

ON A MULTIVARIATE STORAGE PROCESS

UDC 519.21

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ABSTRACT. A multivariate storage process that satisfies the Langevin equation is studied in the paper.

1. INTRODUCTION

Let a process $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T \in \mathbb{R}^n$ satisfy the Langevin equation

$$(1) \quad dx(t) = Ax(t) dt + dz(t),$$

where $z(t) = (z_1(t), z_2(t), \dots, z_n(t))^T \in \mathbb{R}^n$ is a generalized Poisson process with parameter λ and jumps $\eta^1, \eta^2, \dots, \eta^j, \dots$; $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear operator and $\|a_{ij}\|_{i,j=1}^n$ is the matrix of its representation in some basis of \mathbb{R}^n .

Equation (1) with initial data $x(0) = x_0$ has a unique solution in the class of measurable processes. This solution can be written in the following form:

$$(2) \quad x(t) = \exp\{At\}x_0 + \int_0^t \exp\{A(t-u)\} dz(u).$$

It is shown in [1] that the process $x(t)$ has the limit distribution as $t \rightarrow \infty$ and this distribution does not depend on the initial data x_0 if and only if

- a) the eigenvalues of A belong to the left semiplane,
- b) $E(\ln |\eta^1|; |\eta^1| > 1) < \infty$.

It is also proved in [1] that the limit distribution is a unique stationary distribution of the process $x(t)$ if both of the above conditions hold. The characteristic function of the limit distribution is given by

$$(3) \quad \psi(s) = \exp \left\{ -\lambda \int_0^\infty (1 - \varphi(\exp\{A^T u\} s)) du \right\}$$

where $\varphi(s) = E\{\exp i(s, \eta^1)\}$.

As is seen from equality (2), the stationary distribution of $x(t)$ coincides with the distribution of the vector

$$(4) \quad \xi = \int_0^\infty \exp\{Au\} dz(u).$$

Moreover, equality (3) implies that the characteristic function of the stationary distribution of $x(t)$ is of the form $\psi(s) = \exp\{\lambda K(s)\}$ where $K(s)$ does not depend on λ . In the stationary regime, $x(\cdot, \lambda)$ can be viewed as values of a stochastically continuous homogeneous process with independent increments at the moment λ .

2000 *Mathematics Subject Classification.* Primary 60Fxx, 60G10.

2. SETTING OF THE PROBLEM

The limit behavior of the distribution of $x(\cdot, \lambda)$ as $\lambda \rightarrow 0$ is studied in [1] for the case of $A = U\Lambda U^{-1}$ where $\Lambda = \|\delta_{ij}\lambda_i\|_{i,j=1}^n$, λ_i ($i = 1, \dots, n$) are real eigenvalues of the matrix A such that $\lambda_i < 0$ for all i , and $U = \|u_{ij}\|_{i,j=1}^n$ is a nonsingular matrix.

The limit behavior as $\lambda \rightarrow 0$ of the distribution of $x(\cdot, \lambda)$ is obtained in [5] for the case of $A = UJU^{-1}$ where J is a Jordan matrix,¹ $A = \|a_{ij}\|_{i,j=1}^2$, and $U = \|u_{ij}\|_{i,j=1}^2$.

In this paper, we consider the general case of $A = UJU^{-1}$ where J is a Jordan matrix, $U = \|u_{ij}\|_{i,j=1}^n$ is a nonsingular matrix, and $A = \|a_{ij}\|_{i,j=1}^n$. We study the limit behavior as $\lambda \rightarrow 0$ of the vector

$$\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^T = U^{-1}x(\cdot, \lambda)$$

under the assumption that the distribution of $x(\cdot, \lambda)$ is stationary.

Below we show that the components of the vector \tilde{x} are completely determined by the form of the Jordan blocks. Thus we obtain the limit behavior, as $\lambda \rightarrow 0$, of the part of the vector \tilde{x} that corresponds to a Jordan block J_i . In doing so, we consider separately the cases of real and complex eigenvalues λ_i of the matrix A .

3. AUXILIARY RESULTS AND NOTATION

The process $z(t)$ is completely determined by the heights of the jumps η^1, η^2, \dots and by the lengths of the intervals $\lambda^{-1}\tau_1, \lambda^{-1}\tau_2, \dots$ between the jumps. All the random variables η^j , $j = 1, 2, \dots$, and τ_i , $i = 1, 2, \dots$, are independent and $P\{\tau_i > t\} = \exp\{-t\}$ for $t \geq 0$. Thus we obtain from (4) that

$$(5) \quad \begin{aligned} \xi &= \exp\{\lambda^{-1}\tau_1 A\} \eta^1 + \exp\{\lambda^{-1}(\tau_1 + \tau_2)A\} \eta^2 + \dots \\ &+ \exp\left\{\lambda^{-1}A \sum_{k=1}^j \tau_k\right\} \eta^j + \dots \end{aligned}$$

Below we use the following notation: $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^T = U^{-1}x(\cdot, \lambda)$;

$$\tilde{\eta}^j = (\tilde{\eta}_1^j, \dots, \tilde{\eta}_n^j)^T = U^{-1}\eta^j, \quad j = 1, 2, \dots;$$

$$\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_n)^T = U^{-1}\xi; p_r = P\{\tilde{\eta}_r^j = 0\}; p_r^+ = P\{\tilde{\eta}_r^j > 0\};$$

$$\operatorname{sgn} z = (\operatorname{sgn} z_1, \dots, \operatorname{sgn} z_n)^T \quad \text{for } z = (z_1, \dots, z_n)^T \in \mathbb{R}^n;$$

$J = \{J_1, \dots, J_m\}$ where J_i is the Jordan block of order k_i corresponding to the eigenvalue λ_i , $i = 1, \dots, m$, of the matrix A (there could be equal numbers among the λ_i , $i = 1, \dots, m$); $\sum_{i=1}^m k_i = l_m$, $m = 1, \dots, n$; $l_n = n$; $\nu_i = \lambda\lambda_i^{-1}$ for real λ_i and $\kappa_i = \lambda a_i^{-1}$ for complex $\lambda_i = a_i + ib_i$.

