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OPTIMAL FILTRATION FOR SYSTEMS WITH FRACTIONAL BROWNIAN NOISES

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Abstract. We consider the problem of the optimal filtration for systems with the noise being a multivariate fractional Brownian motion. We partially solve the problem of the optimal filtration for nonlinear systems. The system of equations for the optimal filtration is obtained in the case of linear systems.

1. Introduction

Systems governed by the fractional Brownian motion form a generalization of systems governed by the standard Brownian motion.

We consider a real-valued process X_t treated as an input signal and a process Y_t treated as an observation (the signal distorted by the noise). The processes X_t and Y_t are defined by the following system of equations:

$$\begin{cases} X_t = \eta + \int_0^t a(s, X_s)\,ds + \sum_{i=1}^N \int_0^t b_i(s)\,dV_s^i, & t \in [0,T], \\ Y_t = \xi + \int_0^t A(s, X_s)\,ds + \int_0^t B(s)\,dV_s, & t \in [0,T], \end{cases} \tag{1.1}$$

where $V = (V_t, t \in [0,T])$ and $V^i = (V_t^i, t \in [0,T])$, $i = 1, \dots, N$, are fractional Brownian motions with Hurst parameters $H \in (\frac{1}{2}, 1)$ and $H_1, \dots, H_N \in [\frac{1}{2}, 1)$, respectively. The coefficients A and a are continuous functions on $[0,T] \times \mathbb{R}$, while the coefficients B and $b_i, i = 1, \dots, N$, are continuous functions on $[0,T]$ such that B vanishes nowhere. The random initial data (η, ξ) do not depend on the fractional Brownian motions $(V^1, \dots, V^N, V)$, and the pair (X, ξ) has a given distribution $\mu_{(X,\xi)}$. Assume that Y is observed and one wants to estimate the process X. So we deal with the classical problem of filtration of a signal X at a moment t from the values of the process Y observed until the moment t.

It is well known that the conditional expectation $\pi_t(X)$ with respect to the σ-algebra $\mathcal{Y}_t = \sigma(\{Y_s, s \in [0,t]\})$ is the solution of the problem of filtration ($\pi_t(X)$ is called the optimal filter for this problem). The solution of this problem is presented in books [1] and [2] for the case of noise being the standard Brownian motion. The classical theory of filtration is extended in [3] to the case of noise being a fractional Brownian motion. The filtration for linear systems with one-dimensional fractal Brownian noises is studied in [4].

This paper is devoted to the problem of the optimal filtration for systems with a multivariate fractional Brownian noise. The solution of this problem is based on the representation of the process X in terms of the corresponding family of semimartingales.

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The main properties of the fractional Brownian motion are briefly discussed in Section 1. These properties allow one to use classical methods of filtration theory. The problem of the optimal filtration is partially solved in Section 2 for system (1.1). The system of equations for the optimal filter is obtained in Section 3 for linear systems.

2. Main properties

2.1. **Fractional Brownian motion.** In what follows we assume that all random variables and processes are defined on a common stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathsf{P})$ satisfying standard assumptions. The P-completion of the flow of σ-algebras generated by a stochastic process is taken as the natural flow of σ-algebras for this process.

Let $T > 0$ be fixed. A stochastic process $V = (V_t, t \in [0,T])$ is called normalized fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ if

(1) V is a Gaussian process with continuous trajectories and stationary increments;
(2) $V_0 = 0$, $\mathsf{E}\, V_t = 0$, and $\mathsf{E}\, V_t^2 = t^{2H}$ for all $t \in [0,T]$.

The standard Brownian motion corresponds to the case of $H = 1/2$. Any fractional Brownian motion is not a semimartingale; thus one cannot apply the classical theory of the stochastic integral for the fractional Brownian motion. Nevertheless the integration with respect to the fractional Brownian motion can be defined for some deterministic functions.

Let the functions f and g be defined on $[0,T]$. Put

$$\langle\langle f, g\rangle\rangle_H = H(2H-1)\int_0^T\int_0^T f(s)g(t)|s-t|^{2H-2}\,ds\,dt. \tag{2.1}$$

Then the space L_2^H of classes of equivalent measurable functions f on $[0,T]$ such that $\langle\langle f, f\rangle\rangle_H < \infty$ is a Hilbert space equipped with the scalar product $\langle\langle \cdot, \cdot\rangle\rangle_H$. The correspondence $\mathbb{1}_{[0,T]} \longrightarrow V_T$ can be extended to the isometry between L_2^H and the Gaussian space generated by random variables V_t, $t \in [0,T]$. The integral $\int_0^T f(s)\,dV_s$ for $f \in L_H^2$ is defined as the image of f for this isometry. For all $f, g \in L_H^2$, we have

$$\mathsf{E}\left[\left\{\int_0^T f(s)\,dV_s\right\}\left\{\int_0^T g(s)\,dV_s\right\}\right] = \langle\langle f, g\rangle\rangle_H, \tag{2.2}$$

where $\langle\langle f, g\rangle\rangle_H$ is defined by (2.1).

