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THE ERGODICITY AND STABILITY OF QUASI-HOMOGENEOUS MARKOV SEMIGROUPS OF OPERATORS

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Abstract. A nonhomogeneous-in-time semigroup of Markov operators acting in a Banach space is called quasi-homogeneous if the domain of its infinitesimal operator is dense and the operator itself can be represented as the sum of the infinitesimal operator of a homogeneous semigroup and a bounded operator function.

Under the condition that the basic homogeneous semigroup is uniformly ergodic, we prove the uniform ergodicity and strong stability of the nonhomogeneous semigroup and obtain the estimates of the rate of convergence in the corresponding limit theorems.

1. Introduction

The stability of perturbed homogeneous semigroups of operators with discrete time is studied in the author's monograph [1]. The foundation of the theory of nonhomogeneous-in-time Markov processes is developed by Dynkin [2] and Gikhman and Skorokhod [3]. The general theory of perturbed operators is given in the monograph by Kato [4].

The problem of the stability of nonhomogeneous semigroups with continuous time is motivated by some applied models in risk theory, insurance, and financial mathematics, where the data is nonhomogeneous in time (because of the season phenomena, say).

Below we develop an approach proposed in [5]. Note that Theorem 1 in [5] is a generalization of Corollary 1.7 [10, Theorem 1.10] to the case of a nonhomogeneous-in-time semigroup [10], while Theorem 2 in [5] is an analog of the Dyson–Phillips theorem (see, for example, [10]).

Notice also that the proof of these two results given in [5] does not use the Markov property of semigroups.

2. Main definitions

1. Let (E, Ξ) be a measurable space. Denote by $f\Xi$ and $m\Xi$ the classes of all measurable functions and finite charges on (E, Ξ), respectively.

Assume that the Banach subspace $\aleph$ of the space $m\Xi$ is equipped with a norm $\|\cdot\|$ such that

$$(\mathbf{M}) \qquad \mathrm{Var}(\mu) \le c\,\|\mu\|\,, \qquad \|\mu\| \le \|\mu+\nu\| \qquad \text{for all } \mu,\nu \in \aleph,\ \mu,\nu \ge 0,$$

and some fixed constant c.

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Consider the Banach space $\Im \subset f\Xi$ dual to $\aleph$; it consists of functions equipped with the norm

$$\|f\| = \sup(|\mu f|, \|\mu\| \le 1, \mu \in \aleph),$$

where the dual linear form is given by

$$\mu f = \int_E f(x)\,\mu(dx), \qquad \mu \in \aleph,\ f \in \Im,$$

and

$$\|\mu\| = \sup(|\mu f|, \|f\| \le 1, f \in \Im). \tag{1}$$

If condition (**M**) holds, then the space $\Im$ contains all measurable bounded functions. Some examples of such spaces and their dual counterparts are given in [1, Chapter 1].

Every transient kernel $Q = (Q(x,B), x \in E, B \in \Xi)$ on (E,Ξ) generates (see [6]) linear mappings

$$\mu Q(B) = \int_E \mu(dx) Q(x,B) \colon m\Xi \to m\Xi, \tag{2}$$

$$Qf(x) = \int_E Q(x,dy) f(y) \colon f\Xi \to f\Xi. \tag{3}$$

The linear subclasses of these mappings equipped with the finite norms

$$\|Q\| = \sup(\|\mu Q\|, \|\mu\| \le 1) = \sup(\|Qf\|, \|f\| \le 1) < \infty \tag{4}$$

form Banach spaces $L(\aleph)$ and $L(\Im)$ of linear bounded operators, respectively; the product of the corresponding linear operators is generated by the kernel

$$PQ(x,B) = \int_E P(x,dy) Q(y,B). \tag{5}$$

2. Let $(P(s,x,t,B), x \in E, B \in \Xi, 0 \le s \le t)$ be a Markov transient function understood in the wide sense (see [4, Chapter 3]). According to (2) and (3), one can associate the linear mappings

$$P_{st} \colon m\Xi \to m\Xi, \qquad P_{st} \colon f\Xi \to f\Xi$$

with the Markov transient function $P(s,x,t,B)$. If these mappings are bounded, then they form a semigroup with respect to the multiplication (see [2, 3, 6]). The mappings are bounded if $\aleph$ and $\Im$ are the space of all bounded charges equipped with the full variation norm and the space of all bounded measurable functions equipped with the sup-norm, respectively.

