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ON THE PRICING OF EQUITY-LINKED LIFE INSURANCE CONTRACTS IN GAUSSIAN FINANCIAL ENVIRONMENT

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Abstract. The paper deals with the problem of pricing an equity-linked insurance contract based on stock prices. The stock prices are supposed to follow a stochastic exponent model with respect to a given Gaussian martingale. The model gives a possibility to obtain unified formulas for "mean–variance" hedging and the corresponding premium for both natural cases: Black–Scholes and Gaussian discrete time models.

1. Introduction

Suppose that an insurance company has a portfolio of l insurance contracts. Any contract is associated with a random time τ_i, $i = 1, \dots, l$, which indicates the time of incident occurrence. The corresponding premiums should be distributed between financial assets to guarantee the best correspondence between liabilities of the company and its capital V^π. As a criterion of the quality of a financial portfolio π we shall use the mean variance distance

$$\mathsf{E}\big[(V_T^\pi - f_T)^2\big],$$

where f_T represents the claim that should be paid by the company at the terminal time T.

The insurance contract based on the market's price of a given asset S_t is called an *equity-linked life insurance contract*. An appropriate description of such a contract was given, for instance, in [3], in the framework of the Black–Scholes model for S. This paper is devoted to the pricing of such insurance contracts and in a more general case of a financial market driven by Gaussian martingale.

2. Financial market and insurance portfolio

Suppose the company invests in the (B, S)-market with two traded assets: bank account $B_t \equiv 1$ and stock S_t,

$$S_t = S_0 \exp\left\{Y_t - \frac{1}{2}\langle Y\rangle_t\right\},$$

where Y_t is the right continuous Gaussian martingale (see [2]) on a given stochastic basis

$$(\Omega^1, \mathcal{F}^1, P^1, F^1)$$

with filtration F^1 generated by Y.

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Any self-financing trading strategy $\pi = (\beta, \gamma)$ can be determined by its capital

$$V_t^\pi = V_0^\pi + \int_0^t \gamma_u \, dS_u,$$

where γ_t is the number of stocks in the portfolio at time t (see [4]). The number of bank account units β can be identified from the balance equation

$$V_t = \gamma_t S_t + \beta_t.$$

Using the explicit form of S and the Kolmogorov–Itô formula [2], we can derive the following expression for the capital V_t^π:

$$V_t^\pi = V_0^\pi + \int_0^t \gamma_u S_{u^-} \, dY_u + \sum_{u \le t} \gamma_u S_{u^-} \left(\exp\left\{ \Delta Y_u - \frac{1}{2} \Delta \langle Y \rangle_u \right\} - 1 - \Delta Y_u \right). \tag{1}$$

The maturity time of this contract will be denoted by T. We also assume a pure technical condition: $\Delta Y_T = 0$. The corresponding contingent claim has the form $f_T = f(S_T)$, where the Borel function f satisfies the following condition:

$$f(x) \le c \left(1 + x^{p^1}\right) x^{-p^2}, \qquad c \ge 0,\ p^1 \ge 0,\ p^2 \ge 0,\ x \ge 0.$$

In this case (see [1]) we can define the function

$$F(u,\ x) = \frac{1}{\sqrt{2\pi \langle Y \rangle_T - u}} \int_{\mathbf{R}^+} \frac{f(z)}{z} \exp\left\{ -\frac{\left(\ln(x/z) + \frac{1}{2}(\langle Y \rangle_T - u)\right)^2}{2(\langle Y \rangle_T - u)} \right\} dz.$$

It is easy to prove that this function is twice continuously differentiable with respect to both argument u and x and admits the representation

$$F(\langle Y \rangle_t, S_t) = \mathsf{E}\left[f(S_T) \mid \mathcal{F}_t^1 \right] \quad \text{for any } t < T. \tag{2}$$

Using the martingale convergence theorem [2] we can conclude that

$$F(\langle Y \rangle_T, S_T) = f(S_T).$$

The Kolmogorov–Itô formula gives directly that

$$\begin{aligned} F(\langle Y \rangle_t, S_t) = F(0, S_0) &+ \int_0^t F_x(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-} \, dY_u \\ &+ \int_0^t F_u(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-} \, d\langle Y \rangle_u \\ &+ \frac{1}{2} \int_0^t F_{xx}(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-}^2 \, d\langle Y^c \rangle_u \\ &+ \sum_{u \le t} \Big\{ F(\langle Y \rangle_u, S_u) - F(\langle Y \rangle_{u^-}, S_{u^-}) - F_x(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-} \Delta Y_u \\ &\qquad - F_u(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-} \Delta Y_u \Big\}, \end{aligned} \tag{3}$$

