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APPROXIMATION OF A STOCHASTIC INTEGRAL WITH RESPECT TO FRACTIONAL BROWNIAN MOTION BY INTEGRALS WITH RESPECT TO ABSOLUTELY CONTINUOUS PROCESSES

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Abstract. We consider an absolutely continuous process converging in the mean square sense to a fractional Brownian motion. We obtain sufficient conditions that the integral with respect to this process converges to the integral with respect to the fractional Brownian motion.

1. Introduction

We consider the process $B^{H,\alpha} = \left(B_t^{H,\alpha}\right)_{t\geq 0}$ whose derivative is continuous on $(0,\infty)$. The process $B^{H,\alpha}$ approximates the fractional Brownian motion $B^H = (B_t^H)_{t\geq 0}$ in the mean square sense where H is the Hurst parameter, $H \in \left(\frac{1}{2}, 1\right)$. The process $B^{H,\alpha}$ is introduced in [1] for the problem of estimation of the ruin probability in the case where an insurance company buys a share described by a fractional Brownian motion or by a mixed model.

We study the question of sufficient conditions on the process f that guarantee (in a certain sense) the convergence

$$\int_a^b f(u)\, dB_u^{H,\alpha} \to \int_a^b f(u)\, dB_u^H \quad \text{as } \alpha \to 1-. \tag{1.1}$$

We use two approaches to answer this question. In Theorem 3.1, we prove the convergence

$$\mathsf{E}\left\|B^H - B^{H,\alpha}\right\|_{\lambda,1} \to 0$$

in the norm of the Besov space $W^{\lambda,1}[a,b]$ for $\lambda \in \left(0, \frac{1}{2}\right)$. In Section 4 we obtain a corollary that convergence (1.1) holds in probability if f is a Hölder function of order $\frac{1}{2}+\varepsilon$ for some $\varepsilon > 0$ (Theorem 4.1). Moreover, convergence (1.1) holds in L_1 if the Hölder norm of order $\frac{1}{2}+\varepsilon$ of the process f is bounded in $\omega \in \Omega$ for some $\varepsilon > 0$ (Theorem 4.2).

Another approach is to obtain explicitly an estimate of the difference between the integrals in (1.1) and to prove the convergence in probability if f is Hölder continuous of order $2(1-H)+\varepsilon$ (Theorem 5.1). Combining these two results we prove convergence (1.1) in probability if f is Hölder continuous of order

$$\min\left\{\frac{1}{2}, 2(1-H)\right\} + \varepsilon$$

for some $\varepsilon > 0$.

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2. A process of bounded variation that approximates fractional Brownian motion

According to [2], the fractional Brownian motion with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ can be represented in the form

$$B_t^H = \int_0^t s^{H-1/2}\, dY_s,$$

where

$$Y_t = C_0 \int_0^t (t-s)^{H-1/2} s^{1/2-H}\, dW_s, \tag{2.1}$$

$C_0 = C_0(H) > 0$ is a nonrandom constant, and W_t is a Wiener process with respect to the flow of σ-algebras $(\mathcal{F}_t) = (\mathcal{F}\{B_u, u \le t\})$.

Since the fractional Brownian motion is not a semimartingale, it is natural to consider an approximation of the process B_t^H by "nicer" processes than the fractional Brownian motion. For example, it is proposed in [1] to substitute the process

$$Y_t^\alpha = C_0 \left(H - \frac{1}{2}\right) \int_0^t \left[\int_0^{\alpha s} (s-u)^{H-3/2} u^{1/2-H}\, dW_u\right] ds$$

with $\alpha \in (0,1)$ for Y_t and take the process

$$\begin{aligned} B_t^{H,\alpha} &= \int_0^t s^{H-1/2}\, dY_s^\alpha \\ &= C_0 \left(H - \frac{1}{2}\right) \int_0^t s^{H-1/2} \left[\int_0^{\alpha s} (s-u)^{H-3/2} u^{1/2-H}\, dW_u\right] ds \end{aligned} \tag{2.2}$$

as an approximation of B_t^H. One can show that for $B_t^{H,\alpha}$ and $t \in [0,T]$,

$$\operatorname{Var}\left(B_{t+h}^{H,\alpha} - B_t^{H,\alpha}\right) \le K(T,\alpha) \cdot |h|^{2H},$$