The process $\int_0^t f(s)\,dV_s$ is defined for $f \in L_H^2$ as follows:

$$\int_0^t f(s)\,dV_s = \int_0^T 1_{[0,t)}(s)f(s)\,dV_s, \qquad t \in [0,T].$$

The process V is not a semimartingale. Nevertheless, an integral transform of V,

$$V_t^* = \int_0^t k_t^*(s)\,dV_s, \tag{2.3}$$

is a martingale, where k_t^* is a nonrandom kernel such that

$$k_t^*(s) = k_H^{-1} s^{1/2-H}(t-s)^{1/2-H}, \qquad 0 < s < t,\ k_H = 2H\Gamma(3/2-H)\Gamma(H+1/2). \tag{2.4}$$

The covariance matrix of V^* is given by

$$\psi_H(t) = \langle V^*\rangle_t = \frac{\Gamma(3/2-H)}{2H\Gamma(3-2H)\Gamma(H+1/2)}t^{2-2H}. \tag{2.5}$$

Moreover the flow of σ-algebras generated by the process V^* coincides with the flow of σ-algebras generated by the process V up to null sets. The process V^* is called the fundamental martingale of a fractional Brownian motion V (see [5]).

More details concerning the fractional Brownian motion can be found in [5].

2.2. **The optimal filtration for nonlinear systems with fractional Brownian motion.** The problem of filtration is considered in [3] for the case of a fractional Brownian noise. It is shown in [3] that equations for the optimal filter $\pi_t(X) = \mathsf{E}[X_t|\mathcal{Y}_t]$, $\mathcal{Y}_t = \sigma\{Y_s, s \in [0,t]\}$, can be obtained by a method similar to the classical method for the standard Brownian noise. Following this method, one uses the existence and properties of the fundamental martingale for the fractional Brownian motion and introduces the process Z_t as follows:

$$Z_t = \int_0^t k_t^*(s) B^{-1}(s)\, dY_s. \tag{2.6}$$

The process Z_t is a P-semimartingale. It is proved in [3] that the σ-algebras

$$\mathcal{F}^{\xi,Z} = \sigma\{\xi; Z_s, s \in [0,t]\}$$

and $\mathcal{Y}_t$ coincide for all $t \in [0,T]$ up to P-null sets.

The following theorem contains equations for the optimal filter of a special semimartingale.

Theorem 2.1 (Theorem 2 [3])**.** *Let $\xi = (\xi_t, t \in [0,T])$ be an $((\mathcal{F}_t), \mathsf{P})$-semimartingale given by*

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + m_t, \qquad t \in [0,T],$$

where $\mathsf{E}[\xi_0^2] < \infty$, $\mathsf{E}\big[\int_0^T \beta_t^2\, dt\big] < \infty$, and $m = (m_t, t \in [0,T])$ is a square integrable $((\mathcal{F}_t), \mathsf{P})$-martingale such that $\langle\langle m, V^*\rangle\rangle_t = \int_0^t \lambda_s\, d\psi_H(s)$, $t \in [0,T]$. Then the process $\pi(\xi) = \pi_t(\xi)$, $t \in [0,T]$, satisfies the stochastic differential equation

$$\pi_t(\xi) = \pi(\xi_0) + \int_0^t \pi_s(\beta)\, ds + \int_0^t [\pi_s(\lambda) + \pi_s(\xi q(X)) - \pi_s(\xi)\pi_s(q(X))]\, dv_s, \qquad t \in [0,T], \tag{2.7}$$

where

$$q_t(X) = \frac{d}{d\psi_H(t)} \int_0^t k_t^*(s) B^{-1}(s) a(s, X_s)\, ds, \qquad t \in [0,T], \tag{2.8}$$

and

$$v_t = Z_t - \int_0^t \pi_s(q(X))\, d\psi_H(s). \tag{2.9}$$

The process v plays the same role as the innovation process in the classical model. The process v is a Gaussian $(\mathcal{Y}_t)$-martingale with the correlation function ψ_H. Each square integrable $(\mathcal{Y}_t)$-martingale M_t with respect to the measure P with $M_0 = 0$ can be represented as $M_t = \int_0^t P_s\, dv_s$, $t \in [0,T]$, where $P = (P_t, t \in [0,T])$ is a $(\mathcal{Y}_t)$-adapted process such that $\mathsf{E}\big[\int_0^T P_t^2\, d\psi_H(t)\big] < \infty$.