where F_x, F_u, and F_{xx} are the corresponding derivatives. Taking into account that the process $F(\langle Y \rangle_t, S_t)$ should be a martingale, we can reduce (3) to the equality

$$\begin{aligned} F(\langle Y \rangle_t, S_t) = F(0, S_0) &+ \int_0^t F_x(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-} \, dY_u \\ &+ \sum_{\{u \le t\}} \{ F(\langle Y \rangle_u, S_u) - F(\langle Y \rangle_{u^-}, S_{u^-}) - F_x(\langle Y \rangle_{u^-}, S_{u^-}) S_{u^-} \Delta Y_u \}. \end{aligned} \tag{4}$$

The insurance portfolio of l contracts can be characterized by random times

$$\tau_i, \qquad i = 1, \ldots, l,$$

of incident occurrence and claim payment values. We shall assume that τ_i are i.i.d. random variables on some probability space $(\Omega^2, \mathcal{F}^2, P^2)$. The payment function $g(\cdot)$ for the contract indicates the value $g(S_T)$ which should be paid if no incident occurs during the insured period.

Suppose that the distribution of τ_i admits the following representation:

$$P^2(\tau \le t) = 1 - \exp\left\{-\int_0^t \mu_s\, ds\right\},$$

where μ is called a force of mortality. Denote $I_t^k = I_{\{\tau_k \le t\}}$, and $N_t = I_t^1 + \cdots + I_t^l$. The counting process N_t indicates the number of incidents on $[0, t]$. We shall equip $(\Omega^2, \mathcal{F}^2, P^2)$ with a filtration $F^2 = (\mathcal{F}_t^2)_{t\ge 0}$ generated by $(N_t)_{t\ge 0}$. The process

$$\int_0^t (l - N_{s^-})\mu_s\, ds$$

is a compensator of N_t and

$$M_t = N_t - \int_0^t (l - N_{s^-})\mu_s\, ds$$

is a martingale with respect to F^2.

3. Main results and examples

It is quite natural to think that the financial market and the lives of insured are independent. Hence the general probability space for the model can be defined as a product of $(\Omega^1, \mathcal{F}^1, F^1, P^1)$ and $(\Omega^2, \mathcal{F}^2, F^2, P^2)$ with the general filtration F generated by F^1 and F^2:

$$\Omega = \Omega^1 \times \Omega^2, \qquad \mathcal{F} = \mathcal{F}^1 \times \mathcal{F}^2, \qquad \mathsf{P} = P^1 \times P^2,$$
$$F = F^1 \times F^2 = \left\{\mathcal{F}_t = \mathcal{F}_t^1 \times \mathcal{F}_t^2\right\}_{t\ge 0}.$$

The payment function of such a contract has the form

$$f_T = g(S_T)(l - N_T),$$

where the Borel function $g(\cdot)$ satisfies the conditions mentioned above,

$$g(x) \le c(1 + x^{p^1})x^{-p^2}, \qquad c \ge 0,\ p^1 \ge 0,\ p^2 \ge 0,\ x > 0.$$

To optimize its liabilities the company should choose an initial capital $\hat{v}$ and a self-financing trading strategy $\hat{\pi}$ such that

$$\mathsf{E}\left[(V_T^{\hat{\pi}}(\hat{v}) - f_T)^2\right] \le \mathsf{E}\left[(V_T^{\pi}(v) - f_T)^2\right]$$

for any v and any self-financing strategy π.

Denote by

$$V_t^* = \mathsf{E}[f_T \mid \mathcal{F}_t] = \mathsf{E}[g(S_T)(l - N_T) \mid \mathcal{F}_t]$$

the so-called "tracking process" V_t^*; using the independence of S and N we get

$$V_t^* = E^1\left[g(S_T) \mid \mathcal{F}_t^1\right] E^2\left[(l - N_T) \mid \mathcal{F}_t^2\right].$$

