}$ is the mapping $x \mapsto \psi((x - \log 2)^+)$; hence the sequence $(\tilde{X}_n)_n$ has $\tilde{\psi}$ -negligible covariance. Since V_m is equivalent to $(\tilde{X}_1 + \dots + \tilde{X}_m)/m$, the statement follows from Lemmas 2.3 and 2.5 by simply remarking that $\tilde{\psi}$ has the same integrability properties as ψ . \square

Remark 3.1. It may happen that the condition of ψ -negligibility is not satisfied by the whole sequence $(X_n)_n$, but by suitable subsequences $(X_{n_k})_k$ only. This is the case, for instance, for sequences $(X_n)_n$ such that $\liminf_n |\text{Cov}(X_n, X_{2n})| > 0$ but

$$(3.1) \quad |\text{Cov}(X_p, X_q)| \leq r \left(\frac{g(p)}{g(q)} \right),$$

where g is an increasing positive function on \mathbb{N} , with $\sup g = \infty$, and r is an increasing function vanishing at 0. More specifically, if for a subsequence $(n_k)_k$ (induced by an increasing map f defined on \mathbb{R}^+) there exists a constant $c > 0$ such that, for every pair p and q , with $p \leq q$,

$$(3.2) \quad cg(n_q) \geq g(n_{q-p+1})g(n_p),$$

then $(X_{n_k})_k$ has ψ -negligible covariance where $\psi = r \circ \frac{c}{g \circ f} \circ \exp$. So conditions (3.1) and (3.2) yield

$$|\text{Cov}(X_{n_p}, X_{n_q})| \leq \psi(\log(q - p + 1)).$$

In what concerns the integrability of the function ψ , we remark that the condition suggested by Weber, namely

$$(3.3) \quad \sup_m \sum_{h=1}^m r \left(\frac{g(h_h)}{g(n_m)} \right) + \sum_{h=m+1}^{\infty} r \left(\frac{g(n_m)}{g(n_h)} \right) < \infty,$$

assures even the integrability of the function $\psi \circ \log$ on $[e, \infty[$ (i.e., the integrability of $u \mapsto \psi(u)e^u$ on $[1, \infty[$). As a matter of fact, we have

$$\begin{aligned} \int_e^{\infty} \psi(\log t) dt &= \int_e^{\infty} r \left(\frac{c}{g(f(t))} \right) dt \leq \sum_{h=1}^{\infty} r \left(\frac{c}{g(n_h)} \right) \\ &\leq \bar{m}r \left(\frac{c}{g(n_1)} \right) + \sum_{h=\bar{m}+1}^{\infty} r \left(\frac{g(n_{\bar{m}})}{g(n_h)} \right), \end{aligned}$$

where \overline{m} is an integer such that $g(n\overline{m}) \geq c$. In other words, if the variances of $(X_{n_k})_k$ are bounded, conditions (3.1), (3.2), and (3.3) guarantee that $(X_{n_k} - \mathbb{E}[X_{n_k}])_k$ is an almost-orthogonal sequence. Some examples of almost-orthogonal subsequences will be given in Section 4. We remark also that the integrability of ψ is assured by the condition

$$(3.4) \quad \int_1^\infty \frac{dx}{x^2} \int_1^x r \left(\frac{g(f(t))}{g(f(x))} \right) dt < \infty,$$

which is weaker than (3.3). Finally if, instead of (3.2), the condition

$$(3.5) \quad c g(q) \geq g(p) e^{C(\phi(q) - \phi(p) + 1)}$$

holds for a pair c and C of positive numbers and for an increasing positive function ϕ , with $\sup \phi = \infty$, then we get

$$r(g(p)/g(q)) \leq \tilde{\psi}(\log(\phi(q) - \phi(p) + 1)),$$

where $\tilde{\psi} = r \circ c \cdot \exp(-C \exp)$. Hence conditions (3.1) and (3.5) yield that

$$U_n \left[\phi(n) / \log^\beta(\phi(n)) \right]^{1/2}$$

converge almost surely to 0, for every $\beta > 3$, if $\int_0^\infty r(c e^{-Cx}) dx < \infty$.