We assume that the noises $V^1, \dots, V^N$ in the definition of the process X depend on each other in the sense that we know the functions $\psi_{i,j}(t) = \langle V^{i^*}, V^{j^*}\rangle_t$, where V^{i^*}, $i = 1, \dots, N$, are the corresponding fundamental martingales for V_i, $i = 1, \dots, N$, represented in the form of (2.3) with H_i instead of H for any V^i in the definition of $k_t^*(s)$. We assume that $\psi_{i,j} \in C^1[0,T]$, $i = 1, \dots, N$. We seek a solution represented as the optimal filter for the functional $\phi(X_t)$ such that $\phi \in C^2[0,T]$. If the noises in the definition (1.1) of the input signal X are standard Brownian motions, then the Itô formula applied to the representation of the semimartingale $\phi(X_t)$ and Theorem 2.1 solve the filtration problem under consideration (see Chapter 5 of [3]). On the other hand, X

is not a semimartingale. To avoid this problem we apply an approach of [4] based on the representation of the integral with respect to the fractional Brownian motion in terms of the integral of the corresponding kernel with respect to the fundamental martingale of this fractional Brownian motion.

Lemma 2.2 (Lemma 4 [4])**.** *Let V be a fractional Brownian motion with the Hurst parameter $H \in (1/2, 1)$ and let V^* be the corresponding Gaussian martingale given by the integral transform* (2.3)*. Let $B \in L^2_H$ and let the function $K^B_H(t,s)$ be defined by*

$$K^B_H(t,s) = H(2H-1)\int_s^t B(r) r^{H-1/2}(r-s)^{H-3/2}\,dr, \qquad 0 \le s \le t \le T.$$

Then, for $t \in [0,T]$, the function $K^B_H(t,\cdot)$ belongs to the space $L^2([0,t],\mu)$,

$$\mu(A) = \int_A d\psi_H(s), \qquad A \subset [0,t],$$

and

$$\int_0^t B(s)\,dV_s = \int_0^t K^B_H(t,s)\,dV^*_s, \qquad t \in [0,T], \quad \mathsf{P}\text{-}a.s. \tag{2.10}$$

Using Lemma 2.2 we introduce a family of semimartingales X^t_s, $0 \le s \le t$, as follows:

$$X^t_s = \eta + \int_0^s a(u, X_u)\,du + \sum_{i=1}^N \int_0^s K^{b_i}_{H_i}(t,u)\,dV^{i*}_u, \qquad s \in [0,t]. \tag{2.11}$$

It is obvious that $X^t_t = X_t$. The Itô formula implies that the process $\phi(X^t_s)$, $s \in [0,t]$, is a semimartingale represented as follows:

$$\phi(X^t_s) = \phi(\eta) + \int_0^s \mathcal{L}_u(\phi(X^t_u))\,du + \sum_{i=1}^N \int_0^s \phi'(X^t_u) K^{b_i}_{H_i}(t,u)\,dV^{i*}_u, \qquad s \in [0,s], \tag{2.12}$$

where $\mathcal{L}_u(\phi(\cdot))$ is such that

$$\mathcal{L}_u(\phi(\cdot)) = a(u,X_u)\phi'(\cdot) + \sum_{i,j=1}^N \phi''(\cdot)\big(K^{b_i}_{H_i}(t,u)\big)\big(K^{b_j}_{H_j}(t,u)\big)\left(\frac{d\psi_{i,j}(u)}{du}\right).$$

The derivatives

$$\frac{d\psi_{i,j}(u)}{du}, \qquad i,j = 1,\dots,N,\ i \ne j,$$

are well defined by assumption. Moreover, it follows from (2.5) that

$$\frac{d\psi_{H_i}(u)}{du} = \frac{(2-2H_i)\Gamma(3/2-H_i)}{2H_i\Gamma(3-2H_i)\Gamma(H_i+1/2)} u^{1-2H_i}$$

for $i = j$. Applying Theorem 2.1 to the semimartingale $\phi(X^t)$ we obtain the equation for the optimal filter $\pi_s(\phi(X^t))$:

$$\begin{aligned}\pi_s(\phi(X^t)) = \pi(\phi(\eta)) &+ \int_0^s \pi_u\big(\mathcal{L}_u(\phi(X^t))\big)\,du \\ &+ \int_0^s \big[\pi_u\big(\phi(X^t)q\big) - \pi_u\big(\phi(X^t)\big)\pi_u(q)\big]\,dv_u.\end{aligned} \tag{2.13}$$

Putting $s = t$ and using the equality $X^t_t = X_t$, we get the equation for the filter $\pi_t(\phi(X))$:

$$\begin{aligned}\pi_t(\phi(X)) = \pi(\phi(\eta)) &+ \int_0^t \pi_s\big(\mathcal{L}_s(\phi(X^t))\big)\,ds \\ &+ \int_0^t \big[\pi_s\big(\phi(X^t)q\big) - \pi_s\big(\phi(X^t)\big)\pi_s(q)\big]\,dv_s.\end{aligned} \tag{2.14}$$