whence it follows that $B_t^{H,\alpha}$ is a continuous process in view of the Kolmogorov criterion. The process

$$Z_t^\alpha = t^{H-1/2} \int_0^{\alpha t} (t-u)^{H-3/2} u^{1/2-H}\, dW_u$$

is the derivative of $B_t^{H,\alpha}$ within a factor. Moreover

$$\operatorname{Var}\left(Z_{t+h}^\alpha - Z_t^\alpha\right) \le K(\delta,T,\alpha) \cdot |h|^{2-2H}$$

for $t \in [\delta, T]$ and some $\delta > 0$. Since Z_t^α is a Gaussian process, Theorem IV.5.6 in [3] implies that Z_t^α is a continuous process on $[\delta, T)$, and thus on $(0,\infty)$. Therefore $B_t^{H,\alpha}$ has a continuous derivative on $(0,\infty)$, and its variation on $[\delta, T]$, $0 < \delta < T$, is finite.

Proposition 2.1. *We have*

$$\mathsf{E}\left(B_t^H - B_t^{H,\alpha}\right)^2 \le C_1 t^{2H} (1-\alpha)^{2H-1}, \tag{2.3}$$

where $C_1 = C_1(H) > 0$ is a constant that depends neither on t nor on α.

Proof. Using the stochastic Fubini theorem (see Theorem IV.4.5 in [4]) we represent the process Y_t^α as follows:

$$
\begin{aligned}
Y_t^\alpha &= C_0\left(H-\frac{1}{2}\right)\int_0^t\left[\int_0^{\alpha s}(s-u)^{H-3/2}u^{1/2-H}\,dW_u\right]ds \\
&= C_0\left(H-\frac{1}{2}\right)\int_0^{\alpha t}\left[\int_{u/\alpha}^t(s-u)^{H-3/2}\,ds\right]u^{1/2-H}\,dW_u \\
&= C_0\left(\int_0^{\alpha t}(t-u)^{H-1/2}u^{1/2-H}\,dW_u-\left(\frac{1-\alpha}{\alpha}\right)^{H-1/2}W_{\alpha t}\right).
\end{aligned}
\tag{2.4}
$$

Thus

$$
\begin{aligned}
\mathsf{E}\,(Y_t-Y_t^\alpha)^2 &= C_0^2\,\mathsf{E}\left(\int_{\alpha t}^t(t-u)^{H-1/2}u^{1/2-H}\,dW_u+\left(\frac{1-\alpha}{\alpha}\right)^{H-1/2}W_{\alpha t}\right)^2 \\
&= C_0^2\int_{\alpha t}^t(t-u)^{2H-1}u^{1-2H}\,du+C_0^2\left(\frac{1-\alpha}{\alpha}\right)^{2H-1}\cdot(\alpha t) \\
&\le Ct(1-\alpha)^{2H-1}.
\end{aligned}
\tag{2.5}
$$

The integration by parts formula yields

$$
\begin{aligned}
B_t^H-B_t^{H,\alpha} &= t^{H-1/2}\,(Y_t-Y_t^\alpha)-\lim_{\varepsilon\to0}\varepsilon^{H-1/2}\,(Y_\varepsilon-Y_\varepsilon^\alpha) \\
&\quad -\left(H-\frac{1}{2}\right)\int_0^t(Y_s-Y_s^\alpha)\,s^{H-3/2}\,ds,
\end{aligned}
$$

whence

$$
\begin{aligned}
\mathsf{E}\left(B_t^H-B_t^{H,\alpha}\right)^2 &\le 2t^{2H-1}\,\mathsf{E}\,(Y_t-Y_t^\alpha)^2+2\left(H-\frac{1}{2}\right)^2t\int_0^t\mathsf{E}\,(Y_s-Y_s^\alpha)^2\,s^{2H-3}\,ds \\
&\le 2C(1-\alpha)^{2H-1}\left(t^{2H}+\left(H-\frac{1}{2}\right)^2t\int_0^t s^{2H-2}\,ds\right) \\
&= C_1t^{2H}(1-\alpha)^{2H-1}
\end{aligned}
$$

by (2.5). □

3. Convergence $B^{H,\alpha}\to B^H$ in the norm of the space $W^{\lambda,1}[a,b]$

Following [5], we denote by $W^{\lambda,1}[a,b]$, $\lambda\in\left(0,\frac{1}{2}\right)$, the space of measurable functions

$$f\colon[a,b]\to\mathbb{R}$$

such that

$$\|f\|_{\lambda,1}=\int_a^b\frac{|f(s)|}{(s-a)^\lambda}\,ds+\int_a^b\int_a^s\frac{|f(s)-f(y)|}{(s-y)^{\lambda+1}}\,dy\,ds<\infty.$$

