ON A QUEUEING SYSTEM WITH SEQUENTIAL SERVICE

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1. INTRODUCTION

We study a one channel queueing system with sequential service. The system contains \( n \) servers; every customer should be served sequentially by each of the servers. The input for the first server is a generalized Poisson process with parameter \( \lambda \); the speed of service at a server \( k \) is proportional to the amount of unfinished work at this server. Any customer arriving at the system should sequentially be served by every server; the total service time for the \( i \)-th customer is denoted by \( \eta_i \).

Let \( x_k(t), k = 1, \ldots, n \), be the unfinished work at the \( k \)-th server at the moment \( t \), and let \( \mu_kx_k(t), \mu_k > 0 \), be the speed of service at server \( k = 1, \ldots, n \). Assume that the total unfinished work in the system \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n \) satisfies the Langevin equation

\[
\frac{dx(t)}{dt} = Ax(t) dt + dz(t),
\]

where \( z(t) = (z_1(t), 0, \ldots, 0)^T \in \mathbb{R}^n \) is a generalized Poisson process with parameter \( \lambda \) and jumps \( \eta^1, \eta^2, \ldots, \eta^n \), and

\[
A = \begin{bmatrix}
-\mu_1 & 0 & 0 & \cdots & 0 & 0 \\
\mu_1 & -\mu_2 & 0 & \cdots & 0 & 0 \\
0 & \mu_2 & -\mu_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \mu_{n-1} & -\mu_n
\end{bmatrix}
\]

for some \( \mu_k > 0, k = 1, \ldots, n \).

2. SETTING OF THE PROBLEM

A one channel queueing system containing a single server whose unfinished work satisfies the differential equation

\[
\frac{dx(t)}{dt} = -\mu x(t) dt + dz(t)
\]

is considered in [1] for the case where \( z(t) \) is a generalized Poisson process and \( \mu > 0 \). For such systems, the waiting time until the start of the service and the time spent by a
customer in the system are studied in [1] under the condition of busy traffic (that is, as $\lambda \to \infty$).

It is of interest to study the behavior of the time spent by a customer in a one channel queueing system with sequential service, and this is the aim of this paper.

The time spent by a customer in a queueing system depends on the discipline in the system; we consider FIFO systems (“First In First Out”). In what follows, such a system is called a filter system.

3. Conditions for the Existence of a Stationary Regime in a Filter System

It is shown in [1] that the process $x(t)$ has the limit distribution as $t \to \infty$; moreover, the limit distribution does not depend on the initial value $x_0 = x(0)$ if and only if

- a) eigenvalues of $A$ belong to the left half-plane, and
- b) $E(\ln |\eta|; |\eta| > 1) < \infty$.

It is also proved in [1] that the limit distribution is a unique stationary distribution of the process $x(t)$ if assumptions a) and b) hold.

Note that condition a) is equivalent to

$$\mu_1 > 0, \quad \mu_2 > 0, \quad \ldots, \quad \mu_n > 0,$$

as is easily seen from (2).

4. Auxiliary Results

Let the stationary distribution of the process $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ coincide with the distribution of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$. By $F(y_1, y_2, \ldots, y_n)$ we denote the distribution function of $\eta = (\eta_1, 0, \ldots, 0)^T$ and by $E \eta = (E \eta_1, 0, \ldots, 0)^T$ we denote the expectation of $\eta$ where $E \eta_1 = \int_0^\infty \cdots \int_0^\infty y_1 dF(y_1, y_2, \ldots, y_n) = m_1$. Let $\varphi(s) = E\{\exp(is, \eta)\}$ be the characteristic function of $\eta$, where $s = (s_1, \ldots, s_n)$.

**Lemma 4.1.** If the stationary distribution of the process $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ coincides with the distribution of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ and $E \eta_1 = m_1 < \infty$, then the vector $\lambda^{-1}(\xi_1, \xi_2, \ldots, \xi_n)^T$ converges in probability to the vector $m_1(\mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_n^{-1})^T$ as $\lambda \to \infty$.

**Proof.** It is shown in [1] that the characteristic function of $\xi$ is of the form

$$\Xi_\xi(s) = \exp\left\{-\lambda \int_0^\infty (1 - \varphi(\exp\{uA^T\} s)) \, du\right\},$$

whence

$$\ln\Xi_{\lambda^{-1}\xi}(s) = \ln\Xi_\xi(\lambda^{-1}s) = \int_0^\infty \frac{\varphi(\exp\{uA^T\} \lambda^{-1}s) - 1}{\lambda^{-1}} \, du$$

$$= |\lambda^{-1} - l| = \int_0^\infty \frac{\varphi(\exp\{uA^T\} ls) - 1}{l} \, du.$$