It is easy to check that

$$E^2\left[(l - N_T) \mid \mathcal{F}_t^2\right] = E^2\left[\sum_{i=1}^{l}(1 - I_{\{\tau_i \le T\}}) \,\Big|\, \mathcal{F}_t^2\right] = (l - N_t)_t p_T, \tag{5}$$

where ${}_tp_T = E^2[1 - I_{\{t<\tau_i\le T\}} \mid \mathcal{F}_t] = \exp\{-\int_t^T \mu_s\, ds\}$ is the probability of incident occurrence after expiration date T. Using the representation (2) we have

$$V_t^* = F(\langle Y\rangle_t, S_t)(l - N_t)_t p_T.$$

Applying the Kolmogorov–Itô formula to the process V_t^* we obtain the integral representation

$$
\begin{aligned}
V_t^* = V_0^* &+ \int_0^t (l - N_{u_-})_u p_T \, dF(\langle Y\rangle_u, S_u) + \int_0^t F(\langle Y\rangle_{u_-}, S_{u_-})(l - N_{u_-})_u p_T \mu_u \, du \\
&- \int_0^t F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \, dN_u \\
&+ \sum_{u\le t}\Big[F(\langle Y\rangle_u, S_u)(l - N_u)_u p_T - F(\langle Y\rangle_{u_-}, S_{u_-})(l - N_{u_-})_u p_T \\
&\qquad - (l - N_{u_-})_u p_T \Delta F(\langle Y\rangle_u, S_u) + F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \Delta N_u\Big].
\end{aligned} \tag{6}
$$

The equality (6) can be rewritten as

$$
\begin{aligned}
V_t^* = V_0^* &+ \int_0^t (l - N_{u_-})_u p_T \, dF(\langle Y\rangle_u, S_u) - \int_0^t F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \, dM_u \\
&+ \sum_{u\le t}\Big[F(\langle Y\rangle_u, S_u)(l - N_u)_u p_T - F(\langle Y\rangle_{u_-}, S_{u_-})(l - N_{u_-})_u p_T \\
&\qquad - (l - N_{u_-})_u p_T \Delta F(\langle Y\rangle_u, S_u) + F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \Delta N_u\Big],
\end{aligned} \tag{7}
$$

where

$$
M_t = N_t - \int_0^t (l - N_{u_-})\mu_u \, du.
$$

Taking into account the representation (4) for $F(\langle Y\rangle_t, S_t)$ we have from (7) that

$$
\begin{aligned}
V_t^* = V_0^* &+ \int_0^t (l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) S_{u_-} \, dY_{u_-} - \int_0^t F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \, dM_u \\
&+ \sum_{u\le t}\Big[F(\langle Y\rangle_u, S_u)(l - N_u)_u p_T - F(\langle Y\rangle_{u_-}, S_{u_-})(l - N_{u_-})_u p_T \\
&\qquad + F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \Delta N_u - (l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) \Delta Y_u\Big].
\end{aligned}
$$

Note that any capital V_t^π controlled by a self-financing strategy can be represented in the form (1). Consider the mean variance distance between V_T^π and $V_T^* = f_T$:

$$
R(\pi, v) = \mathsf{E}\left[(V_T^\pi - V_T^*)^2\right].
$$

Because of the martingale properties of V_t^π and V_t^*, it is clear that the initial capital $\hat{v}$ for the optimal trading strategy should be equal to V_0^*:

$$
\hat{v} = lE^1[g(S_T)]p(0,T) = lF(0, S_0)\,\mathsf{P}(\{\tau > T\}).
$$

Let the initial capital v of the self-financing strategy π be equal to $\hat{v}$; then

$$
\begin{aligned}
R(\pi, v) = \mathsf{E}\Bigg[\Bigg(&\int_0^t ((l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) - \gamma_u) S_{u_-} \, dY_u \\
&- \int_0^t F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T \, dM_u \\
&+ \sum_{u\le t}\Big[(l - N_u)_u p_T \Delta F(\langle Y\rangle_u, S_u) \\
&\qquad - (l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) S_{u_-} \Delta Y_{u_-} \\
&\qquad\qquad - \gamma_u \Delta S_u + \gamma_u S_{u_-} \Delta Y_u\Big]\Bigg)^2\Bigg].
\end{aligned} \tag{8}
$$

Since the difference

$$\mathcal{M}_t = V_t^* - V_t^\pi$$

is a martingale with respect to F, there is a unique representation of the form

$$\mathcal{M}^c + \mathcal{M}^d,$$

where $\mathcal{M}^c$ and $\mathcal{M}^d$ are respectively the purely continuous and discontinuous parts of $\mathcal{M}_t$, which are orthogonal. Consequently

$$\mathsf{E}\left[\mathcal{M}^2\right] = \mathsf{E}\left[(\mathcal{M}^c)^2\right] + \mathsf{E}\left[\left(\mathcal{M}^d\right)^2\right].$$