It is proved in [3] that there exists a probability measure $\tilde{\mathsf{P}}$ such that the fractional Brownian motion $\tilde{V}$ considered with respect to the measure $\tilde{\mathsf{P}}$ generates the same filtration as the process Y and is independent of the input process X. The process Y considered with respect to this measure satisfies the equation

$$Y_t = \xi + \int_0^t B(s)\, d\tilde{V}_s, \qquad t \in [0,T].$$

This result can be derived from a Girsanov type theorem for the fractional Brownian motion (see, for example, [5]). The relation between $\tilde{\mathsf{P}}$ and the measure P for which Y satisfies system (1.1) is as follows:

$$\mathsf{P} = \Lambda_T \tilde{\mathsf{P}}, \tag{2.15}$$

where

$$\Lambda_t = \exp\left(\int_0^t q_s(X)\, dZ_s - \frac{1}{2}\int_0^t q_s^2(X)\, d\psi_H(s)\right), \qquad t \in [0,T]. \tag{2.16}$$

Then $\sigma(\phi(X^t)) := \tilde{\mathsf{E}}[\phi(X_t)\Lambda_t|\mathcal{Y}_t]$ is called the nonnormalized filter. Its name is explained by the dependence between $\sigma(\phi(X^t))$ and the optimal filter $\pi_t(\phi(X))$ known as the Kallianpur–Striebel formula:

$$\pi_t(\phi(X)) = \frac{\sigma_t(\phi(X))}{\sigma_t(1)}. \tag{2.17}$$

The equation for $\tilde{\Lambda}_t = \sigma_t(1) = \tilde{\mathsf{E}}[\Lambda_t|\mathcal{Y}_t]$ is obtained in [3]; namely,

$$\tilde{\Lambda}_t := 1 + \int_0^t \tilde{\Lambda}_s \pi_s(q)\, dZ_s, \qquad t \in [0,T], \quad \mathsf{P}\text{-a.s.} \tag{2.18}$$

Using this equation and (2.13) we obtain an analog of the classical Zakai equation for the nonnormalized filter σ_t in the case of several fractional Brownian noises distorting the input signal.

Theorem 2.3. *Let σ_t be a nonnormalized filter for the model* (1.1)*. Then*

$$\sigma_t(\phi(X)) = \sigma_0(\phi(\eta)) + \int_0^t \sigma_s\big(\mathcal{L}_s(\phi(X^t))\big)\, ds + \int_0^t \sigma_s\big(\phi(X^t)q\big)\, dZ_s, \qquad t \in [0,T], \tag{2.19}$$

for any twice continuously differentiable function ϕ.

Proof. The Kallianpur–Striebel formula implies that

$$\sigma_s\big(\phi(X^t)\big) = \tilde{\Lambda}_s \pi_s\big(\phi(X^t)\big). \tag{2.20}$$

Representation (2.13) for the optimal filter $\pi_s(\phi(X^t))$, equation (2.18), and the Itô formula for differentials imply that

$$\begin{aligned} d\sigma_s\big(\phi(X^t)\big) &= \tilde{\Lambda}_s\, d\pi_s\big(\phi(X^t)\big) + \pi_s\big(\phi(X^t)\big)\, d\tilde{\Lambda}_s + d\big\langle \tilde{\Lambda}, \pi\big(\phi(X^t)\big)\big\rangle_s \\ &= \tilde{\Lambda}_s \pi_s\big(\mathcal{L}_s\phi(X^t)\big)\, ds + \tilde{\Lambda}_s\left[\pi_s\big(\phi(X^t q)\big) - \pi_s\big(\phi(X^t)\big)\pi_s(q)\right] dv_s \\ &\quad + \pi_s\big(\phi(X^t)\big)\tilde{\Lambda}_s \pi_s(q)\, dv_s + \pi_s\big(\phi(X^t)\big)\tilde{\Lambda}_s \pi_s^2(q)\, d\psi_H(s) \\ &\quad + \tilde{\Lambda}_s \pi_s(q)\left[\pi_s\big(\phi(X^t)q\big) - \pi_s\big(\phi(X^t)\pi_s(q)\big)\right] d\psi_H(s) \\ &= \tilde{\Lambda}_s \pi_s\big(\mathcal{L}_s\phi(X^t)\big)\, dt + \tilde{\Lambda}_s \pi_s\big(\phi(X^t)q\big)\left[dv_s + \pi_s(q)\, d\psi_H(s)\right]. \end{aligned} \tag{2.21}$$

Now we apply (2.20) and (2.9) again and get

$$d\sigma_s\big(\phi(X^t)\big) = \sigma_s\big(\mathcal{L}_s(\phi(X^t))\big)\, ds + \sigma_s\big(\phi(X^t)q\big)\, dZ_t. \tag{2.22}$$

Putting $s = t$ and integrating both sides of equality (2.22) we obtain (2.19). □

Remark 2.4. If some constants H_i are equal to $1/2$, then the corresponding processes V^i are standard Brownian motions and we put $V^{i^*} \equiv V^i$ in this case. The integral terms do not change for these observations V^i, and the proof is similar to that of Theorem 2.3.

