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SOME PROPERTIES OF ASYMPTOTIC QUASI-INVERSE FUNCTIONS AND THEIR APPLICATIONS I

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Abstract. We introduce the notions of asymptotic quasi-inverse functions and asymptotic inverse functions as weaker versions of (quasi-)inverse functions, and study their main properties. Asymptotic quasi-inverse functions exist in the class of so-called *pseudo-regularly varying* (PRV) functions, i.e. functions preserving the asymptotic equivalence of functions and sequences. On the other hand, asymptotic inverse functions exist in the class of so-called POV functions, i.e., functions with *positive order of variation*. In this paper, we obtain some new results about PRV and POV functions. Some further properties of asymptotic (quasi-)inverse functions as well as some applications will be discussed in Part II of this paper to appear in no. 71 of this journal.

1. Introduction

Let a real-valued function $f(\cdot)$ be defined and locally bounded on $[t_0, \infty)$ for some $t_0 \geq 0$, and let $f(t) \to \infty$ as $t \to \infty$. Then its *generalized inverse function*

$$f^{(-1)}(s) = \inf\{t \in [t_0, \infty) \colon f(t) \geq s\}$$

is defined on $[f(t_0), \infty)$, is nondecreasing and tends to ∞ as $s \to \infty$. If $f(\cdot)$ is continuous and strictly increasing then its inverse function $f^{-1}(\cdot)$ exists and coincides with $f^{(-1)}(\cdot)$.

Here are the two determining properties for the *inverse function*:

(i) $f(f^{-1}(t)) = t$, $t \in [f(t_0), \infty)$, and
(ii) $f^{-1}(f(t)) = t$, $t \in [t_0, \infty)$.

But, if either $f(\cdot)$ is discontinuous or $f(\cdot)$ is not strictly increasing then its inverse function $f^{-1}(\cdot)$ does not exist, in which case the generalized inverse function is a natural substitution for the inverse function in many situations. However, generalized inverse functions do not always satisfy the determining properties of inverse functions. For example, if a function f is regularly varying (not necessarily monotone), then its generalized inverse function satisfies (i)–(ii) in an "asymptotic sense", namely

$$f\bigl(f^{(-1)}(t)\bigr) \sim f^{(-1)}(f(t)) \sim t \quad \text{as } t \to \infty;$$

see Bingham et al. [4]. Another example is given by a continuous function f with $f(t) \to \infty$ as $t \to \infty$, in which case (i) holds for all large t, while (ii) is not necessarily satisfied.

Various other definitions of "*quasi-inverse*" functions are known in the literature. Any of these definitions either lacks part of the determining properties or weakens them in one way or another. In the sequel we study *asymptotic quasi-inverse* functions $\tilde{f}^{(-1)}(\cdot)$

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defined, for a given function $f(\cdot)$, by the property $f(\tilde{f}^{(-1)}(t)) \sim t$ as $t \to \infty$, and *asymptotic inverse functions*, satisfying, in addition, $\tilde{f}^{(-1)}(f(t)) \sim t$ as $t \to \infty$. That is, we only keep condition (i) (and (ii)) in an asymptotic sense in the definition of quasi-inverse (inverse) functions. Note that an asymptotic quasi-inverse function is not unique, even when its original function is continuous and strictly increasing. On the other hand, not all functions f have an asymptotic quasi-inverse function, and one of the questions is how to describe an appropriate class of functions f possessing asymptotic quasi-inverses.

Below we consider the following four problems for asymptotic quasi-inverse functions:

(A) to characterize functions for which their generalized inverse functions are also asymptotic quasi-inverse functions;
(B) to characterize functions for which their asymptotic quasi-inverse functions are also asymptotic inverse functions;
(C) to characterize functions for which their asymptotic quasi-inverse functions are asymptotically equivalent;
(D) to characterize functions for which the asymptotic behavior of their asymptotic quasi-inverses can be obtained from the asymptotic behavior of the original functions or *vice versa*.

These four problems are closely connected to each other; we shall solve them for the classes of PRV and POV functions; see Definitions 2.1 and 2.3 below.

PRV functions were studied by Korenblyum [12], Stadtmüller and Trautner [15], Yakymiv [16], Klesov et al. [11], and Buldygin et al. [5].

Korenblyum [12] as well as Stadtmüller and Trautner [15] considered nondecreasing PRV functions and studied their properties in order to obtain analogs of Tauberian theorems for Laplace transforms. In particular, Stadtmüller and Trautner [15] proved that the Tauberian theorem for the Laplace transform of a nondecreasing positive function f is valid if and only if f is a PRV function. A substantial progress has been achieved by Yakymiv [16], who investigated the multivariate PRV-property, but his results are also of interest in the one-dimensional case. Note that PRV functions are called *weakly oscillating* in Yakymiv [16].

Klesov et al. [11] studied the relationship between the strong law of large numbers for sequences of random variables and its counterpart for renewal processes constructed from these sequences. Namely, given a sequence of random variables $\{Z_n, n \geq 0\}$, generalized renewal processes $\{R(t), t > 0\}$ can be defined as follows: either take $R(t) = \sup\{n \geq 0 \colon Z_n \leq t\}$ or $R(t) = \sup\{n \geq 0 \colon \max(Z_0, Z_1, \ldots, Z_n) \leq t\}$ or $R(t) = \sum_{n=1}^{\infty} I(Z_n \leq t)$. If the sequence $\{Z_n, n \geq 0\}$ is strictly increasing, then all three functions coincide. Otherwise they are different, and further "natural" definitions of renewal processes could be given. Under some mild conditions, it was proved by Klesov et al. [11] that, if $Z_n/a_n \to 1$ almost surely (a.s.) as $n \to \infty$, then $R(t)/a^{-1}(t) \to 1$ a.s. as $t \to \infty$, where $a_n = a(n)$, $a^{-1}(\cdot)$ is the inverse to $a(\cdot)$, and $a(\cdot)$ is a continuous, strictly increasing and unbounded function. The main assumption about the function $a(\cdot)$ in [11] is that either $a(\cdot)$ or $a^{-1}(\cdot)$ or both of them (the choice depends on the desired result) satisfy the PRV property. The above results are of the following nature: given an asymptotic behavior for an original function, find the corresponding limit behavior for its inverse function. We study the same problem in this paper, but do not necessarily assume that the original function has an inverse, in which case it is substituted by an asymptotic quasi-inverse.

Some results of Klesov et al. [11] have been extended in Buldygin et al. [5], where the PRV property was studied in more detail. One of the results is that the PRV functions, and only they, preserve the equivalence of functions and sequences. Moreover, the POV property was introduced in [5] as a generalization of RV functions with positive index. In particular, it is proved in Buldygin et al. [5] that strictly increasing, unbounded POV functions and their quasi-inverse functions simultaneously preserve the equivalence of functions and sequences. Moreover, only POV functions possess this property. As an application of the general theory, the asymptotic behavior of generalized renewal sample functions of continuous functions and sequences was studied in [5].

The main aim of this paper is to extend the investigations initiated in Buldygin et al. [5] to asymptotic quasi-inverse functions and to obtain solutions of the problems (A)–(D). In order to pursue this program additional studies of the properties of PRV and POV functions are needed.

The paper is organized as follows. In Section 2, we introduce the necessary definitions and give some typical examples of PRV and POV functions. Some auxiliary results on PRV functions used throughout the paper are also given in this section. In Section 3, we recall the integral representation theorems for PRV functions and obtain some equivalent characterizations of POV functions. Using these results, we get in Section 4 a theorem on increasing versions for POV functions and, in Section 5, a variant of Potter's theorem for PRV functions. In Section 6, we consider asymptotic quasi-inverse and asymptotic inverse functions and investigate the problem of existence of such functions. In Section 7, we discuss conditions under which quasi-inverse functions preserve the equivalence of functions. Main properties and characterizations of POV functions and their asymptotic quasi-inverses are studied in Section 8. In Section 9, the limiting behavior of the ratio of asymptotic quasi-inverse functions is discussed. In Section 10, the limiting behavior of (generalized) renewal functions for nonrandom sequences is investigated. Finally, in Section 11, we present several applications of the general results above, e.g. to the asymptotic stability of a Cauchy problem, to the asymptotics of the solution of a stochastic differential equation, and to the limiting behavior of sojourn times.