Put

$$\exp\{uA^T\} ls = (\theta_1, \theta_2, \ldots, \theta_n) = (lk_1(u), lk_2(u), \ldots, lk_n(u)),$$

where

$$k_i(u) = \sum_{i=1}^n a_{ii} \exp\{-\mu_i u\} \sum_{j=1}^{r_i} c_{ij} u^{j-1}, \quad i = 1, \ldots, n,$$

$r_i$ is the multiplicity of the eigenvalue $\lambda_i = -\mu_i$ of the matrix $A$, and $a_{ii}, c_{ij}$ are some constants. Then

$$\varphi(\theta_1, \theta_2, \ldots, \theta_n) = 1 + (im_1, 0, \ldots, 0) (\theta_1, \theta_2, \ldots, \theta_n) + o_u(\rho)$$

for

$$\rho = \sqrt{\theta_1^2 + \theta_2^2 + \cdots + \theta_n^2} = |l| \sqrt{k_1^2(u) + k_2^2(u) + \cdots + k_n^2(u)}.$$
Thus
\[ \ln \Xi_{\lambda^{-1}\xi}(s) = \int_0^\infty \left( (im_1, 0, \ldots, 0) \exp \left\{ uA^T \right\} s + \frac{o_u(\rho)}{l} \right) du \]
\[ = (im_1, 0, \ldots, 0) \int_0^\infty \exp \left\{ uA^T \right\} du (s_1, s_2, \ldots, s_n)^T + \int_0^\infty \frac{o_u(\rho)}{l} du. \]

Put
\[ \lim_{l \to 0} \left| \int_0^\infty \frac{o_u(\rho)}{l} du \right| = \varepsilon. \]

For an arbitrary \( \delta > 0 \), we choose \( \kappa \) such that \( |o_u(\rho)| < \delta \rho \) if \( \rho < \kappa \). Then
\[ \varepsilon < \lim_{l \to 0} \int_0^\infty \frac{\delta \rho}{l} du = \lim_{l \to 0} \int_0^\infty \frac{|l| \sqrt{k_1^2(u) + k_2^2(u) + \cdots + k_n^2(u)}}{l} du \]
\[ \leq \delta \int_0^\infty (k_1^2(u) + k_2^2(u) + \cdots + k_n^2(u))^{1/2} du \leq \delta \int_0^\infty (n \max_i k_i^2(u))^{1/2} du \]
\[ \leq \delta n^{1/2} \int_0^\infty \max_i |k_i(u)| du. \]

Since \( k_i(u) \leq n^2 \max_{a_{il}} |a_{il}| \max_j |c_{ij}| \exp \{ -\mu_i u \} u^{i-1} \), \( i = 1, \ldots, n \), we have
\[ \varepsilon < \delta n \max_{i,l} |a_{il}| \max_{j} |c_{ij}| \int_0^\infty u^{i-1} \exp \{ -\mu_i u \} du = \delta n \max_{i,l} |a_{il}| \max_{j} |c_{ij}| (j-1)! \mu_i^{-j}, \]
whence \( \varepsilon = 0 \).

Since \( A \) is nonsingular, the eigenvalues of the matrix \( A \) belong to the left half-plane. Hence
\[ \int_0^\infty \exp \left\{ uA^T \right\} du = - (A^T)^{-1} \]
(see [2]) and
\[ \lim_{\lambda \to \infty} \ln \Xi_{\lambda^{-1}\xi}(s) = (im_1, 0, \ldots, 0) (-A)^{-1} (s_1, s_2, \ldots, s_n)^T. \]

It follows from (2) that the matrix \((-A)^{-1}\) is of the form
\[ (-A)^{-1} = \begin{bmatrix}
\mu_1^{-1} & \mu_2^{-1} & \cdots & \mu_n^{-1} \\
0 & \mu_2^{-1} & \cdots & \mu_n^{-1} \\
0 & 0 & \cdots & \mu_n^{-1} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \mu_n^{-1} \\
0 & 0 & 0 & 0 & \mu_n^{-1}
\end{bmatrix}. \]

Thus
\[ \lim_{\lambda \to \infty} \Xi_{\lambda^{-1}\xi}(s) = \exp \left\{ i \left( s_1 m_1 \mu_1^{-1} + s_2 m_1 \mu_2^{-1} + \cdots + s_n m_1 \mu_n^{-1} \right) \right\} \]
and
\[ \lambda^{-1} (\xi_1, \xi_2, \ldots, \xi_n)^T \xrightarrow{P} m_1 \left( \mu_1^{-1}, \mu_2^{-1}, \ldots, \mu_n^{-1} \right)^T. \]