3. A family of operators $(P_{st}, 0 \le s \le t < \infty) \subset L(\Im)$ is called a *bounded semigroup* in $L(\Im)$ if

$$P_{su}P_{ut} = P_{st}, \qquad 0 \le s \le u \le t, \qquad \sup_{0 \le s \le t} \|P_{st}\| < \infty. \tag{**P**}$$

A semigroup of operators $(Q_{st}, 0 \le s \le t < \infty)$ is called *homogeneous in time* if

$$Q_{st} = Q_{t-s}, \qquad 0 \le s \le t, \qquad Q_s Q_{t-s} = Q_t. \tag{6}$$

The limit

$$\lim_{u \uparrow s, v \downarrow s} (v-u)^{-1}(P_{uv}f - f) \equiv A_s f, \qquad s \ge 0, \tag{7}$$

is called the *infinitesimal operator* A_s of a nonhomogeneous semigroup (P_{st}) provided the limit exists in the norm of the space $\Im$.

Definition. A bounded semigroup $(P_{st}, 0 \le s \le t < \infty)$ is called quasi-homogeneous if there exists a nontrivial homogeneous bounded semigroup $(Q_{t-s}, 0 \le s \le t < \infty)$ with the infinitesimal operator A defined on a dense subspace $\Im_0 \subset \Im$ and

$$(\mathbf{A}) \qquad \text{for all } f \in \Im_0, \text{ there exists } \quad \lim_{h\to 0} h^{-1}(Q_h f - f) \equiv Af,$$

and, for some bounded family of operators $(D_s, 0 \le s < \infty) \subset L(\Im)$, the infinitesimal operator A_s of the semigroup (P_{st}) is equal to

$$(\mathbf{AD}) \qquad A_s = A + D_s \quad \text{on } \Im_0.$$

Remark 1. Condition (**AD**) is equivalent to the condition

$$(\mathbf{D}) \qquad \begin{gathered} \text{for all } s \ge 0 \quad \text{and for all } f \in \Im_0, \\ \text{the limit } \lim_{u\uparrow s, v\downarrow s} (v-u)^{-1}(P_{uv}f - Q_{v-u}f) \equiv D_s f \in \Im \text{ exists.} \end{gathered}$$

This assertion is a particular case of a result in [5] by the definition of the infinitesimal operator, since D_s is bounded.

In what follows we refer to the semigroup (Q_{t-s}) as to the *basic* semigroup for (P_{st}).

4. A semigroup (P_{st}) is called *uniformly ergodic* (in the space $\Im$) if

$$(8) \qquad \|P_{st} - \Pi\| \to 0, \qquad t \to \infty, \quad \text{for all } s \ge 0,$$

for some operator $\Pi \in L(\Im)$. If a semigroup is homogeneous, that is $P_{st} = Q_{t-s}$, then the operator Q_{t-s} is called the *stationary projector* [1]:

$$(9) \qquad Q_t\Pi = \Pi Q_t = \Pi = \Pi^2.$$

The rate of convergence in this case is geometric; namely,

$$(10) \qquad \text{there exists } \rho < 1 \quad \text{such that } \|Q_t - \Pi\| = O(\rho^t), \qquad t \to \infty.$$

If for some $s > 0$ the transient stochastic kernel Q_s has a unique invariant probability, that is, if

$$(11) \qquad \text{there exists a unique } \pi \in \aleph_+ \quad \text{such that } \pi = \pi Q_s, \qquad \pi(E) = 1,$$

then the projector is generated by the kernel $\Pi(x, A) \equiv \pi(A)$ (see [1]) that does not depend on x. This property holds for nonreducible Markov processes; some criteria for the existence and uniqueness can be found in [7].