Equations (2.14) and (2.19) imply that the solution of the problem of filtration requires an equation to determine $\pi_t(q)$. We consider this problem in the next section for linear systems and obtain a system of equations for the optimal filter.

2.3. **The optimal filtration in Gaussian linear systems with fractional Brownian noises.** We assume in this section that the input signal X_t, $t \in [0,T]$, and the process Y_t, $t \in [0,T]$, are such that

$$\begin{cases} dX_t = a(t)X_t\,dt + \sum_{i=1}^N b_i(t)\,dV_t^i, & t \in [0,T],\ X_0 = \eta, \\ dY_t = A(t)X_t\,dt + B(t)\,dV_t, & t \in [0,T],\ Y_0 = \xi. \end{cases} \tag{2.23}$$

Similarly to the general case (see system of equations (1.1)), V_i, $i = 1,\dots,N$, and V are fractional Brownian motions with the Hurst parameters $H_i \in [1/2,1)$, $i = 1,\dots,N$, and $H \in (1/2,1)$. We also assume that, for all $i = 1,\dots,N$, the processes V_i and V are independent, while V_i, $i = 1,\dots,N$, depend on each other in the sense that we know the functions $\psi_{i,j}(t) = \langle V^{i^*}, V^{j^*}\rangle_t$, where V^{i^*}, $i = 1,\dots,N$, are corresponding fundamental martingales for V_i, $i = 1,\dots,N$, given by (2.3). The coefficients A, B, a, and b_i, $i = 1,\dots,N$, are bounded nonrandom functions acting from $[0,T]$ to $\mathbb{R}$. The coefficient B does not vanish and B^{-1} is a bounded function. The pair (η,ξ) does not depend on $(V_1,\dots,V_N,V)$ and $\pi_0 = \mathsf{E}[\eta|\xi]$ has the Gaussian distribution with mean m_0 and variance γ_0. Then the system (2.23) has a unique solution with a Gaussian distribution. Nevertheless, it is neither a Markov process nor a semimartingale. The problem is to filtrate values of X at the moment t by using values Y observed up to the moment t. The solution of this problem is the conditional distribution of X_t with respect to the σ-algebra $\mathcal{Y}_t = \sigma\{Y_s, s \in [0,t]\}$ generated by the process Y.

Since the processes in (2.23) are Gaussian, the distribution of the optimal Gaussian filter is Gaussian, too. To completely define the filter one needs to know the mean value $\pi_t(X)$ and the variance $\gamma_{XX}(t) = \mathsf{E}(X_t - \pi_t(X))^2$ of errors of filtration.

To obtain necessary equations we apply the technique proposed in [4]. By ζ we denote the two-dimensional process $(X,q)'$ where X is defined in system (2.23), while q is given by (2.8). Our current aim is to get an equation for the mean value $\pi_t(\zeta)$ and covariance matrix $\gamma_{\zeta\zeta}(t) = \mathsf{E}(\zeta_t - \pi_t(\zeta))(\zeta_t - \pi_t(\zeta))'$. The processes ζ and Y are jointly Gaussian; however, ζ is not a semimartingale. As in the preceding section we apply Lemma 2.2 to solve the problem.

First we introduce the following nonrandom functions:

$$p(t,s) = \frac{d}{d\psi_H(t)} \int_s^t k_t^*(r) \frac{A}{B}(r)\,dr,$$

$$Q(i,t,s) = \frac{d}{d\psi_H(t)} \int_s^t k_t^*(r) K_{H_i}^{b_i}(r,s) \frac{A}{B}(r)\,dr.$$

Given a fixed $t \in [0,T]$ we define a Gaussian semimartingale $\zeta_s^t = ((X_s^t, q_s^t)',\ s \in [0,t])$, as follows:

$$X_s^t = \eta + \int_0^s a(u)X_u\,du + \sum_{i=1}^N \int_0^s K_{H_i}^{b_i}(t,u)\,dV_u^{i^*}, \tag{2.24}$$

$$q_s^t = p(t,0)\eta + \int_0^s a(u)p(t,u)X_u\,du + \sum_{i=1}^N \int_0^s Q(i,t,u)\,dV_u^{i^*}. \tag{2.25}$$