By $W^{1-\lambda,\infty}[a,b]$ we denote the space of measurable functions $g\colon[a,b]\to\mathbb{R}$ such that

$$\|g\|_{1-\lambda,\infty}=\sup_{a<s<t<b}\left(\frac{|g(t)-g(s)|}{(t-s)^{1-\lambda}}+\int_s^t\frac{|g(y)-g(s)|}{(y-s)^{2-\lambda}}\,dy\right)<\infty.$$

We have

$$C^{1-\lambda+\varepsilon}[a,b]\subset W^{1-\lambda,\infty}[a,b]\subset C^{1-\lambda}[a,b]$$

for all $\varepsilon>0$, where $C^\lambda[a,b]$ denotes the space of Hölder continuous functions of order λ equipped with the norm

$$\|f\|_\lambda=\sup_{u\in[a,b]}|f(t)|+\sup_{u,v\in[a,b]}\frac{|f(u)-f(v)|}{|u-v|^\lambda}.$$

It follows from [5] that the generalized Stieltjes integral $\int_a^b f\,dg$ exists and

$$\left|\int_a^b f\,dg\right| \le C(\lambda)\,\|f\|_{\lambda,1}\,\|g\|_{1-\lambda,\infty} \tag{3.1}$$

in the case of $f \in W^{\lambda,1}[a,b]$ and $g \in W^{1-\lambda,\infty}[a,b]$.

Theorem 3.1. *For all* $\lambda \in \left(0,\frac{1}{2}\right)$, $H \in \left(\frac{1}{2},1\right)$, *and all intervals* $[a,b] \subset [0,T]$,

$$\mathsf{E}\left\|B^H - B^{H,\alpha}\right\|_{\lambda,1} \le C(H,\lambda,T)\cdot(1-\alpha)^{H-1/2} \quad \textit{for } \alpha \in \big(1/2,1\big).$$

Proof. Let

$$\Delta B_t^{H,\alpha} = B_t^H - B_t^{H,\alpha}.$$

It follows from the definition of the norm $\|\cdot\|_{\lambda,1}$ that

$$\left\|\Delta B^{H,\alpha}\right\|_{\lambda,1} = \int_a^b \frac{\left|\Delta B_s^{H,\alpha}\right|}{(s-a)^\lambda}\,ds + \int_a^b\int_a^s \frac{\left|\Delta B_s^{H,\alpha} - \Delta B_y^{H,\alpha}\right|}{(s-y)^{\lambda+1}}\,dy\,ds. \tag{3.2}$$

Using estimate (2.3) we get

$$\begin{aligned}\mathsf{E}\int_a^b \frac{\left|\Delta B_s^{H,\alpha}\right|}{(s-a)^\lambda}\,ds &\le \int_a^b \frac{\sqrt{\mathsf{E}\big(\Delta B_s^{H,\alpha}\big)^2}}{(s-a)^\lambda}\,ds \le C(H)(1-\alpha)^{H-1/2}\int_a^b \frac{s^H\,ds}{(s-a)^\lambda}\\ &\le C(H,\lambda,T)(1-\alpha)^{H-1/2}.\end{aligned}$$

Consider the second term in (3.2). We rewrite the numerator as follows:

$$\begin{aligned}\Delta B_s^{H,\alpha} - \Delta B_y^{H,\alpha} &= \big(B_s^H - B_s^{H,\alpha}\big) - \big(B_y^H - B_y^{H,\alpha}\big)\\ &= \big(B_s^H - B_y^H\big) - \big(B_s^{H,\alpha} - B_y^{H,\alpha}\big)\\ &= \int_y^s u^{H-1/2}\,d\left(Y_u - Y_u^\alpha\right).\end{aligned} \tag{3.3}$$