5. If a semigroup is quasi-homogeneous and uniformly ergodic, then the numbers $\varepsilon(D)$, $q(Q)$, $\alpha(Q)$, and σ_δ are positive and finite, where

$$(12) \qquad \varepsilon(D) \equiv \sup_{s\ge 0} \|D_s\| < \infty,$$

$$(\mathbf{Q}) \qquad q(Q) \equiv \sup_{s\ge 0} \|Q_s\| < \infty,$$

$$(13) \qquad \alpha(Q) \equiv -\lim_{t\to\infty} t^{-1} \ln \left\|Q^t - \Pi\right\| > 0,$$

$$(14) \qquad \sigma_\delta \equiv \int_0^\infty \exp(\delta t)\, \|Q_t - \Pi\|\, dt < \infty, \qquad 0 \le \delta < \alpha.$$

In [8], explicit estimates for α are obtained in terms of the generalized potential (defined as the inverse operator to the infinitesimal operator A) of the corresponding process. Note that the function $\|Q_t - \Pi\|$ in the integral on the right-hand side of (14) is Borel, since it is semimultiplicative, and therefore $(\|Q_t - \Pi\|)^{1/t}$ is nonincreasing.

Consider the following measurability condition:

$$(\mathbf{L}) \qquad \begin{gathered} \text{there exists a dense subspace } \aleph_0 \subset \aleph \text{ such that, for all } \mu \in \aleph_0 \text{ and } f \in \Im_0, \\ \text{the functions } \|\mu P_{su} - \mu\Pi\|, \ \|\mu\Pi D_u\|, \text{ and } \mu D_u Q_{t-u} f \text{ are Lebesgue measurable.} \end{gathered}$$

If (**L**) holds, then

$$d_t(\delta) \equiv \sup_{\mu\in\aleph_0:\ \|\mu\|=1} \int_0^t \exp(-\delta(t-s))\,\|\mu\Pi D_s\|\,ds \tag{15}$$

if well defined. If condition (**L**) does not hold, then the measurable function

$$\sup_{u\geq s} \|\mu\Pi D_u\|$$

should be substituted for the second factor in the integral (15).

Remark 2. Condition (**L**) follows from the dual conditions to (**A**), (**D**): there exists a dense subspace $\aleph_0 \subset \aleph$ such that

$$\text{for all } \mu\in\aleph_0,\ \text{there exists } \lim_{h\to 0} h^{-1}(\mu Q_h - \mu) \equiv \mu A; \tag{16}$$

$$\begin{aligned} &\text{for all } s\geq 0,\ \text{and } \mu\in\aleph_0,\\ &\text{there exists } \lim_{u\uparrow s, v\downarrow s} (v-u)^{-1}(\mu P_{uv} - \mu Q_{v-u}) \equiv \mu D_s \in \aleph. \end{aligned} \tag{17}$$

Throughout this paper, all the integrals of operator-valued bounded functions are understood as weak integrals defined by their action on the elements $\mu\in\aleph$ and $f\in\Im$:

$$\mu\left(\int_s^t D_u\,du\right) f = \int_s^t \mu D_u f\,du.$$

3. Main results

The following proposition describes a relationship between the uniform ergodicity and the convergence to zero of the perturbation as $t\to\infty$.

Theorem 1. *Let (P_{st}) be a quasi-homogeneous bounded semigroup. If the basic homogeneous semigroup (Q_{t-s}) is uniformly ergodic, condition* (10) *holds, (Q_{t-s}) has a unique invariant probability measure (in other words, condition* (11) *holds), and the perturbation (D_s) is such that*

$$\varepsilon(D)\sigma_0 < 1, \tag{18}$$

$$\|\Pi D_t\| \to 0, \qquad t\to\infty, \tag{19}$$

then the semigroup (P_{st}) is uniformly ergodic in the sense that (8) *holds.*

The following result claims that condition (19) is necessary.

Theorem 2. *Let (P_{st}) be a quasi-homogeneous bounded semigroup with a two-points phase space $E=\{0,1\}$ and with bounded transient intensities $(A_s)_{ij}$.*

If this semigroup is uniformly ergodic, that is (8) *holds, where Π is the stationary projector of some basic homogeneous semigroup satisfying conditions* (10) *and* (11)*, then the Cesàro means converge:*

$$\left\| t^{-1}\int_0^t \Pi D_s\,ds \right\| \to 0, \qquad t\to\infty. \tag{20}$$

Moreover, for every $\delta>0$, the Abel means also converge:

$$\left\| \int_0^t \exp(-\delta(t-s))\Pi D_s\,ds \right\| \to 0, \qquad t\to\infty. \tag{21}$$

The following result shows that the mean convergence in the Abel sense can be substituted for the ergodicity condition (19) in Theorem 1; namely, one may assume that $d_t(\delta)\to 0$ instead of (19), where d_t is defined by (15). Another goal of the next theorem is to obtain the rate of convergence for relation (8).