**Lemma 4.2.** If \( z(t) = (z_1(t), 0, \ldots, 0)^T \) is a generalized Poisson process and
\[ \mathbb{E} \eta_1 = m_1 < \infty, \]
then the vector \( \lambda^{-1} (z_1(t), 0, \ldots, 0)^T \) converges in probability to the vector \( (m_1 t, 0, \ldots, 0)^T \)
as \( \lambda \to \infty. \)
Proof. The characteristic function of $\lambda^{-1}z(t)$ is given by

$$\Psi_{\lambda^{-1}z(t)}(s) = \mathbb{E}\exp\{is\lambda^{-1}z(t)\} = \exp\{\lambda t (\varphi(\lambda^{-1}s) - 1)\}.$$ 

Therefore,

$$\lim_{\lambda \to \infty} \ln \Psi_{\lambda^{-1}z(t)}(s) = \lim_{\lambda \to \infty} \lambda t (\varphi(\lambda^{-1}s) - 1) = |\lambda^{-1} = l, \lambda^{-1}s_i = \theta_i|$$

whence

$$t(\varphi(t^*) - 1) = \lim_{l \to 0} \frac{d}{dl} \varphi(s(t)) = \lim_{l \to 0} \frac{\partial \varphi}{\partial \theta_i} \frac{d\theta_i}{dl}.$$ 

Since $d\theta_i/dl = s_i$, we get

$$\lim_{\lambda \to \infty} \Psi_{\lambda^{-1}z(t)}(s) = \exp\{im_1ts_1\},$$

which completes the proof of the lemma.

□

5. THE TIME SPENT BY A CUSTOMER IN A FILTER SYSTEM UNDER BUSY TRAFFIC

Assume that a customer arriving at the system at a moment $t_0$ needs $y$ units of service time,

$$x(t_0) = (x_1(t_0), x_2(t_0), \ldots, x_n(t_0)) = (x_1^* + y, x_2^*, \ldots, x_n^*).$$

Let $S$ be the total time spent by the customer in the system and let $W_i$ be the waiting time at the $i$-th server, $i = 1, \ldots, n$. To determine the distributions of these characteristics we consider the amount $\beta_i(t)$ of work finished by the server $i$ until the moment $t \geq t_0$. Without loss of generality let $t_0 = 0$. Then

$$(3) \quad \{ S > t \} = \{ x_1^* + x_2^* + \cdots + x_n^* + y > \beta_i(t) \},$$

$$(4) \quad \{ W_i > t \} = \{ x_1^* + x_2^* + \cdots + x_n^* > \beta_i(t) \}$$

by the definition of the process $\beta_i(t)$, where $d\beta_i(t) = \mu_i x_i(t) dt$, $\beta_i(0) = 0$. Rewriting system (1) as follows:

$$\begin{cases}
dx_1(t) = dz_1(t) - \mu_1 x_1(t) dt, \\
dx_2(t) = \mu_1 x_1(t) dt - \mu_2 x_2(t) dt, \\
\vdots \\
dx_n(t) = \mu_{n-1} x_{n-1}(t) dt - \mu_n x_n(t) dt
\end{cases}$$

and summing up all the equations of system (5), we obtain

$$dx_1(t) + dx_2(t) + \cdots + dx_n(t) = dz_1(t) - \mu_n x_n(t) dt,$$

$$\mu_n x_n(t) dt = dz_1(t) - dx_1(t) - dx_2(t) - \cdots - dx_n(t),$$

$$d\beta_n(t) = dz_1(t) - dx_1(t) - dx_2(t) - \cdots - dx_n(t),$$

$$\int_0^t d\beta_n(u) = \int_0^t dz_1(u) - \int_0^t dx_1(u) - \int_0^t dx_2(u) - \cdots - \int_0^t dx_n(u),$$

$$\beta_n(t) = z_1(t) - x_1(t) + x_1(0) - x_2(t) + x_2(0) - \cdots - x_n(t) + x_n(0).$$

Thus

$$\beta_n(t) = z_1(t) - x_1(t) - x_2(t) - \cdots - x_n(t) + x_1^* + x_2^* + \cdots + x_n^* + y.$$ 

This equality together with (3) implies that

$$\{ S > t \} = \{ x_1^* + \cdots + x_n^* + y > z_1(t) - x_1(t) - \cdots - x_n(t) + x_1^* + \cdots + x_n^* + y \}$$

$$= \{ x_1(t) + x_2(t) + \cdots + x_n(t) > z_1(t) \},$$

whence

$$\begin{align*}
P \{ S > t \} &= P \{ x_1(t) + x_2(t) + \cdots + x_n(t) > z_1(t) \}.
\end{align*}$$
Proceeding in a similar way we get

$$P \{ W_i > t \} = P \{ x_1(t) + x_2(t) + \cdots + x_i(t) > z(t) + y \}. \quad (7)$$

Now we consider the total time $T$ spent by a customer in the system and the waiting time $W_i^s$ at the $i$-th server. In what follows we assume that the system remains in the stationary regime. The distributions of these characteristics are given by equalities (6) and (7), respectively. Moreover, the stationary distribution of

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$$

coinsides with the distribution of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$ (see [3]).