The paper is divided into two parts, Part I containing Sections 1–7, and Part II (see [7]) containing Sections 8–11.

The results of the paper were announced in Buldygin et al. [6].

2. Definitions and preliminaries

Let $\mathbf{R}$ be the set of real numbers, $\mathbf{R}_+$ the set of nonnegative reals, $\mathbf{Z}$ be the set of integers, and $\mathbf{N}$ the set of positive integers. Also let $\mathbb{F} = \mathbb{F}(\mathbf{R}_+)$ be the space of real-valued functions $f(\cdot) = (f(t),\ t \geq 0)$, and $\mathbb{F}_+ = \bigcup_{A>0}\{f(\cdot) \in \mathbb{F} \mid f(t) > 0,\ t \in [A, \infty)\}$. Thus $f(\cdot) \in \mathbb{F}_+$ if and only if $f(\cdot)$ is eventually positive.

Let $\mathbb{F}^{(\infty)}$ be the space of functions $f(\cdot) \in \mathbb{F}_+$ such that

(i) $\sup_{0\leq t\leq T} f(t) < \infty$ for all $T > 0$;
(ii) $\limsup_{t\to\infty} f(t) = \infty$.

Further let $\mathbb{F}^{\infty}$ and $\mathbb{F}^{\infty}_{\mathrm{ndec}}$ be the spaces of functions $f(\cdot) \in \mathbb{F}^{(\infty)}$ such that

$$\lim_{t\to\infty} f(t) = \infty$$

and $f(\cdot)$ are nondecreasing for large t, respectively.

We also use the subspaces $\mathbb{C}^{(\infty)}$, $\mathbb{C}^{\infty}$, and $\mathbb{C}^{\infty}_{\mathrm{ndec}}$ of continuous functions in $\mathbb{F}^{(\infty)}$, $\mathbb{F}^{\infty}$, and $\mathbb{F}^{\infty}_{\mathrm{ndec}}$, respectively.

Finally, the space $\mathbb{C}^{\infty}_{\mathrm{inc}}$ contains all functions $f(\cdot) \in \mathbb{C}^{\infty}$ that are strictly increasing for large t.

Throughout the paper "measurability" means "Lebesgue measurability".

For a given $f(\cdot) \in \mathbb{F}_+$, we introduce the *upper* and *lower limit functions*

$$f^*(c) = \limsup_{t\to\infty} \frac{f(ct)}{f(t)} \quad \text{and} \quad f_*(c) = \liminf_{t\to\infty} \frac{f(ct)}{f(t)}, \qquad c > 0,$$

which take values in $[0, \infty]$.

First, we summarize some elementary properties of f^* and f_*, which are obvious from their definitions, and will be used later on.

Lemma 2.1. *Let $f \in \mathbb{F}_+$. Then*

(i) *for any $c > 0$,*
$$0 \le f_*(c) \le f^*(c) \le \infty;$$
(ii) *for any $c > 0$,*
$$f_*(c) = \frac{1}{f^*(1/c)},$$
where we set $1/\infty = 0$ and $1/0 = \infty$;

(iii) *for any $c_1, c_2 > 0$ with $0 < f_*(c_i) \le f^*(c_i) < \infty$, $i = 1, 2$,*
$$\begin{aligned} f_*(c_1)f_*(c_2) \le f_*(c_1c_2) &\le \min\{f_*(c_1)f^*(c_2),\, f_*(c_2)f^*(c_1)\} \\ &\le \max\{f_*(c_1)f^*(c_2),\, f_*(c_2)f^*(c_1)\} \le f^*(c_1c_2) \\ &\le f^*(c_1)f^*(c_2); \end{aligned}$$
(iv)
$$f_*(1) = f^*(1) = 1.$$

RV and ORV functions. Recall that a measurable function $f(\cdot) \in \mathbb{F}_+$ is called *regularly varying* (RV) if $f_*(c) = f^*(c) = \varkappa(c) \in \mathbf{R}_+$ for all $c > 0$ (see Karamata [9]). In particular, if $\varkappa(c) = 1$ for all $c > 0$, then the function $f(\cdot)$ is called *slowly varying* (SV). For any RV function $f(\cdot)$, $\varkappa(c) = c^\alpha$, $c > 0$, for some number $\alpha \in \mathbf{R}$, which is called the *index* of the function $f(\cdot)$. Moreover, $f(t) = t^\alpha l(t)$, $t > 0$, where $l(\cdot)$ is a slowly varying function.

A measurable function $f(\cdot) \in \mathbb{F}_+$ is called *O-regularly varying* (ORV) if $f^*(c) < \infty$ for all $c > 0$ (see Avakumović [2] and Karamata [10]). It is obvious that any RV function is an ORV function. ORV functions have been further investigated in Aljančić and Arandelović [1]. Independently, Bari and Stechkin [3] studied Avakumović–Karamata (ORV) functions and their applications in the theory of best approximation of functions. These results (together with the theory of RV functions and later extensions and generalizations) turned out to be fruitful in various fields of mathematics (cf. Seneta [14] and Bingham et al. [4] for excellent surveys on this topic, including its history and applications). Various subclasses of ORV functions are known in the literature. For example, Drasin and Seneta [8] studied the so-called O-*slowly varying* (OSV) functions.

PRV functions. For any RV function $f(\cdot)$, we have $f^*(c) \to 1$ as $c \to 1$. In order to generalize this property to a wider class of functions, we introduce the following definition (see Buldygin et al. [5]).

Definition 2.1. A function $f(\cdot) \in \mathbb{F}_+$ is called *weakly pseudo-regularly varying* (WPRV) if
$$\limsup_{c\to 1} f^*(c) = 1. \tag{2.1}$$
A function $f(\cdot) \in \mathbb{F}_+$ is called *pseudo-regularly varying* (PRV) if it is a measurable WPRV function (cf. Buldygin et al. [5]).

Obviously, from (2.1) it follows that every PRV function is an ORV function. Any quickly growing function, e.g. $f(t) = e^t$, $t \ge 0$, is not PRV.

Remark 2.1 (Buldygin et al. [5]). Let $f(\cdot) \in \mathbb{F}_+$. Then

1) condition (2.1) is equivalent to any of the following four conditions:
   (i) $\liminf_{c\to 1} f_*(c) = 1$;

(ii) $\lim_{c\to 1}\limsup_{t\to\infty}|f(ct)/f(t)-1|=0$;
(iii) $\lim_{c\downarrow 1} f^*(c)=\lim_{c\downarrow 1} f_*(c)=1$;
(iv) $\lim_{c\uparrow 1} f^*(c)=\lim_{c\uparrow 1} f_*(c)=1$;

2) condition (2.1) holds if and only if the upper limit function $f^*(\cdot)$ (or the lower limit function $f_*(\cdot)$) is continuous at the point $c=1$, that is, either
$$\lim_{c\to 1} f^*(c)=1$$
or $\lim_{c\to 1} f_*(c)=1$;

3) if $f(\cdot)$ is a function with a nondecreasing upper limit function $f^*(\cdot)$, then condition (2.1) holds if and only if $\lim_{c\downarrow 1} f^*(c)=1$ or $\lim_{c\uparrow 1} f_*(c)=1$; moreover, under these conditions, $f^*(\cdot)$ is continuous at every point $c\in(0,\infty)$.

Example 2.1. Any PRV function is ORV, but not vice versa. For example, the function $f(t)=2+(-1)^{[t]}$, $t\ge 0$, is ORV, but not PRV.

Example 2.2. Any RV function is PRV, but not vice versa. For example, let α be a fixed real number. Then the function
$$f(t)=\begin{cases}0, & \text{for } t=0,\\ t^\alpha\exp\{\sin(\log t)\}, & \text{for } t>0,\end{cases}$$
is PRV, but not RV.

Example 2.3. Also, the function
$$f(t)=\begin{cases}1, & \text{for } t\in[0,1);\\ 2^k, & \text{for } t\in\left[2^{2k},2^{2k+1}\right),\ k=0,1,2\ldots;\\ t/2^{k+1}, & \text{for } t\in\left[2^{2k+1},2^{2k+2)}\right),\ k=0,1,2\ldots;\end{cases}$$
is PRV, but not RV.