4. APPLICATIONS

Many situations in which the above results can be applied are listed in Berkes and Csáki [4]. In this section, we study an example which generalizes the classical *Almost Sure Central Limit Theorem*. More specifically, let $(W_n)_{n \geq 1}$ be a sequence of independent identically distributed random variables, with $\mathbb{E} W_1^2 = 1$ and $\mathbb{E} W_1 = 0$, and put

$$U_n = \frac{W_1 + \dots + W_n}{\sqrt{n}}.$$

Moreover, consider the random variables

$$X_n = \mathbf{1}_{\cap_{k=a_n}^{b_n} \{U_k \in J\}},$$

where J is an interval and (a_n) and (b_n) are two increasing sequences of integers, with $a_n \leq b_n$. In the case $a_n = b_n = n$, the classical ASCLT states that $\sum_{j=1}^n j^{-1} X_j / \log n$ converges almost surely to $\mathcal{N}(0, 1)(J)$. In particular, for any $M > 1$, it follows that $(X_{\lfloor M^j \rfloor} + \dots + X_{\lfloor M^n \rfloor})/n$ converges almost surely to $\mathcal{N}(0, 1)(J)$. In Giuliano Antonini and Weber [8] the classical ASCLT is generalized for $a_n = n$, $b_n = \lfloor Ln \rfloor$, and $L \geq 1$.

In the present situation, we remark that there exists a constant C , depending only on the law of W_1 , such that, for every pair p and q of integers, with $p \leq q$, we have

$$(4.1) \quad |\text{Cov}(X_p, X_q)| \leq C \left(\frac{b_p}{a_q} \right)^{1/4}.$$

(See Giuliano Antonini [7] for a proof.) The above inequality does not guarantee that the sequence (X_n) has ψ -negligible covariance for some function ψ . Nevertheless, by Remark 3.2, if there exist two constants $c_1, c_2 > 0$ such that

$$(4.2) \quad b_p \leq c_1 a_q e^{-c_2(\phi(q) - \phi(p))}$$

(i.e., condition (3.5) is verified), we deduce that

$$U_n = \frac{1}{\sqrt{\phi(n)} \log^{\beta/2}(\phi(n))} \sum_{k=1}^n (\phi(k) - \phi(k-1)) (X_k - \mathbb{E}[X_k])$$

converges to 0 almost surely, for any $\beta > 3$. In particular, we get the following result:

Generalized ASCLT. Assume that condition (4.2) holds and

$$\lim_n b_n/a_n = L, \quad \phi(k) - \phi(k - 1) \sim \phi'(k).$$

Then

$$\frac{1}{\phi(n)} \sum_{k=1}^n \phi'(k) \mathbf{1}_{\cap_{j=a_k}^{b_k} \{U_j \in J\}}$$

almost surely converges to $\mathbb{P}(\cap_{1 \leq t \leq L} \{B_t/\sqrt{t} \in J\})$, where $(B_t)_t$ denotes the standard Brownian motion. In particular, if $\phi = \log$ and $(a_n/n^c)_n$ is an increasing sequence for some positive constant c , condition (4.2) is verified and $\lim_n b_n/a_n = L$ implies that