Consider the following family of 2×2 covariance matrices:

$$\Gamma_{\zeta\zeta}(t,s) = \mathsf{E}\big(\zeta_s^t - \pi_s(\zeta^t)\big)(\zeta_s - \pi_s(\zeta))', \qquad 0 \le s \le t \le T. \tag{2.26}$$

Let

$$\begin{gathered} \tilde{\varepsilon} = (0,1)'; \qquad \tilde{p}(t,u) = (1, p(t,u))'; \\ \tilde{b}(i,t,u) = \big(K_{H_i}^{b_i}(t,u), Q(i,t,u)\big)', \qquad 0 \le u \le t; \end{gathered} \tag{2.27}$$

and

$$\tilde{a}(t,u) = a(u)(\tilde{p}(t,u);\bar{0}) = a(u)\begin{pmatrix} 1 & 0 \\ p(t,u) & 0 \end{pmatrix}, \qquad 0 \le u \le t. \tag{2.28}$$

We need the following auxiliary result.

Lemma 2.5. *Let x, y, and z be Gaussian random variables such that*

$$\mathsf{E}[x|\mathcal{Y}_s] = 0, \qquad \mathsf{E}[y|\mathcal{Y}_s] = 0, \qquad \mathsf{E}[z|\mathcal{Y}_s] = 0 \qquad \mathsf{P}\text{-}a.s. \tag{2.29}$$

for some $s \in [0,T]$. Then

$$\mathsf{E}[xyz|\mathcal{Y}_s] = 0 \quad \mathsf{P}\text{-}a.s.$$

Proof. Since x, y, and z are Gaussian and the σ-algebra $\mathcal{Y}_s$ is generated by the Gaussian process Y, condition (2.29) implies that x, y, and z do not depend on $\mathcal{Y}_s$. Therefore the product xyz does not depend on $\mathcal{Y}_s$ either. Thus the conditional expectation $\mathsf{E}[xyz|\mathcal{Y}_s]$ is equal to the unconditional mean value $\mathsf{E}[xyz]$. To evaluate $\mathsf{E}[xyz]$ we use the following properties of Gaussian distributions:

$$y = \rho x + \alpha, \qquad z = \varrho x + \beta,$$

where $\rho = \operatorname{cov}(x,y)$, $\varrho = \operatorname{cov}(x,z)$, and α and β do not depend on x. We have

$$\begin{aligned} \mathsf{E}[xyz] &= \mathsf{E}[x(\rho x + \alpha)z] = \mathsf{E}[\rho x^2 z] + \mathsf{E}[x\alpha z] = \mathsf{E}[\rho x^2(\varrho x + \beta)] + \mathsf{E}[\alpha x(\varrho x + \beta)] \\ &= \mathsf{E}[\rho\varrho x^3] + \mathsf{E}[\rho x^2\beta] + \mathsf{E}[\varrho x^2\alpha] + \mathsf{E}[x\alpha\beta] \\ &= \mathsf{E}[\rho\varrho x] + \mathsf{E}[\beta]\,\mathsf{E}[\rho x^2] + \mathsf{E}[\alpha]\,\mathsf{E}[\varrho x^2] + \mathsf{E}[x]\,\mathsf{E}[\rho\varrho] = 0, \end{aligned}$$

since $\mathsf{E}[x] = 0$, $\mathsf{E}[x^3] = 0$, $\mathsf{E}[\alpha] = 0$, and $\mathsf{E}[\beta] = 0$. □

Now we obtain the equation for the filtration of the process ζ.

Theorem 2.6. *Let (X,Y) be a solution of system (2.23) and let q and Z be the processes defined by (2.8) and (2.6), respectively. Then the conditional expectation $\pi_t(\zeta)$ and covariance matrix $\gamma_{\zeta\zeta}(t)$ of the errors of filtration satisfy the equations*

$$\pi_t(\zeta) = \tilde{p}(t,0)m_0 + \int_0^t a(s)\tilde{p}(t,s)\pi_s(X)\,ds + \int_0^t \Gamma_{\zeta\zeta}(t,s)\tilde{\varepsilon}\left[dZ_s - \pi_s(q)\,d\psi_H(s)\right], \tag{2.30}$$

$$\gamma_{\zeta\zeta}(t) = \Gamma_{\zeta\zeta}(t,t), \tag{2.31}$$

where the matrix $\Gamma_{\zeta\zeta}(t,s)$, $0 \le s \le t \le T$, is such that

$$\begin{aligned} \Gamma_{\zeta\zeta}(t,s) = \tilde{p}(t,0)\tilde{p}'(s,0)\gamma_0 &+ \int_0^s \left[\tilde{a}(t,u)\Gamma'_{\zeta\zeta}(s,u) + \Gamma_{\zeta\zeta}(t,u)\tilde{a}'(s,u)\right] du \\ &+ \sum_{i,j=1}^N \int_0^s \tilde{b}(i,t,u)\tilde{b}'(j,s,u)\,d\psi_{i,j}(u) \\ &- \int_0^s \Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\tilde{\varepsilon}'\Gamma'_{\zeta\zeta}(s,u)\,d\psi_H(u). \end{aligned} \tag{2.32}$$

Proof. For all $t, r \in [0,T]$, the vector (ζ^t, ζ^r, Y) is Gaussian. Thus the conditional covariance for (ζ^t, ζ^r) with respect to $\mathcal{Y}_s$, $s \in [0, t \wedge r]$, is nonrandom. It is obvious that $\zeta^t_t = \zeta$.