Recall that the matrix $f(A)$ is well defined if $f(t)$ is an analytic function. Since $A = UJU^{-1}$, the matrix $f(J)$ is well defined and, moreover, $f(A) = Uf(J)U^{-1}$. Thus relation (5) can be rewritten in the following form:

$$\begin{aligned} \xi &= U \exp\{\lambda^{-1}\tau_1 J\} \\ &\times U^{-1} \left(\eta^1 + U \exp\{\lambda^{-1}\tau_2 J\} U^{-1}\eta^2 + \dots + U \exp\left\{\lambda^{-1}J \sum_{k=2}^j \tau_k\right\} U^{-1}\eta^j + \dots \right) \end{aligned}$$

or

$$\xi = U \exp\{\lambda^{-1}\tau_1 J\} U^{-1} (\eta^1 + \xi^1),$$

where the random variables τ_1 , ξ^1 , and η^1 are independent and the distributions of ξ and ξ^1 are identical.

¹I.e., a matrix in the Jordan normal form.

Therefore

$$(6) \quad \tilde{\xi} = \exp \{ \lambda^{-1} \tau_1 J \} \left(\tilde{\eta}^1 + \exp \{ \lambda^{-1} \tau_2 J \} \tilde{\eta}^2 + \dots + \exp \left\{ \lambda^{-1} J \sum_{k=2}^j \tau_k \right\} \tilde{\eta}^j + \dots \right)$$

or

$$(7) \quad \tilde{\xi} = \exp \{ \lambda^{-1} \tau_1 J \} \left(\tilde{\eta}^1 + \tilde{\xi}^1 \right),$$

where $\tilde{\xi}^1 = (\tilde{\xi}_1^1, \dots, \tilde{\xi}_n^1) = U^{-1} \xi^1$, the distributions of \tilde{x} , $\tilde{\xi}$, and $\tilde{\xi}^1$ are identical, and

$$(8) \quad \exp \{ \lambda^{-1} \tau_1 J \} = \left\{ \exp \{ \lambda^{-1} \tau_1 J_1 \}, \dots, \exp \{ \lambda^{-1} \tau_1 J_m \} \right\},$$

$$\exp \{ \lambda^{-1} \tau_1 J_i \} = \exp \{ \lambda^{-1} \tau_1 \lambda_i \} \begin{pmatrix} 1 & \frac{\lambda^{-1} \tau_1}{1!} & \dots & \frac{(\lambda^{-1} \tau_1)^{k_i-1}}{(k_i-1)!} \\ 0 & 1 & \dots & \frac{(\lambda^{-1} \tau_1)^{k_i-2}}{(k_i-2)!} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

It is seen from (6) and (7) that the components of the vector $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_n)^T$ as well as those of the vector $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ are determined by the Jordan blocks. Thus, without loss of generality, we restrict our consideration below to the investigation of the part of the vector \tilde{x} that corresponds to the Jordan block J_i of order k_i related to the eigenvector λ_i .

Denote by $(\tilde{x}_{l_{i-1}+1}, \tilde{x}_{l_{i-1}+2}, \dots, \tilde{x}_{l_i})^T$ the part of the vector \tilde{x} that corresponds to the Jordan block J_i and let $(\tilde{\eta}_{l_{i-1}+1}^j, \tilde{\eta}_{l_{i-1}+2}^j, \dots, \tilde{\eta}_{l_i}^j)^T$, $j = 1, 2, \dots$, be the part of the vector $\tilde{\eta}^j$ related to the Jordan block J_i .

We introduce the random events $A_1 = \{ \tilde{\eta}_{l_i}^1 \neq 0 \}$,

$$A_j = \{ \tilde{\eta}_{l_i}^1 = 0, \dots, \tilde{\eta}_{l_i}^{j-1} = 0, \tilde{\eta}_{l_i}^j \neq 0 \},$$

$B_1 = \{ \tilde{\eta}_{l_{i-1}}^1 \neq 0 \}$, $B_j = \{ \tilde{\eta}_{l_{i-1}}^1 = 0, \dots, \tilde{\eta}_{l_{i-1}}^{j-1} = 0, \tilde{\eta}_{l_{i-1}}^j \neq 0 \}$, $j = 2, 3, \dots$, and denote the indicators of events A_j and B_j by $1(A_j)$ and $1(B_j)$, respectively. Let $P\{A_1\} = p$ and $P\{B_1\} = q$. In what follows we assume that all stochastic processes and random variables are defined on the same probability space.

4. MAIN RESULTS

We distinguish between the following two cases.

- I. An eigenvalue $\lambda_i < 0$ of the matrix A is real ($\tilde{x}_{l_{i-1}+1}, \dots, \tilde{x}_{l_i}$ are real in this case).
- II. An eigenvalue $\lambda_i < 0$ of the matrix A is complex; that is,

$$\lambda_i = a_i + ib_i, \quad a_i < 0, \quad b_i \neq 0.$$

In this case, $\tilde{x}_{l_{i-1}+1}, \dots, \tilde{x}_{l_i}$ are complex. We represent these numbers as follows:

$$\tilde{x}_{l_{i-1}+1} = |\tilde{x}_{l_{i-1}+1}| \exp\{i\varphi_{l_{i-1}+1}\}, \quad \dots, \quad \tilde{x}_{l_i} = |\tilde{x}_{l_i}| \exp\{i\varphi_{l_i}\},$$

where $\varphi_{l_{i-1}+1} = \arg \tilde{x}_{l_{i-1}+1}, \dots, \varphi_{l_i} = \arg \tilde{x}_{l_i}$, $\varphi_{l_{i-1}+1}, \dots, \varphi_{l_i} \in (0, 2\pi)$.