PMPV and POV Functions. Next we define further classes of functions playing an important role in the context of this paper (see also Buldygin et al. [5]).

Definition 2.2. A function $f(\cdot)\in\mathbb{F}_+$ is called *weakly pseudo-monotone of positive variation* (WPMPV) if
$$f_*(c)>1\quad\text{for all } c>1, \tag{2.2}$$
or, equivalently, if $f^*(c)<1$ for all $c\in(0,1)$. A function $f(\cdot)\in\mathbb{F}_+$ is called *pseudo-monotone of positive variation* (PMPV) if $f(\cdot)$ is a measurable WPMPV function.

Note that a slowly varying function $f(\cdot)$ *cannot* be a PMPV function. On the other hand, any RV function of positive index as well as any monotone *quickly* increasing function, for example $f(t)=e^t,\ t\ge 0$, is PMPV.

Remark 2.2. Observe that any function $f(\cdot)$ satisfying condition (2.2) belongs to $\mathbb{F}^{(\infty)}$. Indeed, by (2.2), there exists an $r>1$ such that, for large t_0,
$$f(2^m t_0)\ge r^m f(t_0)\quad\text{for all } m\in\mathbf{N}.$$
Therefore, $\limsup_{t\to\infty} f(t)=\infty$.

Using condition (2.2) we introduce a subclass of PRV functions, which is similar to the class of RV functions with positive index (cf. Buldygin et al. [5]).

Definition 2.3. A WPRV (PRV) function $f(\cdot)$ is said to have *positive order of variation* WPOV (POV) if it satisfies condition (2.2).

No slowly varying function $f(\cdot)$ and no *quickly* growing function, e.g. $f(t) = e^t$, $t \geq 0$, is POV. On the other hand, any RV function of positive index is a POV function. Example 2.3 presents a PRV function which is neither an RV function nor a POV function. Example 2.2, with $\alpha \geq 1$, gives a PRV function which is not an RV function, but is a POV function.

Functions preserving asymptotic equivalence. In this subsection, functions $u(\cdot)$ and $v(\cdot)$ are nonnegative and eventually positive.

Two functions $u(\cdot)$ and $v(\cdot)$ are called (*asymptotically*) *equivalent* if

$$u(t) \sim v(t) \quad \text{as } t \to \infty,$$

that is, $\lim_{t\to\infty} u(t)/v(t) = 1$. The equivalence of functions is denoted by $u(\cdot) \sim v(\cdot)$. A function $f(\cdot)$ *preserves the equivalence of functions* if $f(u(t))/f(v(t)) \to 1$ as $t \to \infty$ for all nonnegative functions $u(\cdot)$ and $v(\cdot)$ such that $u(\cdot) \sim v(\cdot)$ and

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) = \infty.$$

In a similar way, one can introduce the notion of functions $f(\cdot)$ preserving the equivalence of sequences. Below, all sequences $\{u_n, n \geq 0\}$ and $\{v_n, n \geq 0\}$ are assumed to be nonnegative and eventually positive.

Two sequences $\{u_n, n \geq 0\}$ and $\{v_n, n \geq 0\}$ are called (*asymptotically*) *equivalent* if $\lim_{n\to\infty} u_n/v_n = 1$. Equivalent sequences $\{u_n, n \geq 0\}$ and $\{v_n, n \geq 0\}$ are denoted by $\{u_n\} \sim \{v_n\}$. A function $f(\cdot)$ *preserves the equivalence of sequences* if $f(u_n)/f(v_n) \to 1$ as $n \to \infty$ for all sequences of positive numbers $\{u_n, n \geq 0\}$ and $\{v_n, n \geq 0\}$ such that $\{u_n\} \sim \{v_n\}$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$.

One of the most important properties of WPRV functions is that they and only they preserve the equivalence of both functions and sequences.

Theorem 2.1 (Buldygin et al. [5])**.** *Let $f(\cdot) \in \mathbb{F}_+$. The following three conditions are equivalent:*

(a) *the function $f(\cdot)$ preserves the equivalence of functions;*
(b) *the function $f(\cdot)$ preserves the equivalence of sequences;*
(c) *the function $f(\cdot)$ is WPRV.*

Uniform convergence theorem. Theorem 2.1 implies the following version of a uniform convergence theorem (see also Yakymiv [16] and Buldygin et al. [5]).

Theorem 2.2. *Let $f(\cdot)$ be a WPRV function. Then*

$$\lim_{a\downarrow 1} \limsup_{t\to\infty} \sup_{a^{-1}\leq c\leq a} \left| \frac{f(ct)}{f(t)} - 1 \right| = 0.$$

Proof of Theorem 2.2. Indeed, let

$$\lim_{a\downarrow 1} \limsup_{t\to\infty} \sup_{a^{-1}\leq c\leq a} \left| \frac{f(ct)}{f(t)} - 1 \right| > 0.$$

This means that there exist two sequences $\{c_n\}$ and $\{t_n\}$ such that $c_n \to 1$ and $t_n \to \infty$ as $n \to \infty$, and

$$\lim_{n\to 1} \frac{f(c_n t_n)}{f(t_n)} \neq 1.$$

Since $\{c_n t_n\} \sim \{t_n\}$, the latter relation contradicts Theorem 2.1, and thus completes the proof of Theorem 2.2. $\square$

3. A representation theorem for PRV functions and some characterizations of POV functions

There is a basic result concerning RV functions, namely the representation theorem, for which several proofs have been given in the literature (see e.g. Karamata [9] and Bingham et al. [4]). For ORV functions, the representation theorem has been proved in Karamata [10] and Aljančić and Arandelović [1]. Here we briefly recall a representation theorem for PRV functions in the manner of Karamata's representation for RV functions. Moreover, as an application, we will obtain some equivalent characterizations of POV functions.

Representation theorem for RV and ORV functions. Recall that a function $f(\cdot)$ is RV if and only if

$$f(t) = \exp\left\{\alpha(t) + \int_{t_0}^{t} \beta(s)\frac{ds}{s}\right\} \tag{3.1}$$

for some $t_0 > 0$ and all $t \geq t_0$, where $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded measurable functions such that the limits

$$\lim_{t\to\infty} \alpha(t) \quad \text{and} \quad \lim_{t\to\infty} \beta(t)$$

exist. For SV functions, $\lim_{t\to\infty} \beta(t) = 0$.

ORV functions have the same characterization representation (3.1), with $\alpha(\cdot)$ and $\beta(\cdot)$ being only bounded measurable functions (see Aljančić and Arandelović [1]).

Note that none of these representations is unique. For example, one can start from a discontinuous function $\beta(\cdot)$ and obtain a similar representation with other functions $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$, where $\tilde{\beta}(\cdot)$ is continuous or even infinitely differentiable.

Representation theorem for PRV functions. The proof of the representation theorem for PRV functions is based on that for ORV functions.

Theorem 3.1 (Yakymiv [16] and Buldygin et al. [5]). *A function $f(\cdot)$ is PRV if and only if it has a representation* (3.1), *where $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded measurable functions such that*

$$\lim_{c\to 1} \limsup_{t\to\infty} |\alpha(ct) - \alpha(t)| = 0. \tag{3.2}$$

Remark 3.1. Condition (3.2) characterizes the so-called *slowly oscillating* functions (see Bingham et al. [4]).

Another representation for PRV functions is based on that for SV functions and the fact that $(f \circ \log)(\cdot)$ is an SV function for any PRV function $f(\cdot)$.

Theorem 3.2 (Buldygin et al. [5]). *A function $f(\cdot)$ is PRV if and only if*

$$f(t) = \exp\left\{a(t) + \int_{t_0}^{t} b(s)\,ds\right\} \tag{3.3}$$

for some $t_0 > 0$ and all $t \geq t_0$, where $a(\cdot)$ and $b(\cdot)$ are measurable functions such that $\lim_{t\to\infty} a(t)$ exists, $\lim_{t\to\infty} b(t) = 0$, and

$$\lim_{c\to 1} \limsup_{t\to\infty} \int_t^{ct} b(s)\,ds = \lim_{c\to 1} \liminf_{t\to\infty} \int_t^{ct} b(s)\,ds = 0.$$

Some characterizations for POV functions. Some conditions which are equivalent to the condition (2.2) for PRV functions are considered below. In view of Definition 2.3, these conditions are necessary and sufficient for a PRV function to be a POV function as well.