To derive equation (2.30) we use (2.27) and (2.28) and represent the semimartingale ζ^t_s in the vector form

$$\zeta^t_s = \tilde{p}(t,0)m_0 + \int_0^s a(u)\tilde{p}(t,u)X_u\,du + \sum_{i=1}^N \int_0^s \tilde{b}(i,t,u)\,dV^{i*}_u. \tag{2.33}$$

Applying Theorem 2.1 we get

$$\pi_s(\zeta^t) = \tilde{p}(t,0)m_0 + \int_0^s a(u)\tilde{p}(t,u)\pi_u(X)\,du + \int_0^s \left[\pi_u(\zeta^t q) - \pi_u(\zeta^t)\pi_u(q)\right]\,dv_u, \tag{2.34}$$

since the noises in the input signal and those in the observation are independent. It follows from definition (2.26) of the matrix $\Gamma_{\zeta\zeta}(t,s)$ that

$$\pi_s(\zeta^t) = \tilde{p}(t,0)m_0 + \int_0^s a(u)\tilde{p}(t,u)\pi_u(X)\,du + \int_0^s \Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\,dv_u. \tag{2.35}$$

Since $\zeta^t_t = \zeta_t$, equation (2.30) follows.

To derive the equation for the covariance matrix $\Gamma_{\zeta\zeta}(t,s)$ we note that $\zeta^t(\zeta^r)'$ is a semimartingale on the interval $[0, t \wedge r]$ and rewrite equality (2.33) as follows:

$$\zeta^t_s = \tilde{p}(t,0)m_0 + \int_0^s \tilde{a}(t,u)\zeta_u\,du + \sum_{i=1}^N \int_0^s \tilde{b}(i,t,u)\,dV^{i*}_u. \tag{2.36}$$

The Itô formula gives us the representation for $\zeta^t(\zeta^r)'$:

$$\begin{aligned}\zeta^t_s(\zeta^r_s)' = \tilde{p}(t,0)\tilde{p}'(r,0)\eta^2 &+ \int_0^s \left[\tilde{a}(t,u)\zeta_u(\zeta^r_u)' + \zeta^t_u(\zeta_u)'\tilde{a}'(r,u)\right]du \\ &+ \sum_{i=1}^N \int_0^s \left[\tilde{b}(i,t,u)(\zeta^r_u)' + \zeta(^t_u)\tilde{b}'(i,r,u)\right]\,dV^{i*}_u \\ &+ \sum_{i,j=1}^N \int_0^s \tilde{b}(i,t,u)\tilde{b}'(j,r,u)\,d\psi_{i,j}(u).\end{aligned} \tag{2.37}$$

Applying Theorem 2.1, we get

$$\begin{aligned}\pi_s(\zeta^t(\zeta^r)') = \tilde{p}(t,0)\tilde{p}'(t,0)\,\mathsf{E}\big(\eta^2|\xi\big) &+ \int_0^s \left[\tilde{a}(t,u)\pi_u(\zeta(\zeta^r)') + \pi_s(\zeta^t\zeta')\tilde{a}'(r,u)\right]du \\ &+ \sum_{i,j=1}^N \int_0^s \tilde{b}(i,t,u)\tilde{b}'(j,r,u)d\psi_{(i,j)}(u) \\ &+ \int_0^s \left[\pi_u(\zeta(\zeta^r)'q) - \pi_u(\zeta(\zeta^r)')\pi_u(q)\right]\,dv_u.\end{aligned} \tag{2.38}$$

For all fixed t and r, the process $\pi_s(\zeta^t)\pi_t(\zeta^r)'$, $0 \le s \le r \wedge t$, is a semimartingale. Applying the Itô formula we get the representation for $\pi_s(\zeta^t)\pi_t(\zeta^r)'$:

$$\begin{aligned}\pi(\zeta^t)\pi(\zeta^r)' = \tilde{p}(t,0)\tilde{p}'(r,0)m_0^2 &\\ &+ \int_0^s \left[\tilde{a}(t,u)\pi_u(\zeta)\pi_u(\zeta^r)' + \pi_u(\zeta^t)\pi_u(\zeta')\tilde{a}'(r,u)\right] du \\ &+ \int_0^s \left[\Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\pi_u(\zeta^r)' + \pi_u(\zeta')\tilde{\varepsilon}'\Gamma_{\zeta\zeta}(r,u)\right] dv_u \\ &+ \int_0^s \Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\tilde{\varepsilon}'\Gamma'_{\zeta\zeta}(r,u)\, d\psi_H(u).\end{aligned} \tag{2.39}$$