Theorem 3. *Let (P_{st}) be a quasi-homogeneous bounded semigroup and the corresponding basic homogeneous semigroup (Q_{t-s}) be uniformly ergodic (see* (10)*) and have a unique invariant probability (see* (11)*).*

If the measurability condition (**L**) *holds and, for some $\delta \in (0, \alpha(Q))$, the perturbation (D_s) is such that*

$$\varepsilon(D)\sigma_\delta < 1, \tag{22}$$

$$d_t(\delta) \to 0, \qquad t \to \infty, \tag{23}$$

then the semigroup (P_{st}) is uniformly ergodic and

$$\|P_{st} - \Pi\| = O(d_t(\delta) + \exp(-\delta t)), \qquad t \to \infty. \tag{24}$$

The following estimate of the stability is obtained in [5].

Theorem 4. *Let (P_{st}) be a quasi-homogeneous bounded semigroup and let the corresponding basic homogeneous semigroup (Q_{t-s}) be uniformly ergodic (see* (10)*) and have a unique invariant probability* (11)*.*

If the perturbation (D_s) is uniformly small, that is,

$$\varepsilon(D) \to 0,$$

then

$$\sup_{0 \le s \le t < \infty} \|P_{st} - Q_{t-s}\| = O(\varepsilon(D)), \qquad \varepsilon(D) \to 0. \tag{25}$$

The above estimate is uniform in the scheme of series if $q(Q)$ in (**Q**) *and σ_0 in* (14) *are bounded.*

Example 1. Let a nonhomogeneous semigroup (P_{st}) with the phase space $\{0, 1\}$ be generated by the infinitesimal operator

$$A_s = (a_{ij}(s), i, j = 0, 1)$$

such that $a_{01}(s) = \alpha_s$ and $a_{10}(s) = \beta_s$. If there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\varepsilon \equiv \sup_{s \ge 0} |\alpha_s - \alpha| + \sup_{s \ge 0} |\beta_s - \beta| < \alpha + \beta \tag{26}$$

and

$$\beta\alpha_t - \alpha\beta_t \to 0, \qquad t \to \infty, \tag{27}$$

then the semigroup (P_{st}) is uniformly ergodic.

If $\varepsilon \to 0$, then stability holds, namely,

$$\sup_{0 \le s \le t < \infty} \|P_{st} - Q_{t-s}\| = O(\varepsilon), \qquad \varepsilon \to 0. \tag{28}$$

Note that the above assumptions do not necessarily imply that the "limit" operator exists as $t \to \infty$ for the infinitesimal operator function (A_t). This is the case, for example, if $\alpha_s \equiv c\beta_s$ is an arbitrary nonnegative measurable function. Note also that the corresponding process can be transformed to a homogeneous process by a nonrandom change of time.

4. Proof

Lemma 1. *Let a homogeneous semigroup* (Q_{t-s}) *be uniformly ergodic (see* (10)*) and have a unique invariant probability (see* (11)*). Then its stationary projector* Π *and the perturbation in* (**AD**) *annihilate the infinitesimal operator* (**A**)*:*

$$\Pi A = A\Pi = 0, \qquad D_s\Pi = 0. \tag{29}$$

Proof. The first two equalities are consequences of Definition (**A**) and properties (9) of the projector that hold for all h:

$$\Pi A f = \lim_{h\to 0} h^{-1}\Pi(Q_h - I)f = 0 \quad \text{for all } f \in \Im_0,$$
$$A\Pi f = \lim_{h\to 0} h^{-1}(Q_h - I)\Pi f = 0.$$

It follows from (9) and the condition of the uniqueness of the invariant probability π (which is, in fact, the left eigenelement corresponding to the eigennumber 1) that $\mu\Pi = c(\mu)\pi$ for all $\mu \in \aleph$. Thus

$$\mu(E) = \mu 1 = \mu Q_t 1 \to \mu\Pi 1 = c(\mu)\pi 1 = c(\mu), \qquad t \to \infty,$$

by (10) for all $\mu \in \aleph$. Hence $\mu\Pi = (\mu 1)\pi$. By the definition of the perturbation we have $D_s 1 = 0$, since the left-hand side of (**D**) vanishes for the function $f \equiv 1$. Thus $\mu D_s \Pi = (\mu D_s 1)\pi = 0$. □