Proposition 3.1. *Let $f(\cdot)$ be a PRV function. Then condition* (2.2) *is equivalent to any of the following two conditions:*

1) *for any sequence of positive numbers $\{c_n\}$ such that $\limsup_{n\to\infty} c_n > 1$ and for any sequence of positive numbers $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, one has*

$$\limsup_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} > 1; \tag{3.4}$$

2) *for any sequence of positive numbers $\{c_n\}$ such that $1 < \limsup_{n\to\infty} c_n < \infty$ and for any sequence of positive numbers $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, condition* (3.4) *holds.*

We need the following auxiliary result.

Lemma 3.1. *Let $f(\cdot)$ be a POV function. Then, for any sequence of positive numbers $\{c_n\}$ and for any sequence of positive numbers $\{t_n\}$ such that $\lim_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} t_n = \infty$, one has*

$$\lim_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} = \infty. \tag{3.5}$$

Proof of Lemma 3.1. Let $\{m_n\} \subset \mathbf{N}$ be such that $2^{m_n} \le c_n < 2^{m_n+1}$, $n \ge 1$. It is clear that $\lim_{n\to\infty} m_n = \infty$.

Observe that

$$\frac{f(c_n t_n)}{f(t_n)} = \frac{f(c_n t_n)}{f(2^{m_n} t_n)} \prod_{k=1}^{m_n} \frac{f(2^k t_n)}{f(2^{k-1} t_n)}.$$

By (2.2), there exists an $r > 1$ such that $f(2t) \ge r f(t)$ for all large t. Therefore, for all large n,

$$\frac{f(c_n t_n)}{f(t_n)} \ge r^{m_n} \frac{f(c_n t_n)}{f(2^{m_n} t_n)}.$$

Moreover, by Theorem 3.1, for all large n,

$$\begin{aligned} \frac{f(c_n t_n)}{f(2^{m_n} t_n)} &= \frac{G(c_n t_n)}{G(2^{m_n} t_n)} \exp\left\{ \int_{2^{m_n} t_n}^{c_n t_n} \beta(u) \frac{du}{u} \right\} \\ &\ge \frac{G(c_n t_n)}{G(2^{m_n} t_n)} \exp\{-\beta \log 3\} \ge \frac{\mu\, 3^{-\beta}}{2}, \end{aligned}$$

where $G(\cdot) = \exp\{\alpha(\cdot)\}$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are bounded measurable functions,

$$\mu = \frac{\liminf_{t\to\infty} G(t)}{\limsup_{t\to\infty} G(t)} > 0 \quad \text{and} \quad \beta = \limsup_{t\to\infty} |\beta(t)| < \infty.$$

Thus

$$\lim_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} = \infty. \qquad \square$$

Proof of Proposition 3.1. The implication 1) $\Longrightarrow$ 2) is trivial as well as the implication 2) $\Longrightarrow$ (2.2). Observe that these implications hold without the assumption that $f(\cdot)$ is a PRV function.

Now, assume that (2.2) holds and let $\{c_n\}$ be a sequence of positive numbers such that $1 < c = \limsup_{n\to\infty} c_n < \infty$. Then there exists a subsequence $\{c_{n'}\} \subset \{c_n\}$ such that

$\lim_{n'\to\infty} c_{n'} = c$. Therefore, by Theorem 2.1, for any sequence of positive numbers $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$, we have

$$\limsup_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} \geq \limsup_{n'\to\infty} \frac{f(c_{n'} t_{n'})}{f(t_{n'})} \geq \liminf_{n'\to\infty} \frac{f(c_{n'} t_{n'})}{f(c t_{n'})} \cdot \liminf_{n'\to\infty} \frac{f(c t_{n'})}{f(t_{n'})}$$
$$= \liminf_{n'\to\infty} \frac{f(c t_{n'})}{f(t_{n'})} \geq \liminf_{t\to\infty} \frac{f(ct)}{f(t)} > 1,$$

that is, 1) holds.

Next, assume that (2.2) holds and let $\{c_n\}$, $\{t_n\}$ be sequences of positive numbers such that $\limsup_{n\to\infty} c_n = \infty$ and $\lim_{n\to\infty} t_n = \infty$. Then there exists a subsequence $\{c_{n'}\} \subset \{c_n\}$ such that $\lim_{n'\to\infty} c_{n'} = \infty$. Consequently, in view of Lemma 3.1,

$$\limsup_{n\to\infty} \frac{f(c_n t_n)}{f(t_n)} = \infty,$$

that is, 1) holds, and the proof of Proposition 3.1 is complete. □

Corollary 3.1. *A function $f(\cdot) \in \mathbb{F}_+$ is a POV function if and only if $f(\cdot)$ is a PRV function and condition* (3.4) *holds for any sequence of positive numbers $\{c_n\}$ such that $\limsup_{n\to\infty} c_n > 1$ and for any sequence of positive numbers $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$.*

Remark 3.2. If $f(\cdot) \in \mathbb{F}_+$ is a POV function, then $f \in \mathbb{F}^\infty$.

Indeed, otherwise there exists a sequence of positive numbers $\{t_n\}$ and a number $p \in [0, \infty)$ such that $t_n \to \infty$ and $\lim_{n\to\infty} f(t_n) = p$. Assume first that $p \in (0, \infty)$. Then there exists a sequence $s_k = t_{n_k}$, $k \geq 1$, such that $\lim_{k\to\infty} s_{k+1}/s_k = \infty$, but $\lim_{k\to\infty} f(s_{k+1})/f(s_k) = 1$. Now, assume that $p = 0$. Then, as above, there exists a sequence $s_k = t_{n_k}, k \geq 1$, such that $\lim_{k\to\infty} s_{k+1}/s_k = \infty$, but

$$\lim_{k\to\infty} f(s_{k+1})/f(s_k) = 0.$$

Thus, we have a contradiction with Lemma 3.1 in both cases, which proves that $f \in \mathbb{F}^\infty$.

Remark 3.3. Lemma 3.1 and Remark 3.2 hold for ORV functions $f(\cdot)$ such that

$$\sup_{c>1} f_*(c) > 1.$$

4. Increasing versions for POV functions

Many problems related to POV functions become easier if the functions are monotone. Hence, we next consider the problem of existence of strictly increasing versions for POV functions.

Theorem 4.1. *Assume that $f(\cdot)$ is a POV function. Then there exists a strictly increasing and continuous POV function $f_1(\cdot)$ that tends to ∞ and satisfies $f(\cdot) \sim f_1(\cdot)$.*

Proof of Theorem 4.1. We use the representation theorem for the function f as given in Theorem 3.2. By the POV property (2.2), we get

$$\liminf_{t\to\infty} \int_t^{ct} b(s)\,ds > 0 \quad \text{for all } c > 1.$$

Therefore, for every $c > 1$, there exists $t_c > t_0$ such that

$$\int_t^{ct} b(s)\,ds > 0 \quad \text{for all } t \geq t_c. \tag{4.1}$$

Choose a sequence $\{c_n, n \geq 1\}$ such that

(i) $c_n \downarrow 1$, $n \to \infty$;
(ii) $c_1 c_2 \cdots c_n \to \infty$, $n \to \infty$.

Put $\tau_n = t_{c_n}$, with t_c as given in (4.1), and define a sequence of positive integers $\{m_n, n \ge 1\}$ such that

$$\begin{aligned}
&m_1 \ge 1\colon \quad c_1^{m_1} \ge \tau_2;\\
&m_2 \ge 1\colon \quad c_1^{m_1} c_2^{m_2} \ge \tau_3;\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&m_n \ge 1\colon \quad c_1^{m_1} c_2^{m_2} \dots c_n^{m_n} \ge \tau_{n+1};\\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Now, set

$$T_k = c_1^{m_1} c_2^{m_2} \cdots c_k^{m_k}, \qquad k \in \mathbf{N};$$
$$T_{k+1,l} = T_k c_{k+1}^l, \qquad 0 \le l \le m_{k+1}.$$

Observe that $T_k = T_{k+1,0} < \cdots < T_{k+1,m_{k+1}} = T_{k+1}$ and, by (ii), $T_k \to \infty$ as $k \to \infty$. Moreover, $T_{k+1,l+1} = c_{k+1} T_{k+1,l}$.