4.1. Case I.

Theorem 1. *If $p_{l_i} = 0$ and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$\left(|\tilde{x}_{l_{i-1}+1}|^{-\nu_i}, \dots, |\tilde{x}_{l_i}|^{-\nu_i}, \operatorname{sgn}(\tilde{x}_{l_{i-1}+1}, \dots, \tilde{x}_{l_i}) \right)$$

converges weakly as $\lambda \rightarrow 0$ to the distribution of $(\alpha, \dots, \alpha, \operatorname{sgn}(\tilde{\eta}_{l_i}^1, \dots, \tilde{\eta}_{l_i}^1))$, where α has the uniform distribution on the interval $(0, 1)$ and does not depend on $\tilde{\eta}_{l_i}^1$.

Proof. Since $p_{l_i} = 0$ and the distributions of the vectors \tilde{x} and $\tilde{\xi}$ are identical, we use relations (7) and (8) and obtain

$$(9) \quad \begin{aligned} \left(\tilde{\xi}_{l_{i-1}+1}, \dots, \tilde{\xi}_{l_i}\right)^T &= \exp\{\lambda^{-1}\tau_1 J_i\} \left(\tilde{\xi}_{l_{i-1}+1}^1 + \tilde{\eta}_{l_{i-1}+1}^1, \dots, \tilde{\xi}_{l_i}^1 + \tilde{\eta}_{l_i}^1\right)^T \\ &= \exp\{\lambda^{-1}\tau_1 \lambda_i\} \left(\tilde{\xi}_{l_{i-1}+1}^1, \tilde{\xi}_{l_{i-1}+2}^1, \dots, \tilde{\xi}_{l_i}^1\right)^T, \end{aligned}$$

where

$$\begin{aligned} \tilde{\xi}_{l_{i-1}+1}^1 &= \sum_{m=0}^{k_i-1} \frac{(\lambda^{-1}\tau_1)^m}{m!} \left(\tilde{\xi}_{l_{i-1}+m+1}^1 + \tilde{\eta}_{l_{i-1}+m+1}^1\right), \\ \tilde{\xi}_{l_{i-1}+2}^1 &= \sum_{m=0}^{k_i-2} \frac{(\lambda^{-1}\tau_1)^m}{m!} \left(\tilde{\xi}_{l_{i-1}+m+2}^1 + \tilde{\eta}_{l_{i-1}+m+2}^1\right), \quad \dots, \quad \tilde{\xi}_{l_i}^1 = \tilde{\xi}_{l_i}^1 + \tilde{\eta}_{l_i}^1. \end{aligned}$$

According to Lemma 6.4 in [1], $\tilde{\xi}^1 \xrightarrow{\mathbb{P}} \bar{0}$ as $\lambda \rightarrow 0$ and

$$\begin{aligned} &\mathbb{P} \left\{ \tilde{\eta}_{l_{i-1}+1}^1 + \sum_{m=1}^{k_i-1} \frac{(\lambda^{-1}\tau_1)^m}{m!} \tilde{\eta}_{l_{i-1}+m+1}^1 = 0 \right\} \\ &= \mathbb{P} \left\{ \sum_{m=1}^{k_i-1} \frac{(\lambda^{-1}\tau_1)^m}{m!} \tilde{\eta}_{l_{i-1}+m+1}^1 = -\tilde{\eta}_{l_{i-1}+1}^1 \right\} = 0, \end{aligned}$$

since the random variables τ_1 and $\tilde{\eta}_r^1$ ($r = l_{i-1} + 1, \dots, l_i$) are independent and

$$\sum_{m=1}^{k_i-1} \frac{(\lambda^{-1}\tau_1)^m}{m!} \tilde{\eta}_{l_{i-1}+m+1}^1$$

has an absolutely continuous distribution. Thus

$$\begin{aligned} &\lim_{\lambda \rightarrow 0} \ln \left| \sum_{m=0}^{k_i-1} \frac{(\lambda^{-1}\tau_1)^m}{m!} \left(\tilde{\xi}_{l_{i-1}+m+1}^1 + \tilde{\eta}_{l_{i-1}+m+1}^1\right) \right|^{-\nu_i} \\ &= -\lambda_i^{-1} \lim_{\lambda \rightarrow 0} \frac{\tau_1^{k_i-1} \tilde{\eta}_{l_i}^1}{\tau_1^{k_i-2} \tilde{\eta}_{l_{i-1}}^1 + \lambda^{-1} \tau_1^{k_i-1} \tilde{\eta}_{l_i}^1} = 0. \end{aligned}$$

Therefore relation (9) implies that

$$\begin{aligned} \left|\tilde{\xi}_{l_{i-1}+1}\right|^{-\nu_i} &= \exp\{-\tau_1\} \left| \sum_{m=0}^{k_i-1} \frac{(\lambda^{-1}\tau_1)^m}{m!} \left(\tilde{\xi}_{l_{i-1}+m+1}^1 + \tilde{\eta}_{l_{i-1}+m+1}^1\right) \right|^{-\nu_i} \xrightarrow{\mathbb{P}} \exp\{-\tau_1\}, \\ \left|\tilde{\xi}_{l_{i-1}+2}\right|^{-\nu_i} &\xrightarrow{\mathbb{P}} \exp\{-\tau_1\}, \quad \dots, \quad \left|\tilde{\xi}_{l_i}\right|^{-\nu_i} \xrightarrow{\mathbb{P}} \exp\{-\tau_1\}, \\ \operatorname{sgn} \tilde{\xi}_{l_{i-1}+1} &\xrightarrow{\mathbb{P}} \operatorname{sgn} \tilde{\eta}_{l_i}^1, \quad \operatorname{sgn} \tilde{\xi}_{l_{i-1}+2} \xrightarrow{\mathbb{P}} \operatorname{sgn} \tilde{\eta}_{l_i}^1, \quad \dots, \quad \operatorname{sgn} \tilde{\xi}_{l_i} \xrightarrow{\mathbb{P}} \operatorname{sgn} \tilde{\eta}_{l_i}^1 \end{aligned}$$

as $\lambda \rightarrow 0$. Moreover the random variable $\exp\{-\tau_1\} = \alpha$ has the uniform distribution on the interval $(0, 1)$ and does not depend on $\tilde{\eta}_{l_i}^1$.