Proof of Remark 2. As in the proof of Lemma 3 in [5] we derive from conditions (17) and (16) that for all $\mu_0 \in \aleph_0$ the function $\mu_0 P_{su}$ is strongly continuous in u. Thus the function $\|\mu_0(P_{su} - \Pi)\|$ is continuous in u and, for all $\mu \in \aleph$, is Borel, since the norm of the function $\|\mu(P_{su} - \Pi)\| = \lim_{\mu_0\to\mu} \|\mu_0(P_{su} - \Pi)\|$ is continuous.

Furthermore, from

$$\mu_0 P_{u+h,u+a+h} - \mu_0 P_{u,u+a} = -\mu_0(P_{u,u+h} - I)P_{u+h,u+a+h} + \mu_0(P_{u,u+a+h} - P_{u,u+a})$$

we deduce that, for $\mu_0 \in \aleph_0$, the function $\mu P_{u,u+a}$ is strongly right continuous in u. According to (17) the function

$$\|\mu_0 D_u\| = \lim_{h\downarrow 0} h^{-1} \|\mu_0(P_{u,u+h} - Q_h)\|$$

is Borel as a limit of continuous functions. Since the norm is continuous, $\|\mu D_u\|$ is Borel for all $\mu \in \aleph$.

Similarly, for $f \in \Im_0$,

$$\mu_0 D_u Q_{t-u} f = \lim_{h\downarrow 0} h^{-1} \left(\mu_0(P_{u-h,u} - Q_h)Q_{t-u} f\right)$$

is a Borel function of u, since the prelimit expression is a dual linear form of strongly continuous functions; thus it is continuous, too. □

Proof of Theorem 1. Condition (10) implies that the function σ_δ is continuous in a neighborhood of the origin. Thus it follows from (18) that there exists $\delta > 0$ such that (22) holds. The Abel convergence (23) follows from (19):

$$\begin{aligned}
&\sup_{\mu\in\aleph_0\colon \|\mu\|=1} \int_0^t \exp(-\delta(t-s)) \sup_{u\ge s} \|\mu\Pi D_u\| \, ds \\
&\quad \le \int_0^{t/2} \exp(-\delta(t-s))\, ds \, \|\Pi\| \, \varepsilon(D) \\
&\qquad + \int_{t/2}^t \exp(-\delta(t-s))\, ds \sup_{u\ge t/2} \|\Pi D_u\| \to 0, \qquad t \to \infty.
\end{aligned}$$

Thus all the assumptions of Theorem 3 follow from those of Theorem 1. Therefore Theorem 1 is a corollary of Theorem 3. □

Proof of Theorem 2. We use the system of forward Kolmogorov equations [3, Section 1.1]:

$$\frac{\partial}{\partial t} P_{st} = P_{st} A_t. \tag{30}$$

By Lemma 1, $\Pi D_u = \Pi(A + D_u) = \Pi A_u$, whence

$$\frac{d}{du}(P_{0u} - \Pi) = \Pi D_u + (P_{0u} - \Pi) A_u. \tag{31}$$

Integrating in the interval $[0, t]$ and dividing by t we get

$$t^{-1} \int_0^t \Pi D_u \, du = t^{-1}(P_{0t} - I) - t^{-1} \int_0^t (P_{0u} - \Pi) A_u \, du \to 0, \qquad t \to \infty,$$

since $P_{0u} - \Pi \to 0$, $u \to \infty$, and the function A_u is bounded.

To prove (21) we multiply both sides of equality (31) by $\exp(-\delta(t-u))$, integrate with respect to u on the interval $[0, t]$, and evaluate by parts the integral on the left-hand side. We obtain

$$\begin{aligned} &\int_0^t \exp(-\delta(t-u)) \Pi D_u \, du \\ &\qquad = P_{0t} - \Pi - \exp(-\delta t)(I - \Pi) - \int_0^t \exp(-\delta(t-u))(P_{0u} - \Pi)(\delta I + A_u) \, du. \end{aligned} \tag{32}$$

Since A_u is bounded and $\|P_{0t} - \Pi\| \to 0$, we get (21) in the same way as in the proof of Theorem 1. □