Next, define the functions b_1 and g_1 as follows:

$$b_1(t) = \frac{1}{T_{k+1,l+1} - T_{k+1,l}} \int_{T_{k+1,l}}^{T_{k+1,l+1}} b(s)\, ds, \qquad T_{k+1,l} \le t < T_{k+1,l+1};$$

$$g_1(t) = \begin{cases} 1, & 0 \le t < T_1, \\ \exp\left\{ \int_{T_1}^t b_1(s)\, ds \right\}, & t \ge T_1. \end{cases}$$

Obviously, for every $t \ge T_1$, there exist k and l such that $T_{k+1,l} \le t < T_{k+1,l+1}$ and, by (4.1),

$$b_1(t) = \frac{1}{T_{k+1,l+1} - T_{k+1,l}} \int_{T_{k+1,l}}^{c_{k+1} T_{k+1,l}} b(s)\, ds > 0.$$

Therefore, g_1 is an increasing function for $t > T_1$. Moreover, it is clear that this function is continuous.

Put

$$g(t) = \begin{cases} 1, & 0 \le t < T_1, \\ \exp\left\{ \int_{T_1}^t b(s)\, ds \right\}, & t \ge T_1, \end{cases}$$

and show that $g(t) \sim g_1(t)$ as $t \to \infty$, that is,

$$\int_{T_1}^t (b_1(s) - b(s))\, ds \to 0 \quad \text{as } t \to \infty. \tag{4.2}$$

Indeed, we have by definition

$$\int_{T_{k+1,l}}^{T_{k+1,l+1}} (b_1(s) - b(s))\, ds = 0 \quad \text{for all } k \text{ and } l.$$

Now, for a given t, choose k and l such that $T_{k+1,l} \le t < T_{k+1,l+1}$. Then,

$$\begin{aligned}
\int_{T_1}^t (b_1(s) - b(s))\, ds &= \int_{T_{k+1,l}}^t (b_1(s) - b(s))\, ds \\
&= \frac{t - T_{k+1,l}}{T_{k+1,l+1} - T_{k+1,l}} \int_{T_{k+1,l}}^{T_{k+1,l+1}} b(s)\, ds - \int_{T_{k+1,l}}^t b(s)\, ds,
\end{aligned}$$

and therefore

$$\left| \int_{T_1}^t (b_1(s) - b(s))\, ds \right| \le 2 \sup_{T_{k+1,l} \le t < T_{k+1,l+1}} \left| \int_{T_{k+1,l}}^t b(s)\, ds \right|.$$

Since, by (3.3), g is a PRV function, we can use Theorem 2.2 to prove that

$$\sup_{l} \sup_{T_{k+1,l} \le t < T_{k+1,l+1}} \left| \frac{g(t)}{g(T_{k+1,l})} - 1 \right| \to 0, \qquad k \to \infty,$$

which implies (4.2).

Now, for $t > \max\{ t_0, T_1\}$, we have

$$f(t) = h(t)g(t), \qquad h(t) = \exp\left\{ a(t) + \int_{t_0}^{T_1} b(s)\, ds \right\}.$$

Put

$$h_1(t) = \exp\left\{ \lim_{t\to\infty} a(t) + \int_{t_0}^{T_1} b(s)\, ds \right\} \cdot \left(1 - e^{-t}\right), \qquad t \ge 0,$$

and

$$f_1(t) = h_1(t) g_1(t), \qquad t \ge 0.$$

Observe that, by (3.3) and (4.2), one has $f(\cdot) \sim f_1(\cdot)$. Moreover, the function $f_1(\cdot)$ is continuous and strictly increasing. This function is a POV function, since $f(\cdot)$ is a POV function and $f(\cdot) \sim f_1(\cdot)$. By Remark 3.2, $f_1(\cdot)$ increases to ∞ as $t \to \infty$, which completes the proof of Theorem 4.1 □

5. Potter bounds for PRV functions

Potter's theorem [13] is well known in the RV theory (see, e.g., Bingham et al. [4, p. 25]). Due to the representation theorem and the uniform convergence theorem, we can prove a variant of Potter's theorem for PRV functions. This theorem improves Corollary 4 of Yakymiv [16] and will have some applications in the analysis of asymptotic quasi-inverse functions in Section 6.

Theorem 5.1. *Let $f(\cdot)$ be a PRV function. Then there exists some $b > 0$ and, for any $A > 1$, there exist $\lambda_A > 1$ and $t_A > 0$ such that, for all $\lambda \in (1, \lambda_A]$,*

$$A^{-1}\lambda^{-b} f(s) \le f(t) \le A\lambda^b f(s) \tag{5.1}$$

for all $t \ge t_A$ and all $s \in [\lambda^{-1}t, \lambda t]$.

Proof of Theorem 5.1. By Theorem 3.1, for some $t_0 > 0$ and all $t \ge t_0$,

$$f(t) = G(t) \exp\left\{ \int_{t_0}^{t} \beta(u) \frac{du}{u} \right\}, \tag{5.2}$$

where $G(\cdot) = \exp\{\alpha(\cdot)\}$ is a PRV function and $\alpha(\cdot)$, $\beta(\cdot)$ are bounded measurable functions. Moreover, by Theorem 2.2, for any $A > 1$, there exist $\lambda_A > 1$ and $\tau_A > 0$ such that, for all $t \ge \tau_A$ and all $s \in [\lambda_A^{-1} t, \lambda_A t]$,

$$\frac{G(t)}{G(s)} < A. \tag{5.3}$$

By (5.2) and (5.3),

$$\frac{f(t)}{f(s)} \le A \exp\left\{ \int_{s}^{t} \beta(u) \frac{du}{u} \right\} \tag{5.4}$$

for all $t \ge t_A = \lambda_A \max\{t_0, \tau_A\}$, all $\lambda \in (1, \lambda_A]$, and all $s \in [\lambda^{-1}t, \lambda t]$.

Put $b = \sup_{t \ge t_0} |\beta(t)|$. Note that

$$\exp\left\{ \int_{s}^{t} \beta(u) \frac{du}{u} \right\} \le \exp\left\{ b \int_{s}^{t} \frac{du}{u} \right\} \le \lambda^b$$

for $s \in [\lambda^{-1}t, t]$, and

$$\exp\left\{\int_s^t \beta(u)\frac{du}{u}\right\} = \exp\left\{-\int_t^s \beta(u)\frac{du}{u}\right\} \le \exp\left\{b\int_t^s \frac{du}{u}\right\} \le \lambda^b$$

for $s \in [t, \lambda t]$. Thus, by (5.4),

$$\frac{f(t)}{f(s)} \le A\lambda^b \tag{5.5}$$

for all $t \ge t_A$, all $\lambda \in (1, \lambda_A]$, and all $s \in [\lambda^{-1}t, \lambda t]$. Analogously to (5.5), we also have

$$\frac{f(s)}{f(t)} \le A\lambda^b$$

for all $t \ge t_A$, all $\lambda \in (1, \lambda_A]$, and all $s \in [\lambda^{-1}t, \lambda t]$. □

Remark 5.1. Let $f(\cdot) \in \mathbb{F}_+$. If condition (5.1) holds then $f(\cdot)$ is a WPRV function.

Indeed, if (5.1) holds then, for any $A > 1$, there exists some $\lambda_A > 1$ such that, for all $\lambda \in (1, \lambda_A]$,

$$A^{-1}\lambda^{-b} \le f^*(c) \le A\lambda^b$$

for all $c \in [\lambda^{-1}, \lambda]$. Therefore,

$$A^{-1}\lambda^{-b} \le \liminf_{c\to 1} f^*(c) \le \limsup_{c\to 1} f^*(c) \le A\lambda^b.$$

On letting $\lambda \downarrow 1$ and then $A \downarrow 1$, one has $\lim_{c\to 1} f^*(c) = 1$, that is, $f(\cdot)$ is a WPRV function.