Now Theorem 1 follows, since the convergence in probability of the corresponding coordinates of the vectors implies the weak convergence of the distributions of the vectors. \square

Theorem 2. *If $0 < p_{l_i} < 1$ and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$\left(|\tilde{x}_{l_{i-1}+1}|^{-\nu_i}, \dots, |\tilde{x}_{l_i}|^{-\nu_i}, \operatorname{sgn} \tilde{x}_{l_{i-1}+1}, \dots, \operatorname{sgn} \tilde{x}_{l_i}\right)$$

weakly converges as $\lambda \rightarrow 0$ to the distribution of $(\alpha^{1/p}, \dots, \alpha^{1/p}, \gamma, \dots, \gamma)$, where the random variable α has the uniform distribution on the interval $(0, 1)$, while the random variable γ assumes the values 1 and -1 with the probabilities $p_{l_i}^+$ and $(1 - p_{l_i}^+)$, respectively, and does not depend on α .

Proof. If $0 < p_{l_i} < 1$, then we get from relations (6) and (8) that

$$\left(\tilde{\xi}_{l_{i-1}+1}, \tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i}\right)^T = \exp\{\lambda^{-1}\tau_1\lambda_i\} \left(\tilde{\zeta}_{l_{i-1}+1}, \tilde{\zeta}_{l_{i-1}+2}, \dots, \tilde{\zeta}_{l_i}\right)^T,$$

where

$$\begin{aligned} \tilde{\zeta}_{l_{i-1}+1} &= \sum_{m=1}^{k_i} \left(\frac{(\lambda^{-1}\tau_1)^{k_i-m}}{(k_i-m)!} \tilde{\eta}_{l_{i-1}+1}^1 \right. \\ &\quad \left. + \sum_{j=2}^{\infty} \exp\left\{\lambda^{-1}\lambda_i \sum_{k=2}^j \tau_k\right\} \frac{(\lambda^{-1}\sum_{k=1}^j \tau_k)^{k_i-m}}{(k_i-m)!} \tilde{\eta}_{l_{i-1}+1}^j \right), \\ \tilde{\zeta}_{l_{i-1}+2} &= \sum_{m=2}^{k_i} \left(\frac{(\lambda^{-1}\tau_1)^{k_i-m}}{(k_i-m)!} \tilde{\eta}_{l_{i-1}+2}^1 \right. \\ &\quad \left. + \sum_{j=2}^{\infty} \exp\left\{\lambda^{-1}\lambda_i \sum_{k=2}^j \tau_k\right\} \frac{(\lambda^{-1}\sum_{k=1}^j \tau_k)^{k_i-m}}{(k_i-m)!} \tilde{\eta}_{l_{i-1}+2}^j \right), \quad \dots, \\ \tilde{\zeta}_{l_i} &= \tilde{\eta}_{l_i}^1 + \sum_{j=2}^{\infty} \exp\left\{\lambda^{-1}\lambda_i \sum_{k=2}^j \tau_k\right\} \tilde{\eta}_{l_i}^j. \end{aligned}$$

Consider the random events

$$\begin{aligned} A_1 &= \{\tilde{\eta}_{l_i}^1 \neq 0\}, \quad A_2 = \{\tilde{\eta}_{l_i}^1 = 0, \tilde{\eta}_{l_i}^2 \neq 0\}, \quad \dots, \\ A_j &= \{\tilde{\eta}_{l_i}^1 = 0, \dots, \tilde{\eta}_{l_i}^{j-1} = 0, \tilde{\eta}_{l_i}^j \neq 0\}, \quad \dots \end{aligned}$$

One can treat $\Omega = \{A_1, A_2, \dots, A_j, \dots\}$ as the space of elementary events. Moreover $\mathbf{P}\{A_j\} = p(1-p)^{j-1}$, where $\mathbf{P}\{A_1\} = p$.

The restriction of the random variable $\tilde{\xi}_r$, $r = l_{i-1} + 1, \dots, l_i$, on the elementary event A_j , $j = 1, 2, \dots$, is denoted by $\tilde{\xi}_r|_{A_j}$. Then

$$\begin{aligned} \tilde{\xi}_{l_{i-1}+1}|_{A_1} &\equiv \exp\{\lambda^{-1}\tau_1\lambda_i\} \frac{(\lambda^{-1}\tau_1)^{k_i-1}}{(k_i-1)!} \tilde{\eta}_{l_i}^1, \\ (10) \quad \tilde{\xi}_{l_{i-1}+1}|_{A_2} &\equiv \exp\{\lambda^{-1}(\tau_1 + \tau_2)\lambda_i\} \frac{(\lambda^{-1}(\tau_1 + \tau_2))^{k_i-1}}{(k_i-1)!} \tilde{\eta}_{l_i}^2, \quad \dots, \\ \tilde{\xi}_{l_{i-1}+1}|_{A_j} &\equiv \exp\left\{\lambda^{-1}\lambda_i \sum_{k=1}^j \tau_k\right\} \frac{(\lambda^{-1}\sum_{k=1}^j \tau_k)^{k_i-1}}{(k_i-1)!} \tilde{\eta}_{l_i}^j, \quad \dots \end{aligned}$$