Using (2.38) and (2.39) we obtain

$$\begin{aligned}\Gamma(t,r,s) &= \pi_s(\zeta^t(\zeta^r)') - \pi_s(\zeta^t)\pi_t(\zeta^r)' \\ &= \tilde{p}(t,0)\tilde{p}'(r,0)\gamma_0 + \int_0^s \left[\tilde{a}(t,u)\Gamma(u,r,u) + \Gamma(t,u,u)\tilde{a}'(r,u)\right] du \\ &\quad + \sum_{i,j=1}^N \int_0^s \tilde{b}(i,t,u)\tilde{b}'(j,r,u)\, d\psi_{i,j}(u) \\ &\quad - \int_0^s \Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\tilde{\varepsilon}'\Gamma'_{\zeta\zeta}(r,u)\, d\psi_H(u) \\ &\quad + \int_0^s \Big[\pi_u(\zeta(\zeta^r)'q) - \pi_u(\zeta(\zeta^r)')\pi_u(q) - \Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\pi_u(\zeta^r)' \\ &\qquad\qquad - \pi_u(\zeta')\tilde{\varepsilon}'\Gamma_{\zeta\zeta}(r,u)\Big] dv_u,\end{aligned} \tag{2.40}$$

where $\Gamma(t,r,s) = \mathsf{E}(\zeta_s^t - \pi_s(\zeta'))(\zeta_s^r - \pi_s(\zeta^r))'$, $s \in [0, t \wedge r]$. The definitions of π and Γ together with Lemma 2.5 imply that

$$\begin{aligned}\pi_u\big(\zeta(\zeta^r)'q\big) - \pi_u\big(\zeta(\zeta^r)'\big)\pi_u(q) - \Gamma_{\zeta\zeta}(t,u)\tilde{\varepsilon}\pi_u(\zeta^r)' &\\ - \pi_u(\zeta')\tilde{\varepsilon}'\Gamma_{\zeta\zeta}(r,u) = 0 \quad \mathsf{P}\text{-a.s}, &\qquad 0 \le u \le s,\end{aligned} \tag{2.41}$$

since all the processes are Gaussian. Putting $s = r$ in (2.40) we make sure that equation (2.32) holds, since $\Gamma_{\zeta\zeta}(t,s) = \Gamma(t,s,s) = \Gamma'(s,t,s)$. □

It is obvious that $\pi_t(X)$ is the first component of the vector $\pi_t(\xi)$, while $\gamma_{XX}(t)$ is the $(1,1)$-entry of the matrix $\gamma_{\zeta\zeta}(t)$.

Remark 2.7. Similarly to the general case, the terms corresponding to V^i with $H_i = 1/2$ do not change in the definition of the process X^t.

3. Concluding remarks

We considered the problem of filtration for systems governed by multivariate fractional Brownian noises. We derived the equation for the optimal filter $\pi_t(\phi(X))$ and nonnormalized filter $\sigma_t(\phi(X))$ for the nonlinear case.

The complete solution of the problem of filtration requires an equation for the process $\pi_t(q)$. This equation is obtained for linear systems. In the linear case, we completely solved the problem of filtration in the sense that we derived the system of equations for the expectation $\pi_t(X)$ and covariance $\gamma_{XX}(t)$ of the errors of filtration.

Bibliography

1. R. Sh. Liptser and A. N. Shiryaev, *Statistics of Random Processes*, "Nauka", Moscow, 1974; English transl., Springer-Verlag, Berlin, 1977. MR0431365 (55:4365)
2. G. Kallianpur, *Stochastic Filtering Theory*, Springer-Verlag, Berlin, 1980. MR0583435 (82f:60089)
3. M. L. Kleptsyna, A. Le Breton, and M. C. Roubaud, *An elementary approach to filtering in systems with fractional Brownian observation noise*, Probability Theory and Mathematical Statistics, Proceeding of the 7th Vilnius Conference, VSP/TEV, Utrecht/Vilnius, 2000, 373–392.
4. ———, *General approach to filtering with fractional Brownian noises—application to linear systems*, Stoch. Stoch. Rep. **71** (2000), 119–140. MR1813509 (2001k:93131)
5. I. Norros, E. Valkeila, and J. Virtamo, *An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions*, Bernoulli **5(4)** (1999), 571–587. MR1704556 (2000f:60053)

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