The following result provides a characterization of PRV functions in terms of Potter bounds.

Corollary 5.1. *Let $f(\cdot) \in \mathbb{F}_+$ be measurable. Then $f(\cdot)$ is a PRV function if and only if condition* (5.1) *holds.*

6. Asymptotic quasi-inverse and asymptotic inverse functions

In this section, we study asymptotic quasi-inverse and asymptotic inverse functions and investigate the problem of their existence. Theorem 6.1 below shows that any PRV function $f(\cdot) \in \mathbb{F}^\infty$ has an asymptotic quasi-inverse function, and Theorem 6.2 proves that any POV function has an asymptotic inverse function.

First, we recall the definition of a *quasi-inverse function* which will be useful for our considerations below (cf. Buldygin et al. [5]).

Definition 6.1. Let $f(\cdot) \in \mathbb{F}^{(\infty)}$. A function $f^{(-1)}(\cdot) \in \mathbb{F}^\infty$ is called a *quasi-inverse function* for $f(\cdot)$ if $f\big(f^{(-1)}(s)\big) = s$ for all large s.

For any $f(\cdot) \in \mathbb{C}^{(\infty)}$, a quasi-inverse function exists, but may be nonunique. If $f(\cdot) \in \mathbb{C}^\infty_{\text{inc}}$, then its *inverse function* $f^{-1}(\cdot)$ exists, that is, $f(f^{-1}(s)) = s$ and

$$f^{-1}(f(t)) = t$$

for all sufficiently large s and t.

Example 6.1. Let $x(\cdot) \in \mathbb{C}^{(\infty)}$. Put

$$x_1^{(-1)}(s) = \inf\{t \ge 0 \colon x(t) = s\} = \inf\{t \ge 0 \colon x(t) \ge s\},$$

for $s \ge s_0 = x(0)$, and $x_1^{(-1)}(s) = 0$ for $0 \le s < s_0$ if $s_0 > 0$. The function $x_1^{(-1)}(\cdot)$ is a quasi-inverse function for $x(\cdot)$. If $x(\cdot) \in \mathbb{C}^\infty_{\text{inc}}$, then $x_1^{(-1)}(\cdot) = x^{-1}(\cdot)$.

Example 6.2. Let $x(\cdot) \in \mathbb{C}^{\infty}$. Put

$$x_2^{(-1)}(s) = \sup\{t \geq 0\colon x(t) = s\} = \sup\{t \geq 0 : x(t) \leq s\},$$

for $s \geq s_0 = x(0)$, and $x_2^{(-1)}(s) = 0$ for $0 \leq s < s_0$ if $s_0 > 0$. The function $x_2^{(-1)}(\cdot)$ is a quasi-inverse function for $x(\cdot)$. Observe that $x_1^{(-1)}(s) \leq x_2^{(-1)}(s)$, $s > 0$, and in general $x_1^{(-1)}(\cdot) \neq x_2^{(-1)}(\cdot)$. If $x(\cdot) \in \mathbb{C}^{\infty}_{\text{inc}}$, then $x_2^{(-1)}(\cdot) = x_1^{(-1)}(\cdot) = x^{-1}(\cdot)$.

Example 6.3. Let $f(\cdot)$ be a POV function. Then, by Theorem 4.1, $f(\cdot)$ has a strictly increasing and continuous version $g(\cdot)$ which possesses an inverse function $g^{-1}(\cdot)$.

Next we introduce the notion of an *asymptotic quasi-inverse function.*

Definition 6.2. Let $f(\cdot) \in \mathbb{F}^{(\infty)}$. A function $\tilde{f}^{(-1)}(\cdot)$ is called an *asymptotic quasi-inverse function* for $f(\cdot)$ if

(i) $\tilde{f}^{(-1)}(\cdot) \in \mathbb{F}^{\infty}$;
(ii) $f(\tilde{f}^{(-1)}(s)) \sim s$ as $s \to \infty$.

Of course, every quasi-inverse function is an *asymptotic quasi-inverse* function. Also recall (cf. Bingham et al. [4]) that a function $\tilde{f}^{-1}(\cdot)$ is called an *asymptotic inverse function* for $f(\cdot) \in \mathbb{F}^{(\infty)}$ if $\tilde{f}^{-1}(\cdot)$ is an *asymptotic quasi-inverse function* and

(iii) $\tilde{f}^{-1}(f(s)) \sim s$ as $s \to \infty$.

It is clear that any inverse function is an *asymptotic inverse function.* If $f(\cdot) \in \mathbb{F}^{\infty}$ and if an asymptotic inverse function $\tilde{f}^{-1}(\cdot)$ exists, then $f(\cdot)$ is also an asymptotic inverse function for $\tilde{f}^{-1}(\cdot)$.

Remark 6.1. Let $f(\cdot) \in \mathbb{F}^{(\infty)}$ and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$.

(A) If a function $g(\cdot)$ is asymptotically equivalent to $f(\cdot)$, then $\tilde{f}^{(-1)}(\cdot)$ is an asymptotic quasi-inverse function for $g(\cdot)$.
(B) If $f(\cdot)$ is a WPRV function and $g(\cdot)$ is asymptotically equivalent to $\tilde{f}^{(-1)}(\cdot)$, then, by Theorem 2.1, $g(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$.

Example 6.4. Let $f(\cdot)$ be an RV function with positive index α. Then (see, e.g., Bingham et al. [4], p. 28) there exists an asymptotic inverse function $\tilde{f}^{-1}(\cdot)$ which is an RV function with index $1/\alpha$, and one version of this function is

$$\tilde{f}^{-1}(s) = \inf\{t \geq 0\colon f(t) > s\}.$$

In this case, the asymptotic inverse function $\tilde{f}^{-1}(\cdot)$ is uniquely determined up to asymptotic equivalence.

Next we consider some important examples of asymptotic quasi-inverse functions for PRV functions.

Lemma 6.1. *Let $f(\cdot)$ be a PRV function.*

1) *If $f(\cdot) \in \mathbb{F}^{(\infty)}$, then*

$$\varphi(s) = \inf\{t \geq 0\colon f(t) > s\} \quad \textit{and} \quad \varphi_1(s) = \inf\{t \geq 0\colon f(t) \geq s\}$$

are nondecreasing asymptotic quasi-inverse functions for $f(\cdot)$.

2) *If $f(\cdot) \in \mathbb{F}^{\infty}$, then $\varphi(\cdot)$, $\varphi_1(\cdot)$ and*

$$\psi(s) = \sup\{t \geq 0\colon f(t) < s\}, \qquad \psi_1(s) = \sup\{t \geq 0\colon f(t) \leq s\}$$

are nondecreasing asymptotic quasi-inverse functions for $f(\cdot)$.

Proof of Lemma 6.1. We make use of Theorem 5.1 and some ideas of the proof of Theorem 1.5.12 in Bingham et al. [4].

Consider the function $\varphi(\cdot)$. It is clear that $\varphi(\cdot)$ is nondecreasing and that condition (i) of Definition 6.2 holds for $\tilde{f}^{(-1)}(\cdot) = \varphi(\cdot)$. So, it remains to verify condition (ii).

Choose $A > 1$. Then, by Theorem 5.1, there exist $b > 0$, $\lambda_A > 1$ and $u_A > 0$ such that, for every $\lambda \in (1, \lambda_A)$,

$$A^{-1}\lambda^{-b} f(v) \le f(u) \le A\lambda^b f(v)$$

for all $u \ge u_A$ and all $v \in [\lambda^{-1}u, \lambda u]$. Fix $\lambda \in (1, \lambda_A)$ and choose s such that $\varphi(s) > u_A$. By the definition of $\varphi(\cdot)$, there exists some $v' \in [\varphi(s), \lambda\varphi(s)]$ such that $f(v') > s$. In addition, $f(v'') \le s$ for all $v'' \in [\lambda^{-1}\varphi(s), \varphi(s)]$. On choosing $u = \varphi(s)$, and v to be v' and then v'', we get

$$A^{-1}\lambda^{-b}s \le A^{-1}\lambda^{-b} f(v') \le f(\varphi(s)) \le A\lambda^b f(v'') \le A\lambda^b s.$$

Hence

$$A^{-1}\lambda^{-b} \le \liminf_{s\to\infty} \frac{f(\varphi(s))}{s} \le \limsup_{s\to\infty} \frac{f(\varphi(s))}{s} \le A\lambda^b.$$

Let $\lambda \downarrow 1$ and then $A \downarrow 1$. Consequently, $f(\varphi(s)) \sim s$ as $s \to \infty$, so $\varphi(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$.