Thus we have

$$\begin{aligned} \left|\tilde{\xi}_{l_{i-1}+1}|_{A_1}\right|^{-\nu_i} &\xrightarrow{\text{a.s.}} \exp\{-\tau_1\}, \quad \left|\tilde{\xi}_{l_{i-1}+1}|_{A_2}\right|^{-\nu_i} \xrightarrow{\text{a.s.}} \exp\{-(\tau_1 + \tau_2)\}, \quad \dots, \\ \left|\tilde{\xi}_{l_{i-1}+1}|_{A_j}\right|^{-\nu_i} &\xrightarrow{\text{a.s.}} \exp\left\{-\sum_{k=1}^j \tau_k\right\}, \quad \dots, \end{aligned}$$

as $\lambda \rightarrow 0$; that is,

$$\left| \tilde{\xi}_{l_{i-1}+1} \right|^{-\nu_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp \left\{ - \sum_{k=1}^j \tau_k \right\} 1(A_j) = \chi.$$

A similar reasoning for $\tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i}$ shows that

$$\begin{aligned} \left| \tilde{\xi}_{l_{i-1}+2} \right|^{-\nu_i} &\xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp \left\{ - \sum_{k=1}^j \tau_k \right\} 1(A_j), \quad \dots, \\ \left| \tilde{\xi}_{l_i} \right|^{-\nu_i} &\xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp \left\{ - \sum_{k=1}^j \tau_k \right\} 1(A_j). \end{aligned}$$

Analogously we obtain that

$$\begin{aligned} \operatorname{sgn} \tilde{\xi}_{l_{i-1}+1} &= \sum_{j=1}^{\infty} \operatorname{sgn} \tilde{\eta}_{l_i}^j 1(A_j) = \gamma, \\ \operatorname{sgn} \tilde{\xi}_{l_{i-1}+2} &= \sum_{j=1}^{\infty} \operatorname{sgn} \tilde{\eta}_{l_i}^j 1(A_j), \quad \dots, \quad \operatorname{sgn} \tilde{\xi}_{l_i} = \sum_{j=1}^{\infty} \operatorname{sgn} \tilde{\eta}_{l_i}^j 1(A_j) \end{aligned}$$

as $\lambda \rightarrow 0$.

Since the random variables

$$\exp\{-\tau_j\} = \alpha_j, \quad j = 1, 2, \dots,$$

have the uniform distribution on the interval $(0,1)$, the Laplace transform of χ is given by

$$\begin{aligned} \mathbb{E} \exp\{-s\chi\} &= \mathbb{E} \exp \left\{ -s \sum_{j=1}^{\infty} \alpha_1 \dots \alpha_j 1(A_j) \right\} \\ &= \sum_{k=1}^{\infty} \mathbb{P}(A_k) \mathbb{E}_{A_k} \exp \left\{ -s \sum_{j=1}^{\infty} \alpha_1 \dots \alpha_j 1(A_j) \right\} \\ &= \sum_{k=1}^{\infty} \mathbb{P}(A_k) \mathbb{E} \exp \{-s\alpha_1 \dots \alpha_k\}, \end{aligned}$$

where

$$\mathbb{E} \exp\{-s\alpha_1 \dots \alpha_k\} = \int_0^1 \dots \int_0^1 \exp\{-sx_1 \dots x_k\} dx_1 \dots dx_k.$$

The series

$$\exp\{-sx_1 \dots x_k\} = \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} x_1^j \dots x_k^j$$

converges by the d'Alembert criterion for any $x^0 = x_1^0 \dots x_k^0$; thus it converges on $(0,1)$. By the Weierstrass criterion this series is uniformly convergent on $(0,1)$. Since the terms of this series are continuous functions on $(0,1)$, we get

$$\mathbb{E} \exp\{-s\alpha_1 \dots \alpha_k\} = \sum_{j=0}^{\infty} \int_0^1 \dots \int_0^1 \frac{(-s)^j}{j!} x_1^j \dots x_k^j dx_1 \dots dx_k = \sum_{j=0}^{\infty} \frac{(-s)^j}{j! (j+1)^k}.$$

Hence

$$\begin{aligned} \mathbb{E} \exp\{-s\chi\} &= \sum_{k=1}^{\infty} p(1-p)^{k-1} \sum_{j=0}^{\infty} \frac{(-s)^j}{j!(j+1)^k} = p \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} \sum_{k=1}^{\infty} \frac{(1-p)^{k-1}}{(j+1)^k} \\ &= p \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} \frac{1}{(j+1)} \sum_{k=1}^{\infty} \left(\frac{1-p}{j+1}\right)^{k-1} = p \sum_{j=0}^{\infty} \frac{(-s)^j}{j!} \frac{1}{j+p} = \phi(s) \end{aligned}$$

or

$$\begin{aligned} \phi(s)(-1)^p s^p &= p \sum_{j=0}^{\infty} \frac{(-s)^{j+p}}{j!(j+p)} = \int_0^s d_u (\phi(u)(-u)^p) = -p \int_0^s \sum_{j=0}^{\infty} \frac{(-u)^{j+p-1}}{j!} du \\ &= -p \int_0^s (-u)^{p-1} \sum_{j=0}^{\infty} \frac{(-u)^j}{j!} du = p(-1)^p \int_0^s u^{p-1} \exp\{-u\} du \\ &= p(-1)^p \gamma(p, s), \end{aligned}$$

where

$$(-1)^p = \exp\{(2n+1)p\pi i\} = \cos(2n+1)p\pi + i \sin(2n+1)p\pi, \quad n = 0, \pm 1, \pm 2, \dots,$$

and $\gamma(p, s)$ is the incomplete gamma function.