Proof of Theorem 3. We apply Theorem 1 of [5]. Note that conditions (Q, A, P, D) of [5] are the same as those in the current paper, while condition (T) of [5] follows from (12). According to Theorem 1 of [5],

$$\mu P_{st} f = \mu Q_{t-s} f + \int_s^t \mu P_{su} D_u Q_{t-u} f \, du$$

for all $\mu \in \aleph$, $f \in \Im$, and all $0 \le s \le t$, whence

$$\begin{aligned} \mu(P_{st} - \Pi) f = \mu(Q_{t-s} - \Pi) f &+ \int_s^t \mu \Pi D_u (Q_{t-u} - \Pi) f \, du \\ &+ \int_s^t \mu (P_{su} - \Pi) D_u (Q_{t-u} - \Pi) f \, du \end{aligned} \tag{33}$$

by Lemma 1 for $f \in \Im_0$ (the measurability of integrands follows from condition (**L**) and Theorem 1 of [5]).

Since $\delta < \alpha(Q)$, Definition (13) implies that there exists a constant $C_\delta < \infty$ such that

$$\|Q_s - \Pi\| \le C_\delta \exp(-\delta s), \qquad s \ge 0. \tag{34}$$

Fix $\mu \in \aleph$. The following functions:

$$V(t) = \|Q_t - \Pi\|, \tag{35}$$

$$p_{st} = \|\mu(P_{st} - \Pi)\|, \tag{36}$$

$$q_{st} = \|\mu\| \exp(-\delta(t-s)) + \int_s^t m(u) \exp(-\delta(t-u)) \, du \tag{37}$$

are measurable in t, where

$$m(u) = \begin{cases} \|\mu\Pi D_u\| & \text{if } (\mathbf{L}) \text{ holds,} \\ \sup_{v\geq u} \|\mu\Pi D_u\| & \text{otherwise.} \end{cases} \tag{38}$$

The function q_{st} in (37) is well defined regardless of whether condition ($\mathbf{L}$) holds or not. Moreover, the norm of the first two terms on the right-hand side of (33) does not exceed $C_\delta q_{st} \|f\|$:

$$\left\| \mu(Q_{t-s} - \Pi)f + \int_s^t \mu\Pi D_u(Q_{t-u} - \Pi)f \, du \right\| \leq C_\delta q_{st} \|f\| . \tag{39}$$

The following estimate for q_{st} defined in (37) holds in view of the definition of $d_t(\delta)$ (see (15)) and Remark 2:

$$q_{st} \leq \|\mu\| \left(\exp(-\delta(t-s)) + d_t(\delta)\right). \tag{40}$$

Using the above notation we obtain from (39) and (33) that

$$\|\mu(P_{st} - \Pi)f\| \leq C_\delta q_{st} \|f\| + \int_s^t \|\mu(P_{su} - \Pi)\| \, \varepsilon(D)V(t-u) \, du \, \|f\|$$

for all $\mu \in \aleph$ and $f \in \Im_0$.

Since $\Im_0$ is dense, we use the definition of p_{st} in (36) and obtain the inequality

$$p_{st} \leq C_\delta q_{st} + \varepsilon(D) \int_s^t p_{su} V(t-u) \, du. \tag{41}$$

Condition (22) implies that the right-hand side of (41) is a contraction linear operator in the space of measurable functions $\|f\| = \sup |f|$, since

$$\varepsilon(D) \int_s^t V(t-u) \, du \leq \varepsilon(D)\sigma_0 \leq \varepsilon(D)\sigma_\delta < 1.$$

Using the contraction mapping theorem and the sequential approximation method, we obtain from (41) that

$$p_{st} \leq C_\delta q_{st} + C_\delta \int_s^t q_{su} W(t-u) \, du, \tag{42}$$

where

$$W(t) = \sum_{n\geq 1} \varepsilon^n(D) V^{*n}(t) \tag{43}$$

and the function V^{*n} is the n-fold convolution of the function V with itself.

Furthermore, condition (22) implies that

$$\begin{aligned} \int_0^\infty \exp(\delta u) W(u) \, du &= \sum_{n\geq 1} \varepsilon^n(D) \int_0^\infty \exp(\delta u) V^{*n}(u) \, du = \sum_{n\geq 1} \varepsilon^n(D)(\sigma_\delta)^n \\ &= \varepsilon(D)\sigma_\delta/(1 - \varepsilon(D)\sigma_\delta), \end{aligned} \tag{44}$$

where σ_δ is defined by (14).