A similar reasoning applies to the functions $\varphi_1(\cdot)$, $\psi(\cdot)$ and $\psi_1(\cdot)$. □

Lemma 6.2. *Let $f(\cdot) \in \mathbb{F}^\infty$. If $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ of Lemma* 6.1 *are asymptotically equivalent, then $f(\cdot)$ is an asymptotic quasi-inverse function for both $\varphi_1(\cdot)$ and $\psi_1(\cdot)$.*

Proof of Lemma 6.2. Indeed, by the definition of the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$, one has, for all large t,

$$\varphi_1(f(t)) \le t \le \psi_1(f(t)).$$

Therefore

$$\frac{\varphi_1(f(t))}{\psi_1(f(t))} \le \frac{\varphi_1(f(t))}{t} \le \frac{\psi_1(f(t))}{t} \le \frac{\psi_1(f(t))}{\varphi_1(f(t))}$$

and

$$\lim_{t\to\infty} \frac{\varphi_1(f(t))}{t} = \lim_{t\to\infty} \frac{\psi_1(f(t))}{t} = 1. \qquad \square$$

On combining the above lemmas we get the following result.

Theorem 6.1. *Let $f(\cdot) \in \mathbb{F}^\infty$ be a PRV function. Then the functions*

$$\varphi_1(\cdot) \le \varphi(\cdot) \le \psi(\cdot) \le \psi_1(\cdot)$$

of Lemma 6.1 *are nondecreasing asymptotic quasi-inverse functions for $f(\cdot)$. Moreover, if the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotically equivalent, then all these four functions are asymptotically equivalent and are nondecreasing asymptotic inverse functions for $f(\cdot)$.*

The following result demonstrates that new asymptotic (quasi-)inverse functions can be characterized via the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ above.

Corollary 6.1. *Let $f(\cdot) \in \mathbb{F}^\infty$ be a PRV function, and let $f(\cdot) \sim h(\cdot) \in \mathbb{F}^\infty_{\text{ndec}}$. Assume that $q(\cdot) \in \mathbb{F}_+$ and there exists $s_0 > 0$ such that $\varphi_1(s) \le q(s) \le \psi_1(s)$ for $s \ge s_0$, where $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are the functions of Lemma* 6.1. *Then:*

1) *$q(\cdot)$ is an asymptotic quasi-inverse function for $f(\cdot)$;*
2) *if $\varphi_1(\cdot) \sim \psi_1(\cdot)$, then $q(\cdot)$ is an asymptotic inverse function for $f(\cdot)$ and*

$$q(\cdot) \sim \varphi_1(\cdot) \sim \psi_1(\cdot).$$

Proof of Corollary 6.1. By Theorem 6.1 and Remark 6.1, the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotic quasi-inverse functions for $h(\cdot)$, that is, $h(\varphi_1(s)) \sim s \sim h(\psi_1(s))$ as $s \to \infty$. Therefore $h(q(s)) \sim s$ as $s \to \infty$, since $h(\varphi_1(s)) \le h(q(s)) \le h(\psi_1(s))$ for all large s. Hence $q(\cdot)$ is an asymptotic quasi-inverse function for $h(\cdot)$ and, by Remark 6.1, it is an asymptotic quasi-inverse function for $f(\cdot)$. This proves statement 1).

Next, assume that $\varphi_1(\cdot) \sim \psi_1(\cdot)$. Then $\varphi_1(\cdot) \sim q(\cdot) \sim \psi_1(\cdot)$, since

$$\varphi_1(s) \le q(s) \le \psi_1(s)$$

for all large s. Hence, by Theorem 6.1, we get that $q(f(s)) \sim s$ as $s \to \infty$. Clearly, 2) follows from 1) now. □

The following result shows that every POV function has an asymptotic inverse function.

Theorem 6.2. *Let $f(\cdot)$ be a POV function. Assume that $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are the functions of Lemma* 6.1 *and $q(\cdot) \in \mathbb{F}_+$. Then:*

1) *the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotically equivalent and are nondecreasing asymptotic inverse functions for $f(\cdot)$;*
2) *if there exists $s_0 > 0$ such that $\varphi_1(s) \le q(s) \le \psi_1(s)$ for $s \ge s_0$, then $q(\cdot)$ is an asymptotic inverse function for $f(\cdot)$ and $q(\cdot) \sim \varphi_1(\cdot) \sim \psi_1(\cdot)$.*

Proof of Theorem 6.2. First recall that, by Remark 3.2, $f(\cdot) \in \mathbb{F}^\infty$. By Theorem 4.1, there exists a strictly increasing POV function $h(\cdot)$ such that $f(\cdot) \sim h(\cdot)$. In view of Corollary 6.1, it is sufficient to prove that $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotically equivalent. Note that, by Corollary 3.1, condition (3.4) holds for all sequences of positive numbers $\{c_n\}$ and $\{t_n\}$ such that $\limsup_{n\to\infty} c_n > 1$ and $\lim_{n\to\infty} t_n = \infty$.

Assume that $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are not asymptotically equivalent. This means that there exists a sequence $\{s_n\}$ such that $\lim_{n\to\infty} s_n = \infty$ and

$$\limsup_{n\to\infty} \frac{\psi_1(s_n)}{\varphi_1(s_n)} > 1. \tag{6.1}$$

By the definition of $\varphi_1(\cdot)$ and $\psi_1(\cdot)$, there exist two sequences $\{a_n\}$, $\{b_n\}$ such that $a_n \uparrow 1$, $b_n \downarrow 1$ and $f(a_n\psi_1(s_n)) \le s_n$, $f(b_n\varphi_1(s_n)) \ge s_n$. Therefore

$$\limsup_{n\to\infty} \frac{f(a_n\psi_1(s_n))}{f(b_n\varphi_1(s_n))} \le 1.$$

However, in view of inequality (6.1) and condition (3.4),

$$\limsup_{n\to\infty} \frac{f(a_n\psi_1(s_n))}{f(b_n\varphi_1(s_n))} > 1.$$

This contradiction proves Theorem 6.2. □

In general, the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ of Lemma 6.1 are neither asymptotically equivalent nor quasi-inverse to the original function f.

Example 6.5. Let $f(t) = [\log t]$, $t \ge 1$, where $[x]$ denotes the integer part of a real number x. The function $f(\cdot)$ is an SV function, and hence is not POV, but is PRV. By Lemma 6.1, the functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotic quasi-inverse functions for $f(\cdot)$. Nevertheless, $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are not asymptotically equivalent. Moreover, the function $(e^t, t \ge 0)$ is also an asymptotic quasi-inverse function for $f(\cdot)$. Observe that the function $f(\cdot)$ has no asymptotic inverse function. Indeed, if such a function $g(\cdot)$ exists, then $g(f(t)) \sim t$ as $t \to \infty$. Thus, for any $\theta \in (0,1)$ and $t_n = \exp\{n+\theta\}$, we have $g(f(t_n)) = g(n) \sim \exp\{n+\theta\}$ as $n \to \infty$, which is impossible, since θ is arbitrary.

Note, that $[\log t] \sim \log t$ as $t \to \infty$, and that $(\log t, t \geq 1)$ is not a POV function. But, for the latter function, $(e^t, t \geq 0)$ is the inverse function and $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are equal to $(e^t, t \geq 0)$.

Example 6.6. The functions $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ of Lemma 6.1 are not asymptotic quasi-inverse functions for $f(t) = e^{[t]}$, $t \geq 0$. Nevertheless, $\varphi_1(\cdot)$ and $\psi_1(\cdot)$ are asymptotically equivalent. Observe that $f(\cdot)$ is not PRV.