Therefore

$$\phi(s) = ps^{-p} \gamma(p, s).$$

Using corresponding relations for the inversion of the Laplace–Carson transform [3], we evaluate the density of χ :

$$g(t) = \begin{cases} pt^{p-1}, & 0 < t < 1, \\ 0, & t > 1, \end{cases}$$

and the corresponding distribution function

$$G(t) = \begin{cases} t^p, & 0 < t < 1, \\ 0, & t > 1. \end{cases}$$

If α has the uniform distribution on $(0, 1)$, then $\chi = \alpha^{1/p}$. The theorem is proved. \square

Theorem 3. *If $p_{l_i} = 1$, $p_{l_{i-1}} = 0$, and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of $(|\tilde{x}_{l_{i-1}+1}|^{-\nu_i}, \dots, |\tilde{x}_{l_i-1}|^{-\nu_i}, |\tilde{x}_{l_i}|, \text{sgn } \tilde{x}_{l_{i-1}+1}, \dots, \text{sgn } \tilde{x}_{l_i-1}, \text{sgn } \tilde{x}_{l_i})$ weakly converges as $\lambda \rightarrow 0$ to the distribution of $(\alpha, \dots, \alpha, 0, \text{sgn } \tilde{\eta}_{l_{i-1}}^1, \dots, \text{sgn } \tilde{\eta}_{l_i-1}^1, 0)$, where α has the uniform distribution on $(0, 1)$ and does not depend on $\tilde{\eta}_{l_{i-1}}^1$.*

Proof. The proof of Theorem 3 is similar to that of Theorem 1. \square

Theorem 4. *If $p_{l_i} = 1$, $0 < p_{l_{i-1}} < 1$, and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$(|\tilde{x}_{l_{i-1}+1}|^{-\nu_i}, \dots, |\tilde{x}_{l_i-1}|^{-\nu_i}, |\tilde{x}_{l_i}|, \text{sgn } \tilde{x}_{l_{i-1}+1}, \dots, \text{sgn } \tilde{x}_{l_i-1}, \text{sgn } \tilde{x}_{l_i})$$

weakly converges as $\lambda \rightarrow 0$ to the distribution of $(\alpha^{1/q}, \dots, \alpha^{1/q}, 0, \gamma, \dots, \gamma, 0)$, where α has the uniform distribution on $(0, 1)$, while the random variable γ assumes the values 1 and -1 with probabilities $p_{l_{i-1}}^+$ and $1 - p_{l_{i-1}}^+$, respectively, and does not depend on α .

Proof. The proof of Theorem 4 is analogous to that of Theorem 2. Note however that the random events $B_1 = \{\tilde{\eta}_{l_{i-1}}^1 \neq 0\}$, $B_j = \{\tilde{\eta}_{l_{i-1}}^1 = 0, \dots, \tilde{\eta}_{l_{i-1}}^{j-1} = 0, \tilde{\eta}_{l_{i-1}}^j \neq 0\}$, $j = 2, 3, \dots$, should be substituted for the random events A_j , $j \geq 1$, in the proof of Theorem 4. \square

4.2. Case II.

Theorem 5. *If $p_{l_i} = 0$ and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$\left(|\tilde{x}_{l_{i-1}+1}|^{-\kappa_i}, \dots, |\tilde{x}_{l_i}|^{-\kappa_i}, \varphi_{l_{i-1}+1}, \dots, \varphi_{l_i} \right)$$

weakly converges as $\lambda \rightarrow 0$ to the distribution of $(\alpha, \dots, \alpha, \beta, \dots, \beta)$, where the distribution of the random variable α is uniform on the interval $(0, 1)$, the distribution of the random variable β is uniform on the interval $(0, 2\pi)$, and β does not depend on α .

Proof. Since $p_{l_i} = 0$, $\lambda_i = a_i + ib_i$, and the matrix U is complex, the vector

$$\left(\tilde{\xi}_{l_{i-1}+1}, \tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i} \right)^T$$

is of the form

$$(11) \quad \begin{aligned} & \left(\tilde{\xi}_{l_{i-1}+1}, \tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i} \right)^T \\ & = \exp \{ \lambda^{-1} \tau_1 a_i \} \exp \{ i \lambda^{-1} \tau_1 b_i \} \left(\zeta_{l_{i-1}+1}, \zeta_{l_{i-1}+2}, \dots, \zeta_{l_i} \right)^T \end{aligned}$$

in view of relations (7) and (8), where

$$\zeta_r = |\zeta_r| \exp \{ i \gamma_r \}, \quad \gamma_r = \arg \zeta_r, \quad \gamma_r \in (0, 2\pi), \quad r = l_{i-1} + 1, \dots, l_i.$$

Therefore (11) implies that

$$\begin{aligned} \left| \tilde{\xi}_{l_{i-1}+1} \right|^{-\kappa_i} &= \exp \{ -\tau_1 \} \left| \exp \{ i (\gamma_{l_{i-1}+1} + \lambda^{-1} \tau_1 b_i) \} \right|^{-\kappa_i} \xrightarrow{P} \exp \{ -\tau_1 \}, \\ \left| \tilde{\xi}_{l_{i-1}+2} \right|^{-\kappa_i} &\xrightarrow{P} \exp \{ -\tau_1 \}, \quad \dots, \quad \left| \tilde{\xi}_{l_i} \right|^{-\kappa_i} \xrightarrow{P} \exp \{ -\tau_1 \} \end{aligned}$$

as $\lambda \rightarrow 0$. Moreover the distribution of the random variable $\exp \{ -\tau_1 \} = \alpha$ is uniform on the interval $(0, 1)$, and

$$\begin{aligned} \arg \tilde{\xi}_{l_{i-1}+1} &\equiv (\gamma_{l_{i-1}+1} + \lambda^{-1} \tau_1 b_i) \pmod{2\pi}, \\ \arg \tilde{\xi}_{l_{i-1}+2} &\equiv (\gamma_{l_{i-1}+2} + \lambda^{-1} \tau_1 b_i) \pmod{2\pi}, \quad \dots, \\ \arg \tilde{\xi}_{l_i} &\equiv (\gamma_{l_i} + \lambda^{-1} \tau_1 b_i) \pmod{2\pi} \end{aligned}$$

as $\lambda \rightarrow 0$.