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \mathbf{1}_{\cap_{j=a_k}^{b_k} \{U_j \in J\}}$$

converges to $\mathbb{P}(\cap_{1 \leq t \leq L} \{B_t/\sqrt{t} \in J\})$ almost surely; moreover, the random measure

$$\frac{1}{\log n} \sum_{k=1}^n \frac{1}{k} \delta_{\sup_{a_k \leq j \leq b_k} U_j}$$

strictly converges to μ almost surely, where μ denotes the distribution of

$$\sup_{1 \leq t \leq L} \{B_t/\sqrt{t}\}.$$

Proof. It is immediate to note that $\phi = \log$ satisfies condition (4.2) if $(a_n/n^c)_n$ is an increasing sequence. Hence, by the remarks of the previous section, it is enough to prove that

$$(4.3) \quad \lim_n \mathbb{P}\left(\bigcap_{k=a_n}^{b_n} \{U_k \in J\}\right) = \mathbb{P}\left(\bigcap_{1 \leq t \leq L} \{B_t/\sqrt{t} \in J\}\right).$$

To this end, for every real number $t \geq 1$, let

$$V_n(t) = \frac{1}{\sqrt{[a_n t]}} \sum_{k=1}^{[a_n t]} X_k.$$

By Donsker's functional central limit theorem, $(V_n(t))_{t \geq 1}$ converges in distribution (according to Skorokhod's distance) to the Gaussian process

$$V(t) = \frac{B_t}{\sqrt{t}}.$$

Let ϵ be a fixed positive number, with $\epsilon < L$, and consider the functions g_1 and g_2 , defined on Skorokhod's space by

$$g_1(x) = I_{\{\sup_{t \in [1, L-\epsilon]} (-x(t)) \leq -c, \sup_{t \in [1, L-\epsilon]} x(t) \leq d\}},$$

$$g_2(x) = I_{\{\sup_{t \in [1, L+\epsilon]} (-x(t)) \leq -c, \sup_{t \in [1, L+\epsilon]} x(t) \leq d\}},$$

where c and d are the endpoints of the interval J . Put $Y = \sup_{1 \leq t \leq L+\epsilon} V(t)$. Then Y has a density with respect to Lebesgue measure (see Nualart [12], Proposition 2.1.4), hence g_1 and g_2 are almost surely continuous with respect to $Y(P)$. From Donsker's functional central limit theorem we deduce that

$$\lim_n \mathbb{E}[g_1(V_n)] = \mathbb{E}[g_1(V)], \quad \lim_n \mathbb{E}[g_2(V_n)] = \mathbb{E}[g_2(V)],$$

so that

$$\begin{aligned} \mathbb{E}[g_1(V)] &= \liminf_n \mathbb{E}[g_1(V_n)] \leq \liminf_n \mathbb{P}\left(\bigcap_{k=a_n}^{b_n} \{U_k \in J\}\right) \leq \limsup_n \mathbb{P}\left(\bigcap_{k=a_n}^{b_n} \{U_k \in J\}\right) \\ &\leq \limsup_n \mathbb{E}[g_2(V_n)] = \mathbb{E}[g_2(V)]. \end{aligned}$$

Since the difference $\mathbb{E}[g_2(V)] - \mathbb{E}[g_1(V)]$ converges to 0 as ϵ goes to 0 (the sample paths of V are continuous), from the above relations we get (4.3), hence the theorem. \square

Remark 4.1. Let $(n_k)_k$ be the subsequence defined by $n_k = \lfloor a_k \rfloor$ and assume that b_n/a_n converges to a real number L . From inequality (4.1) and relations (3.1) and (3.2), it is not difficult to deduce that $(X_{n_k} - \mathbb{E}[X_{n_k}])_k$ is an almost-orthogonal sequence if there exists a constant $c > 0$ such that, for any pair p and q , with $p \leq q$,

$$ca_q \geq a_{q-p+1}a_p$$

and

$$\sup_m \sum_{h=1}^m \sqrt[4]{\frac{a_h}{a_m}} + \sum_{h=m+1}^{\infty} \sqrt[4]{\frac{a_m}{a_h}} < \infty.$$

In particular, if $a_n = M^n$ and $b_n = LM^n$, with $L \geq 1$ and $M > 1$, the above conditions are verified and

$$\frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\bigcap_{j=a_k}^{b_k} \{U_j \in J\}}$$

converges to $\mathbb{P}(\bigcap_{1 \leq t \leq L} \{B_t/\sqrt{t} \in J\})$ almost surely. We can analogously get the result for the random measures $n^{-1} \sum_{k=1}^n \delta_{\sup_{a_k \leq j \leq b_k} U_j}$.