Let us rewrite $\mathcal{M}^c$ in the form

$$\mathcal{M}^c = \int_0^t \big((l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) - \gamma_u\big) S_{u_-}\, dY_u^c - \int_0^t F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T\, dM_u^c.$$

Since Y_u and M_u are independent, we get

$$\begin{aligned}\mathsf{E}\left[(\mathcal{M}^c)^2\right] = \mathsf{E}\bigg[&\int_0^t \big((l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) - \gamma_u\big)^2 S_{u_-}^2\, d\langle Y^c\rangle_u\bigg] \\ &+ \mathsf{E}\bigg[\int_0^t (F(\langle Y\rangle_{u_-},\ S_{u_-})p_T)^2\, d\langle M^c\rangle_u\bigg].\end{aligned}$$

In the case of the purely discontinuous part $\mathcal{M}^d$, we have the formula

$$\mathcal{M}^d = \sum_{u\le t}\left[-_u p_T F(\langle Y\rangle_u,\ S_u)\Delta N_u + (l - N_{u_-})_u p_T \Delta F(\langle Y\rangle_u, S_u) - \gamma_u \Delta S_u\right],$$

and therefore

$$\mathsf{E}\left[(\mathcal{M}^d)^2\right] = \mathsf{E}\bigg[\bigg(\sum_{u\le t}\Big[-_u p_T F(\langle Y\rangle_u, S_u)\Delta N_u + (l - N_{u_-})_u p_T \Delta F(\langle Y\rangle_u, S_u) - \gamma_u \Delta S_u\Big]\bigg)^2\bigg].$$

It is well known (see [2]) that the times of the jumps of the Gaussian martingale are deterministic. Denote the corresponding set by A. Define the processes $\overline{\gamma}$ and $\tilde{\gamma}$ by the following formulas:

$$\tilde{\gamma}_s = \gamma_s \chi_{\overline{A}}, \qquad \overline{\gamma}_s = \gamma_s \chi_A,$$

where χ_A is the indicator function of the set A. Take

$$\gamma_s = \tilde{\gamma}_s + \overline{\gamma}_s.$$

Since A is a countable set,

$$\begin{aligned}\mathsf{E}\left[(\mathcal{M}^c)^2\right] = \mathsf{E}\bigg[&\int_0^t ((l - N_{u_-})_u p_T F_x(\langle Y\rangle_{u_-}, S_{u_-}) - \tilde{\gamma}_u)^2 S_{u_-}^2\, d\langle Y^c\rangle_u\bigg] \\ &+ \mathsf{E}\bigg[\int_0^t (F(\langle Y\rangle_{u_-}, S_{u_-})_u p_T)^2\, d\langle M^c\rangle_u\bigg].\end{aligned} \tag{9}$$

On the other hand it is clear that

$$
\begin{aligned}
\mathsf{E}\left[(\mathcal{M}^d)^2\right] &= \mathsf{E}\Bigg[\Bigg(\sum_{u\in A}\Big[-{}_u p_T F(\langle Y\rangle_u, S_u)\Delta N_u \\
&\qquad + (l-N_{u_-})_u p_T \Delta F(\langle Y\rangle_u, S_u) - \gamma_u \Delta S_u\Big] \\
&\qquad - \sum_{u\notin A}[{}_u p_T F(\langle Y\rangle_u, S_u)\Delta N_u]\Bigg)^2\Bigg] \\
&= \mathsf{E}\Bigg[\Bigg(\sum_{u\in A}\Big[-{}_u p_T F(\langle Y\rangle_u,\ S_u)\Delta N_u \\
&\qquad + (l-N_{u_-})_u p_T \Delta F(\langle Y\rangle_u,\ S_u) - \overline{\gamma}_u \Delta S_u\Big]\Bigg)^2\Bigg] \\
&\quad + \mathsf{E}\Bigg[\Bigg(\sum_{u\notin A}[{}_u p_T F(\langle Y\rangle_u, S_u)\Delta N_u]\Bigg)^2\Bigg].
\end{aligned}
\tag{10}
$$

Equations (9) and (10) give us the explicit forms of $\tilde{\gamma}$ and $\overline{\gamma}$:

$$
\tilde{\gamma} = (l-N_{t_-})_t p_T F_x(\langle Y\rangle_{t_-}, S_{t_-})\chi_{\overline{A}} \tag{11}
$$

and

$$
\overline{\gamma}_t =_t p_T (l-N_{t_-})\frac{\mathsf{E}\left[\Delta F(\langle Y\rangle_t, S_t)\Delta S_t \mid \mathcal{F}_{t_-}\right]}{\mathsf{E}\left[\Delta(S_t)^2 \mid \mathcal{F}_{t_-}\right]}, \tag{12}
$$

where $0/0$ is supposed to be equal to 0.