Let $f_{l_{i-1}+1}(t), f_{l_{i-1}+2}(t), \dots, f_{l_i}(t), t \in (0, 2\pi)$, be the densities of the random variables

$$\begin{aligned} & (\gamma_{l_{i-1}+1} + \lambda^{-1} \tau_1 b_i) \pmod{2\pi}, \\ & (\gamma_{l_{i-1}+2} + \lambda^{-1} \tau_1 b_i) \pmod{2\pi}, \quad \dots, \\ & (\gamma_{l_i} + \lambda^{-1} \tau_1 b_i) \pmod{2\pi}, \end{aligned}$$

respectively, defined on a circle of length 2π , while $f(t)$ is the density of the random variable $\lambda^{-1} \tau_1 b_i$. The random variable $\lambda^{-1} \tau_1 b_i \pmod{2\pi}$ is defined on a circle, and its density \tilde{f} is given by

$$\begin{aligned} \tilde{f}(t) &= \sum_{k=0}^{\infty} f(t + 2\pi k) = \lambda b_i^{-1} \sum_{k=0}^{\infty} \exp \{ -\lambda b_i^{-1} (t + 2\pi k) \} = \frac{\lambda \exp \{ -\lambda b_i^{-1} t \}}{b_i (1 - \exp \{ -\lambda b_i^{-1} 2\pi \})}, \\ \lim_{\lambda \rightarrow 0} \tilde{f}(t) &= \lim_{\lambda \rightarrow 0} \frac{\exp \{ -\lambda b_i^{-1} t \} - \lambda b_i^{-1} t \exp \{ -\lambda b_i^{-1} t \}}{2\pi \exp \{ -\lambda b_i^{-1} 2\pi \}} = \frac{1}{2\pi}. \end{aligned}$$

It is known [2] that the convolution of the uniform density on a circle with an arbitrary density on a circle is the density of the uniform distribution. Thus

$$\lim_{\lambda \rightarrow 0} f_{l_{i-1}+1}(t) = \frac{1}{2\pi}, \quad \lim_{\lambda \rightarrow 0} f_{l_{i-1}+2}(t) = \frac{1}{2\pi}, \quad \dots, \quad \lim_{\lambda \rightarrow 0} f_{l_i}(t) = \frac{1}{2\pi}$$

as $\lambda \rightarrow 0$. Therefore the distributions of the random variables

$$(\gamma_{l_{i-1}+1} + \lambda^{-1}\tau_1 b_i) \pmod{2\pi},$$

$$(\gamma_{l_{i-1}+2} + \lambda^{-1}\tau_1 b_i) \pmod{2\pi}, \quad \dots, \quad (\gamma_{l_i} + \lambda^{-1}\tau_1 b_i) \pmod{2\pi}$$

as $\lambda \rightarrow 0$ coincide with the uniform distribution on the interval $(0, 2\pi)$, and these random variables do not depend on α .

Now Theorem 5 follows from the properties of the weak convergence and convergence in probability. \square

Theorem 6. *If $0 < p_i < 1$ and the distribution of $x(\cdot, \lambda)$ is stationary, then the distribution of*

$$(|\tilde{x}_{l_{i-1}+1}|^{-\kappa_i}, \dots, |\tilde{x}_{l_i}|^{-\kappa_i}, \varphi_{l_{i-1}+1}, \dots, \varphi_{l_i})$$

weakly converges as $\lambda \rightarrow 0$ to the distribution of $(\alpha^{1/p}, \dots, \alpha^{1/p}, \beta, \dots, \beta)$ where the random variable α has the uniform distribution on the interval $(0, 1)$, the random variable β has the uniform distribution on the interval $(0, 2\pi)$, and β does not depend on α .

Proof. The proof of Theorem 6 is analogous to that of Theorem 2.

Since $\lambda_i = a_i + ib_i$, relation (10) can be rewritten as follows:

$$\tilde{\xi}_{l_{i-1}+1} \Big|_{A_1} \equiv \exp \{ \lambda^{-1} \tau_1 a_i \} \exp \{ i \lambda^{-1} \tau_1 b_i \} \frac{(\lambda^{-1} \tau_1)^{k_i-1}}{(k_i-1)!} \tilde{\eta}_{l_i}^1,$$

$$\tilde{\xi}_{l_{i-1}+1} \Big|_{A_2} \equiv \exp \{ \lambda^{-1} (\tau_1 + \tau_2) a_i \} \exp \{ i \lambda^{-1} (\tau_1 + \tau_2) b_i \} \frac{(\lambda^{-1} (\tau_1 + \tau_2))^{k_i-1}}{(k_i-1)!} \tilde{\eta}_{l_i}^2, \dots,$$

$$\tilde{\xi}_{l_{i-1}+1} \Big|_{A_j} \equiv \exp \left\{ \lambda^{-1} a_i \sum_{k=1}^j \tau_k \right\} \exp \left\{ i \lambda^{-1} b_i \sum_{k=1}^j \tau_k \right\} \frac{(\lambda^{-1} \sum_{k=1}^j \tau_k)^{k_i-1}}{(k_i-1)!} \tilde{\eta}_{l_i}^j, \dots,$$