**Theorem 5.1.** If $E \eta_1 = m_1 < \infty$, then

$$(\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1})^{-1} T \xrightarrow{m} 1 \quad as \quad \lambda \to \infty.$$

**Proof.** Equality (6) implies that

$$P \{ T > t \} = P \{ \xi_1 + \xi_2 + \cdots + \xi_n > z(t) \} = P \{ \lambda^{-1} (\xi_1 + \xi_2 + \cdots + \xi_n) > \lambda^{-1} z(t) \}. \quad (8)$$

We obtain from Lemmas 4.1 and 4.2 that

$$P \{ T > t \} \xrightarrow{m} P \{ m_1 \mu_1^{-1} + m_1 \mu_2^{-1} + \cdots + m_1 \mu_n^{-1} > m_1 t \}$$

$$= P \{ \mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1} > t \} = \begin{cases} 1, & t < t_{cr}^1, \\ 0, & t \geq t_{cr}^1, \end{cases}$$

as $\lambda \to \infty$, where $t_{cr}^1 = \mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_n^{-1}$. This proves Theorem 5.1. \hfill \Box

**Theorem 5.2.** If $E \eta_1 = m_1 < \infty$, then

$$(\mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_i^{-1})^{-1} W_i^s \xrightarrow{m} 1 \quad as \quad \lambda \to \infty.$$\hfill \Box

**Proof.** Equality (7) implies that

$$P \{ W_i^s > t \} = P \{ \xi_1 + \xi_2 + \cdots + \xi_i > z_i(t) + y \}$$

$$= P \{ \lambda^{-1} (\xi_1 + \xi_2 + \cdots + \xi_i) > \lambda^{-1} z_i(t) + \lambda^{-1} y \}. \quad (9)$$

Lemmas 4.1 and 4.2 imply that

$$P \{ W_i^s > t \} \xrightarrow{m} P \{ m_1 \mu_1^{-1} + m_1 \mu_2^{-1} + \cdots + m_1 \mu_i^{-1} > m_1 t \}$$

$$= P \{ \mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_i^{-1} > t \} = \begin{cases} 1, & t < t_{cr}^i, \\ 0, & t \geq t_{cr}^i, \end{cases}$$

as $\lambda \to \infty$, since $\lambda^{-1} y \xrightarrow{P} 0$ as $\lambda \to \infty$, where $t_{cr}^i = \mu_1^{-1} + \mu_2^{-1} + \cdots + \mu_i^{-1}$. This proves Theorem 5.2. \hfill \Box

6. **Corollaries**

Using Theorems 5.1 and 5.2 one can study the following characteristics of queueing systems under busy traffic:

- the time $T_i$ spent by a customer in the system until the service at the $i$-th server is terminated;
- the time $T_i^l$ spent by a customer in the system after arriving at the $i$-th server;
- the total time $V_i$ spent by a customer in the $i$-th server;
- the waiting time $K_i$ at the $i$-th server;
- the service time $S_i$ at the $i$-th server

provided that the system remains in the stationary regime.
Theorem 6.1. If \( E \eta_1 = m_1 < \infty \), then \( (\mu_1^{-1} + \cdots + \mu_n^{-1})^{-1} T_i \overset{w}{\rightarrow} 1 \) as \( \lambda \to \infty \).

*Proof.* Since one starts to count the time \( T'_i \) at the moment when the \((i - 1)\)-th server begins the service, we have
\[
P \{ T > t \} = P \{ W_{i-1}^s + T'_i > t \}.
\]
Assuming that all random variables and stochastic processes are defined on the common probability space we use Theorems 5.1 and 5.2 and complete the proof of Theorem 6.1. \( \square \)

Theorem 6.2. If \( E \eta_1 = m_1 < \infty \), then \( \mu_i K_i \overset{w}{\rightarrow} 1 \) as \( \lambda \to \infty \).

*Proof.* The proof follows from Theorems 5.1, 5.2, and 6.1, since
\[
P \{ T > t \} = P \{ W_{i-1}^s + K_i + T'_{i+1} > t \}.
\]

Theorem 6.3. If \( E \eta_1 = m_1 < \infty \), then \( \mu_i V_i \overset{w}{\rightarrow} 1 \) as \( \lambda \to \infty \).

*Proof.* Since \( T_i \) coincides with \( T \) if the system contains \( i \) servers, and
\[
P \{ T_i > t \} = P \{ W_i^s + S_i > t \},
\]
we have
\[
S_i \overset{w}{\rightarrow} 0.
\]
Now the proof follows from Theorem 6.2 in view of
\[
P \{ V_i > t \} = P \{ K_i + S_i > t \}. \quad \square
\]

**Bibliography**


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