For non-PRV functions, asymptotic quasi-inverse functions may not exist at all.

Example 6.7. For the non-PRV function $f(t) = e^{[t]}$, $t \geq 0$, there is no asymptotic quasi-inverse function. Indeed, for any $\varphi(\cdot) \in \mathbb{F}_+$ and for $t_n = \exp\{n + \frac{1}{2}\}$, $n \geq 1$, we have that either $[\varphi(t_n)] \geq n+1$ or $[\varphi(t_n)] \leq n$, which implies that either

$$\begin{aligned}\limsup_{t\to\infty} \frac{\exp\{[\varphi(t)]\}}{t} &\geq \limsup_{n\to\infty} \frac{\exp\{[\varphi(t_n)]\}}{t_n} \\ &= \exp\left\{\limsup_{n\to\infty}\left([\varphi(t_n)] - n - \frac{1}{2}\right)\right\} \geq \exp\left\{\frac{1}{2}\right\} > 1,\end{aligned}$$

or

$$\begin{aligned}\liminf_{t\to\infty} \frac{\exp\{[\varphi(t)]\}}{t} &\leq \liminf_{n\to\infty} \frac{\exp\{[\varphi(t_n)]\}}{t_n} \\ &= \exp\left\{\liminf_{n\to\infty}\left([\varphi(t_n)] - n - \frac{1}{2}\right)\right\} \leq \exp\left\{-\frac{1}{2}\right\} < 1.\end{aligned}$$

Example 6.8. In view of Example 6.7 and Remark 6.1, the function $f(t) = e^{[t]} \cdot t/[t]$, $t \geq 1$, does not have an asymptotic quasi-inverse function, since $f(t) \sim e^{[t]}$ as $t \to \infty$. Observe, that the function $f(\cdot)$ is strictly increasing.

Example 6.9. Let $a(\cdot) \in \mathbb{F}_+$. By the same arguments as in Example 6.7, the function $(e^{[a(t)]}, t > 0)$ does not have an asymptotic quasi-inverse function. Hence, by Lemma 6.1, the function $(e^{[a(t)]}, t > 0)$ is not PRV for any $a(\cdot) \in \mathbb{F}^{(\infty)}$.

7. The PRV property for asymptotic quasi-inverse functions

In this section, we discuss conditions under which quasi-inverse functions preserve the equivalence of functions, i.e., in view of Theorem 2.1, are WPRV (or PRV).

Lemma 7.1. *Let $f(\cdot) \in \mathbb{F}^{(\infty)}$. Then an asymptotic quasi-inverse function $\tilde{f}^{(-1)}(\cdot)$ for $f(\cdot)$ is WPRV (and thus preserves the equivalence of functions) if and only if*

$$\limsup_{c\to 1}\limsup_{t\to\infty} \frac{\tilde{f}^{(-1)}(ct)}{\tilde{f}^{(-1)}(t)} = 1. \tag{7.1}$$

Proof of Lemma 7.1. It follows from Definition 2.1 and Theorem 2.1. □

Proposition 7.1. *Let $f(\cdot) \sim h(\cdot) \in \mathbb{F}^{\infty}_{\mathrm{ndec}}$ and let $\tilde{f}^{(-1)}(\cdot)$ be an asymptotic quasi-inverse function for $f(\cdot)$. If (2.2) holds, then $\tilde{f}^{(-1)}(\cdot)$ is WPRV (and thus preserves the equivalence of functions).*

Proof of Proposition 7.1. First, assume that condition (2.2) holds, but condition (7.1) does not. Then, there exist a number $\delta > 0$ and sequences $\{c_n\}$ and $\{s_n\}$ such that $c_n \to 1$ and $s_n \to \infty$ as $n \to \infty$, and

$$\tilde{f}^{(-1)}(c_n s_n) \geq (1+\delta)\tilde{f}^{(-1)}(s_n) \quad \text{for all } n \geq 1.$$

Hence, by condition (2.2),

$$\begin{aligned}
1 &= \lim_{n\to\infty} \frac{f\big(\tilde{f}^{(-1)}(c_n s_n)\big)}{c_n s_n} = \lim_{n\to\infty} \frac{f\big(\tilde{f}^{(-1)}(c_n s_n)\big)}{s_n} = \lim_{n\to\infty} \frac{f\big(\tilde{f}^{(-1)}(c_n s_n)\big)}{f\big(\tilde{f}^{(-1)}(s_n)\big)} \\
&= \lim_{n\to\infty} \frac{h\big(\tilde{f}^{(-1)}(c_n s_n)\big)}{h\big(\tilde{f}^{(-1)}(s_n)\big)} \geq \liminf_{n\to\infty} \frac{h\big((1+\delta)\tilde{f}^{(-1)}(s_n)\big)}{h\big(\tilde{f}^{(-1)}(s_n)\big)} \\
&\geq \liminf_{t\to\infty} \frac{h((1+\delta)t)}{h(t)} = \liminf_{t\to\infty} \frac{f((1+\delta)t)}{f(t)} > 1.
\end{aligned}$$

This contradiction proves the implication (2.2) $\Rightarrow$ (7.1). An application of Lemma 7.1 completes the proof of Proposition 7.1. □

Theorem 7.1. *Let $f(\cdot) \sim h(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$ and let $\tilde{f}^{-1}(\cdot)$ be an asymptotic inverse function for $f(\cdot)$. Then $\tilde{f}^{-1}(\cdot)$ is WPRV (and thus preserves the equivalence of functions) if and only if condition* (2.2) *holds.*

Proof of Theorem 7.1. Assume that condition (2.2) holds. Then, by Proposition 7.1, $\tilde{f}^{-1}(\cdot)$ is WPRV and preserves the equivalence of functions.

Now, assume that $\tilde{f}^{-1}(\cdot)$ is a WPRV function. If, at the same time, (2.2) does not hold, there exists a number $c_0 > 1$ such that

$$\liminf_{t\to\infty} \frac{f(c_0 t)}{f(t)} = 1,$$

since $f(\cdot) \sim h(\cdot)$ and $h(\cdot)$ is nondecreasing. Hence, there is a sequence $\{t_n\}$ such that $t_n \uparrow \infty$ as $n \to \infty$, and

$$\lim_{n\to\infty} \frac{f(c_0 t_n)}{f(t_n)} = 1.$$

This implies that the sequences $u_n = f(c_0 t_n)$ and $v_n = f(t_n)$ are equivalent and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \infty$. By Theorem 2.1, $\tilde{f}^{-1}(\cdot)$ preserves the equivalence of sequences and therefore

$$1 = \lim_{n\to\infty} \frac{\tilde{f}^{-1}(u_n)}{\tilde{f}^{-1}(v_n)} = \lim_{n\to\infty} \frac{c_0 t_n}{t_n} = c_0 > 1.$$

This contradiction proves (2.2), which completes the proof of Theorem 7.1. □

Corollary 7.1. *Let $f(\cdot) \in \mathbb{F}^{\infty}$ and let $\tilde{f}^{-1}(\cdot)$ be an asymptotic inverse function for $f(\cdot)$ such that $\tilde{f}^{-1}(\cdot) \sim h(\cdot) \in \mathbb{F}^{\infty}_{\text{ndec}}$. Then $f(\cdot)$ is WPRV (and preserves the equivalence of functions) if and only if the following condition holds:*

$$\big(\tilde{f}^{-1}\big)_*(c) = \liminf_{t\to\infty} \frac{\tilde{f}^{-1}(ct)}{\tilde{f}^{-1}(t)} > 1 \quad \textit{for all } c > 1. \tag{7.2}$$

Corollary 7.2 (Buldygin et al. [5])**.** *Let $f(\cdot) \in \mathbb{C}^{\infty}_{\text{inc}}$. Then the inverse function $f^{-1}(\cdot)$ is PRV (and thus preserves the equivalence of functions) if and only if condition* (2.2) *holds. Moreover, $f(\cdot)$ is a PRV function if and only if*

$$(f^{-1})_*(c) = \liminf_{t\to\infty} \frac{f^{-1}(ct)}{f^{-1}(t)} > 1 \quad \textit{for all } c > 1.$$

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