BIBLIOGRAPHY

1. M. Atlagh and M. Weber, *Une nouvelle loi forte des grandes nombres*, Convergence in Ergodic Theory and Probability (V. Bergelson, P. March, and J. Rosenblatt, eds.), W. de Gruyter, Berlin, 1996, pp. 41–62. MR1412596 (97i:60034)
2. I. Berkes and H. Dehling, *Some limit theorems in log density*, Ann. Prob. **21** (1993), 1640–1670. MR1235433 (94h:60026)
3. I. Berkes and E. Csáki, *A universal result in almost sure central limit theory*, Stoch. Proc. Appl. **94** (2001), 105–134. MR1835848 (2002j:60033)
4. G. A. Brosamler, *An almost everywhere central limit theorem*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 561–574. MR0957261 (89i:60045)
5. M. Csörgö and L. Horváth, *Invariance principle for logarithmic averages*, Math. Proc. Cambridge Philos. Soc. **112** (1992), 195–205. MR1162944 (93e:60057)
6. M. Dudziński, *A note on the almost sure central limit theorem for some dependent random variables*, Statist. Probab. Letters **61** (2003), 31–40. MR1950451 (2003j:60038)
7. R. Giuliano Antonini, *On the Rosenblatt coefficient for normalized sums of real random variables*, Rend. Acc. Naz. XL Mem. Mat. Appl. (5) **24** (2000), 111–120. MR1827009 (2002c:60073)
8. R. Giuliano Antonini and M. Weber, *The intersective ASCLT*, Stochastic Anal. Appl. **22** (2004), no. 4, 1009–1025. MR2062956 (2005h:60088)
9. I. A. Ibragimov and M. Lifshits, *On the convergence of generalized moments in almost sure central limit theorem*, Statist. Probab. Letters **40** (1998), 343–351. MR1664544 (99m:60032)
10. M. Kac, R. Salem, and A. Zygmund, *A gap theorem*, Trans. Amer. Math. Soc. **63** (1948), 235–243. MR0023937 (9:426a)
11. M. Lacey and W. Philipp, *A note on the almost sure central limit theorem*, Statist. Probab. Letters **9** (1990), 201–205. MR1045184 (91e:60100)
12. D. Nualart, *The Malliavin Calculus and Related Topics*, Springer, 1995. MR1344217 (96k:60130)
13. M. Peligrad and P. Révész, *On the almost sure central limit theorem*, Almost Everywhere Convergence II (A. Bellow and R. Jones, eds.), Academic Press, Boston, 1991, pp. 209–225. MR1131793 (92k:60067)

14. M. Peligrad and Q. Shao, *A note on the almost sure central limit theorem for weakly dependent random variables*, *Statist. & Probab. Letters* **22** (1995), 131–136. MR1327738 (96b:60057)
15. P. Schatte, *On strong versions of the almost sure central limit theorem*, *Math. Nachr.* **137** (1988), 249–256. MR0968997 (89i:60070)
16. M. Weber, *Entropie Métrique et Convergence Presque Partout*, Coll. “Travaux en Cours”, vol. 58, Herman, Paris, 1998. MR1663938 (2000b:60084)

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI PISA, VIA BUONARROTI 2, 56100 PISA, ITALY
E-mail address: `giuliano@dm.unipi.it`

ACCADEMIA NAVALE DI LIVORNO, VIALE ITALIA 72, 57127 LIVORNO, ITALY
E-mail address: `pratel@mail.dm.unipi.it`

Received 14/JUL/2003
 Originally published in English