Put

$$\Delta Y_u^\alpha = Y_u - Y_u^\alpha.$$

Using the integration by parts formula we obtain from (3.3) that

$$\begin{aligned}&\int_a^b\int_a^s \frac{\left|\Delta B_s^{H,\alpha} - \Delta B_y^{H,\alpha}\right|}{(s-y)^{\lambda+1}}\,dy\,ds\\ &\quad= \int_a^b\int_a^s \frac{\left|s^{H-1/2}\Delta Y_s^\alpha - y^{H-1/2}\Delta Y_y^\alpha - (H-\frac{1}{2})\int_y^s \Delta Y_u^\alpha u^{H-3/2}du\right|}{(s-y)^{\lambda+1}}\,dy\,ds\\ &\quad\le \int_a^b\int_a^s \frac{s^{H-1/2}\left|\Delta Y_s^\alpha - \Delta Y_y^\alpha\right|}{(s-y)^{\lambda+1}}\,dy\,ds + \int_a^b\int_a^s \frac{\left(s^{H-1/2} - y^{H-1/2}\right)\left|\Delta Y_y^\alpha\right|}{(s-y)^{\lambda+1}}\,dy\,ds\\ &\qquad+ \left(H-\frac{1}{2}\right)\int_a^b\int_a^s \frac{\int_y^s \left|\Delta Y_u^\alpha\right| u^{H-3/2}\,du}{(s-y)^{\lambda+1}}\,dy\,ds\\ &\quad=: I_1(\alpha) + I_2(\alpha) + \big(H-1/2\big)\cdot I_3(\alpha).\end{aligned}$$

We estimate the expectation of $I_2(\alpha)$ with the help of bound (2.5):

$$\begin{aligned}\mathsf{E}\int_a^b\int_a^s \frac{\left(s^{H-1/2} - y^{H-1/2}\right)\left|\Delta Y_y^\alpha\right|}{(s-y)^{\lambda+1}}\,dy\,ds &\le \int_a^b\int_a^s \frac{(H-1/2)y^{H-3/2}\sqrt{\mathsf{E}\big(\Delta Y_y^\alpha\big)^2}}{(s-y)^\lambda}\,dy\,ds\\ &\le \int_a^b\int_a^s \frac{Cy^{H-1}(1-\alpha)^{H-1/2}}{(s-y)^\lambda}\,dy\,ds \le C(H,\lambda,T)(1-\alpha)^{H-1/2}.\end{aligned}$$

Similarly

$$
\begin{aligned}
\mathsf{E}\, I_3(\alpha) &\le \int_a^b\int_a^s \frac{\int_y^s \sqrt{\mathsf{E}\big(\Delta Y_u^\alpha\big)^2}\, u^{H-3/2} du}{(s-y)^{\lambda+1}}\, dy\, ds \\
&\le C(1-\alpha)^{H-1/2} \int_a^b\int_a^s \frac{\int_y^s u^{H-1}\, du}{(s-y)^{\lambda+1}}\, dy\, ds \\
&\le C(H,\lambda,T)(1-\alpha)^{H-1/2}.
\end{aligned}
$$

Using representations (2.1) and (2.4) for Y_t and Y_t^α, we get an estimate for $I_1(\alpha)$:

$$
\begin{aligned}
I_1(\alpha) &= C_0 \int_a^b\int_a^s \frac{s^{H-1/2}\left|\Big(F(s)+\big(\frac{1-\alpha}{\alpha}\big)^{H-1/2} W_{\alpha s}\Big) - \Big(F(y)+\big(\frac{1-\alpha}{\alpha}\big)^{H-1/2} W_{\alpha y}\Big)\right|}{(s-y)^{\lambda+1}} \\
&\le C_0 \int_a^b\int_a^s \frac{s^{H-1/2}\,|F(s)-F(y)|}{(s-y)^{\lambda+1}}\, dy\, ds \\
&\quad + C_0\left(\frac{1-\alpha}{\alpha}\right)^{H-1/2} \int_a^b\int_a^s \frac{s^{H-1/2}\,|W_{\alpha s}-W_{\alpha y}|}{(s-y)^{\lambda+1}}\, dy\, ds \\
&=: I_4(\alpha) + I_5(\alpha),
\end{aligned}
$$

where

$$
F(x) = \int_{\alpha x}^x (x-u)^{H-1/2} u^{1/2-H}\, dW_u.
$$

Furthermore,

$$
\begin{aligned}
\mathsf{E}\, I_5(\alpha) &= C_0\left(\frac{1-\alpha}{\alpha}\right)^{H-1/2} \int_a^b\int_a^s \frac{s^{H-1/2}\,\mathsf{E}\,|W_{\alpha s}-W_{\alpha y}|}{(s-y)^{\lambda+1}}\, dy\, ds \\
&\le C(1-\alpha)^{H-1/2} \int_a^b\int_a^s \frac{s^{H-1/2}}{(s-y)^{\lambda+1/2}}\, dy\, ds \\
&\le C(H,\lambda,T)\,(1-\alpha)^{H-1/2}.
\end{aligned}
$$