So, we get the following main result of the paper:

Theorem 1. *For the (B,S)-market controlled by a Gaussian martingale and a portfolio of l homogeneous unit-linked pure endowment insurance contracts, there is an optimal mean-variance hedging strategy $\hat{\gamma} = \tilde{\gamma} + \overline{\gamma}$, where the continuous part $\tilde{\gamma}$ and the purely discontinuous part $\overline{\gamma}$ are defined by* (11) *and* (12)*, respectively.*

The initial capital $\hat{v}$ of the strategy can be calculated by

$$
\hat{v} = l\,\mathsf{E}[g(S_T)]\,\mathsf{P}(\tau > T),
$$

where $g(S_t)$ is a payment function for one insurance contract, T is a terminal time, and τ is a random time with distribution of the incident occurrence.

Example 1. The model investigated above includes the model considered by T. Møller (see [3]) when Y_t is Brownian motion. Taking, in our setting,

$$
\overline{\gamma}_t \equiv 0, \qquad \hat{\gamma}_t = \tilde{\gamma}_t = (l-N_{t_-})_t P_T F_x(\langle Y\rangle_t, S_t)
$$

gives the model and the result presented in [3].

Example 2. Another interesting example is the discrete model

$$
S_n = S_0 \exp\left\{M_n - \frac{1}{2}\langle M\rangle_n\right\},
$$
$$
B_n \equiv 1,
$$

where $M_n = h_1 + \cdots + h_n$, $h_i = \sigma\epsilon_i$, $\epsilon_i \sim N(0,1)$, and ϵ_i are i.i.d. random variables on the probability space $(\Omega^1, \mathcal{F}^1, P^1)$ with filtration $F^1 = \{\tilde{\mathcal{F}}^1_n\}$, $\tilde{\mathcal{F}}^1_n = \sigma\{\epsilon_i, i \le n\}$. It is clear that $\langle M\rangle_n = \sigma^2 n$ and

$$
S_n = S_0 \exp\left\{h_1 + \cdots + h_n - \frac{\sigma^2}{2}n\right\}.
$$

Using the independence of ϵ_i, we have

$$\mathsf{E}[S_n \mid \mathcal{F}_{n-1}] = S_{n-1}\,\mathsf{E}\left[\exp\left\{h_n - \sigma^2/2\right\} \mid \mathcal{F}_{n-1}\right] = S_{n-1}.$$

We can embed the discrete model to our general model by the following standard way:

$$Y_t = \begin{cases} 0, & t \in [0,1), \\ M_n, & t \in [n, n+1),\ n < N, \\ M_N, & t \in [N, N+\delta),\ N+\delta = T, \end{cases}$$

$\mathcal{F}^1_t = \mathcal{F}^1_{[t]}$, $F^1 = \{\mathcal{F}^1_t\}_{t\geq 0}$.

It is clear that Y is a Gaussian martingale on the standard stochastic basis

$$\left(\Omega^1, \mathcal{F}^1, F^1, P^1\right)$$

satisfying the technical condition $\Delta Y_T = 0$. In view of the theorem $\tilde{\gamma}_n = 0$ because there is no continuous part of Y. Regarding the other part of a hedging strategy $\overline{\gamma}_n$ we have

$$\begin{aligned}\mathsf{E}[\Delta F(\langle Y\rangle_n, S_n)\Delta S_n \mid \mathcal{F}_{n-1}] &= \mathsf{E}[(F(\langle Y\rangle_n, S_n) - F(\langle Y\rangle_{n-1}, S_{n-1}))\Delta S_n \mid \mathcal{F}_{n-1}] \\ &= \mathsf{E}[F(\langle Y\rangle_n, S_n)\Delta S_n | \mathcal{F}_{n-1}] = \mathsf{E}[g(S_T\Delta S_n) \mid \mathcal{F}_{n-1}].\end{aligned}$$

Hence we get

$$\tilde{\gamma}_t = \gamma_n =_n p_T(l - N_{n-1})\frac{\mathsf{E}\left[g(S_T\Delta S_n) \mid \mathcal{F}_{n-1}\right]}{\mathsf{E}\left[(\Delta S_n)^2 \mid \mathcal{F}_{n-1}\right]},$$

where $S_T = S_N$.

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