Using (37) in the right-hand side of inequality (42), we obtain

$$\begin{aligned} p_{st} &\le C_\delta q_{st} + C_\delta \|\mu\| \int_s^t \exp(-\delta(u-s)) W(t-u)\, du \\ &\quad + C_\delta \int_s^t \int_s^u m(v) \exp(-\delta(u-v))\, dv W(t-u)\, du \\ &= C_\delta q_{st} + C_\delta \|\mu\| \int_s^t \exp(-\delta(t-s)) \exp(\delta u) W(u)\, du \\ &\quad + C_\delta \int_s^t m(v) \exp(-\delta(t-v))\, dv \int_v^t \exp(-\delta(t-u)) W(t-u)\, du, \end{aligned} \tag{45}$$

where we changed the variable and the order of integration. Taking into account (44) we get

$$\begin{aligned} p_{st} &\le C_\delta q_{st} + C_\delta \|\mu\| \exp(-\delta(t-s)) \varepsilon(D)\sigma_\delta/(1-\varepsilon(D)\sigma_\delta) \\ &\quad + C_\delta \int_s^t m(v) \exp(-\delta(t-v))\, dv \cdot \varepsilon(D)\sigma_\delta/(1-\varepsilon(D)\sigma_\delta) \\ &= C_\delta q_{st}/(1-\varepsilon(D)\sigma_\delta). \end{aligned} \tag{46}$$

Together with (36) and (40), this yields

$$\|\mu(P_{st} - \Pi)\| \le C_\delta \|\mu\| \big(\exp(-\delta(t-s)) + d_t(\delta)\big)/(1-\varepsilon(D)\sigma_\delta). \tag{47}$$

Since $\mu \in \aleph$ is arbitrary, we derive estimate (24) from (47). □

Proof of Theorem 4. The following inequality is proved in Theorem 3 of [5]:

$$\sup_{0\le s\le t<T} \|P_{st} - Q_{t-s}\| \le \frac{\varepsilon(T)\sigma(T)}{1-\varepsilon(T)\sigma(T)} q(T), \tag{48}$$

where, by definition, $\varepsilon(T)$ for $T=\infty$ coincides with $\varepsilon(D)$ defined in (12), $\sigma(T)$ for $T=\infty$ coincides with σ_0 defined in (14) for $\delta = 0$, and $q(T)$ for $T=\infty$ coincides with $q(Q)$ defined in (**Q**). It is obvious that the right-hand side of (48) is equal to $O(\varepsilon(D))$.

The estimate is uniform in view of the latter inequality where $\sigma(T)$ and $q(T)$ are uniformly bounded by the assumptions of the theorem. □

Proof of assertions of Example 1. Consider the dual norms

$$\|(\mu_0, \mu_1)\| = \max(|\mu_0|, |\mu_1|), \qquad \|(f_0, f_1)\| = |f_0| + |f_1|$$

in the spaces $\aleph$ and $\Im$, respectively.

Define the infinitesimal operator $A = (a_{ij}, i, j = 0, 1)$ with $a_{01} = \alpha, a_{10} = \beta$, and the perturbation $D_s = (d_{ij}(s), i, j = 0, 1)$ with $d_{01}(s) = \alpha_s - \alpha, d_{10}(s) = \beta_s - \beta$.

Simple calculations show that the corresponding homogeneous semigroup is of the form

$$Q_t = \Pi + (I - \Pi)\exp(-\gamma t),$$

where $\gamma = \alpha + \beta$ and the matrix Π contains equal rows $(\beta, \alpha)\gamma^{-1}$.

Then

$$\begin{aligned} \sigma_0 \equiv \int_0^\infty \|Q_t - \Pi\|\, dt &= \gamma^{-1} \|I - \Pi\| = \gamma^{-1}, \\ \|\Pi D_t\| &= \gamma^{-1} |\beta\alpha_t - \alpha\beta_t|, \\ \|D_t\| &= |\alpha_t - \alpha| + |\beta_t - \beta|. \end{aligned} \tag{49}$$

Thus the assumptions of Example 1 imply the assumptions of both Theorems 1 and 4. □

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