where

$$\tilde{\eta}_{l_i}^j = \left| \tilde{\eta}_{l_i}^j \right| \exp \{ i \phi_j \}, \quad \phi_j = \arg \tilde{\eta}_{l_i}^j, \quad j = 1, 2, \dots, \quad \phi_j \in (0, 2\pi),$$

and

$$\arg \tilde{\xi}_{l_{i-1}+1} \Big|_{A_1} \equiv (\phi_1 + \lambda^{-1} \tau_1 b_i) \pmod{2\pi},$$

$$\arg \tilde{\xi}_{l_{i-1}+1} \Big|_{A_2} \equiv (\phi_2 + \lambda^{-1} \tau_1 b_i + \lambda^{-1} \tau_2 b_i) \pmod{2\pi}, \quad \dots,$$

$$\arg \tilde{\xi}_{l_{i-1}+1} \Big|_{A_j} \equiv \left(\phi_j + \lambda^{-1} \tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^j \tau_k \right) \pmod{2\pi}, \quad \dots$$

Then

$$\arg \tilde{\xi}_{l_{i-1}+1} = \sum_{j=1}^{\infty} \left(\phi_j + \lambda^{-1} \tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^j \tau_k \right) \pmod{2\pi} 1(A_j).$$

Therefore

$$\left| \tilde{\xi}_{l_{i-1}+1} \Big|_{A_1} \right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \exp \{ -\tau_1 \}, \quad \left| \tilde{\xi}_{l_{i-1}+1} \Big|_{A_2} \right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \exp \{ -(\tau_1 + \tau_2) \}, \quad \dots,$$

$$\left| \tilde{\xi}_{l_{i-1}+1} \Big|_{A_j} \right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \exp \left\{ -\sum_{k=1}^j \tau_k \right\}, \quad \dots,$$

as $\lambda \rightarrow 0$; that is,

$$\left| \tilde{\xi}_{l_{i-1}+1} \right|^{-\kappa_i} \xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp \left\{ - \sum_{k=1}^j \tau_k \right\} 1(A_j) = \chi.$$

As in the proof of Theorem 2, $\chi = \alpha^{1/p}$ where the random variable α has the uniform distribution on the interval $(0, 1)$.

A similar reasoning for $\tilde{\xi}_{l_{i-1}+2}, \dots, \tilde{\xi}_{l_i}$ yields

$$\begin{aligned} \left| \tilde{\xi}_{l_{i-1}+2} \right|^{-\kappa_i} &\xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp \left\{ - \sum_{k=1}^j \tau_k \right\} 1(A_j), \quad \dots, \\ \left| \tilde{\xi}_{l_i} \right|^{-\kappa_i} &\xrightarrow{\text{a.s.}} \sum_{j=1}^{\infty} \exp \left\{ - \sum_{k=1}^j \tau_k \right\} 1(A_j), \\ \arg \tilde{\xi}_{l_{i-1}+2} &= \sum_{j=1}^{\infty} \left(\phi_j + \lambda^{-1} \tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^j \tau_k \right) \pmod{2\pi} 1(A_j), \quad \dots, \\ \arg \tilde{\xi}_{l_i} &= \sum_{j=1}^{\infty} \left(\phi_j + \lambda^{-1} \tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^j \tau_k \right) \pmod{2\pi} 1(A_j). \end{aligned}$$

Since the random variable $\lambda^{-1} \tau_1 b_i \pmod{2\pi}$ has the uniform density on $(0, 2\pi)$ as $\lambda \rightarrow 0$ (see Theorem 5) and the convolution of the uniform density on a circle with an arbitrary density on a circle is again uniform, the densities of the random variables

$$\gamma_j = \left(\phi_j + \lambda^{-1} \tau_1 b_i + \lambda^{-1} b_i \sum_{k=2}^j \tau_k \right) \pmod{2\pi}, \quad j = 1, 2, \dots,$$

are again uniform on $(0, 2\pi)$ as $\lambda \rightarrow 0$. Thus

$$\begin{aligned} \arg \tilde{\xi}_{l_{i-1}+1} &= \sum_{j=1}^{\infty} \gamma_j 1(A_j) = \beta, \\ \arg \tilde{\xi}_{l_{i-1}+2} &= \sum_{j=1}^{\infty} \gamma_j 1(A_j), \quad \dots, \quad \arg \tilde{\xi}_{l_i} = \sum_{j=1}^{\infty} \gamma_j 1(A_j) \end{aligned}$$

as $\lambda \rightarrow 0$.

The Laplace transform of the random variable β is given by

$$\mathbf{E} \exp \{-s\beta\} = \mathbf{E} \exp \left\{ -s \sum_{j=1}^{\infty} \gamma_j 1(A_j) \right\} = \sum_{k=1}^{\infty} \mathbf{P}(A_k) \mathbf{E} \exp \{-s\gamma_k\},$$

where

$$\mathbf{E} \exp \{-s\gamma_k\} = \int_0^{2\pi} \exp \{-sx_k\} d\left(\frac{x_k}{2\pi}\right) = \frac{1 - \exp \{-2\pi s\}}{2\pi s}.$$

Therefore

$$\mathbf{E} \exp \{-s\beta\} = \frac{1}{2\pi} \frac{1 - \exp \{-2\pi s\}}{s} \sum_{k=1}^{\infty} p(1-p)^{k-1} = \frac{1}{2\pi} \frac{1 - \exp \{-2\pi s\}}{s}$$

and thus the random variable β has the uniform distribution on $(0, 2\pi)$ and does not depend on α .

Theorem 6 is proved. □

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Received 1/SEP/2003

Translated by S. V. KVASKO