Note that the bound $\lambda < \frac{1}{2}$ is essential for the latter estimate. To estimate $\mathsf{E}\, I_4(\alpha)$ we split the integral with respect to the variable y into two parts:

$$
\begin{aligned}
\mathsf{E}\, I_4(\alpha) &= \int_a^b\int_a^s \frac{s^{H-1/2}\,\mathsf{E}\left|\int_{\alpha s}^s \big(\frac{s-u}{u}\big)^{H-1/2}\, dW_u - \int_{\alpha y}^y \big(\frac{y-u}{u}\big)^{H-1/2}\, dW_u\right|}{(s-y)^{\lambda+1}}\, dy\, ds \\
&\le \int_a^b\int_a^s \frac{s^{H-1/2}\sqrt{\int_0^s \big(G(s,u)-G(y,u)\big)^2 u^{1-2H}\, du}}{(s-y)^{\lambda+1}}\, dy\, ds \\
&= \int_a^b s^{H-1/2}\left(\int_a^{(\alpha s)\vee a} + \int_{(\alpha s)\vee a}^s\right) \frac{\sqrt{\xi(y,s,\alpha)}}{(s-y)^{\lambda+1}}\, dy\, ds \\
&=: J_1(\alpha) + J_2(\alpha),
\end{aligned}
$$

where

$$
G(x,u) = (x-u)^{H-1/2}\, \mathbb{1}[\alpha x, x](u).
$$

On the corresponding intervals of integration, the function $\xi(y,s,\alpha)$ can be represented as follows:

$$\xi(y,s,\alpha)=\begin{cases}\displaystyle\int_{\alpha y}^{y}(y-u)^{2H-1}u^{1-2H}\,du+\int_{\alpha s}^{s}(s-u)^{2H-1}u^{1-2H}\,du\\ \displaystyle\qquad=(y+s)\int_{\alpha}^{1}(1-u)^{2H-1}u^{1-2H}\,du, & \text{for } y\le\alpha s;\\ \displaystyle\int_{\alpha y}^{\alpha s}(y-u)^{2H-1}u^{1-2H}\,du\\ \displaystyle\qquad+\int_{\alpha s}^{y}\left((s-u)^{H-1/2}-(y-u)^{H-1/2}\right)^2u^{1-2H}\,du\\ \displaystyle\qquad+\int_{y}^{s}(s-u)^{2H-1}u^{1-2H}\,du\\ \qquad=:\tilde{\xi}(y,s,\alpha), & \text{for } y>\alpha s.\end{cases}$$

The term $J_1(\alpha)$ is estimated as follows:

$$\begin{aligned}J_1(\alpha)&=\int_a^b s^{H-1/2}\int_a^{(\alpha s)\vee a}\frac{\sqrt{(y+s)\int_\alpha^1(1-u)^{2H-1}u^{1-2H}\,du}}{(s-y)^{\lambda+1}}\,dy\,ds\\&=\sqrt{\int_\alpha^1(1-u)^{2H-1}u^{1-2H}\,du}\int_a^b s^{H-\lambda}\,ds\int_{a/s}^{\alpha\vee(a/s)}\frac{\sqrt{y+1}}{(1-y)^{\lambda+1}}\,dy\\&\le C(H,\lambda,T)(1-\alpha)^{H-\lambda}\quad\text{as }\alpha\in(1/2,1).\end{aligned}$$

Now we show that

$$\tilde{\xi}(y,s,\alpha)\le C(H)(1-\alpha)^{2H-1}(s-y)\tag{3.4}$$

for $y>\alpha s$. This inequality is implied by the following estimates for the terms of $\tilde{\xi}(y,s,\alpha)$:

$$\begin{aligned}\int_{\alpha y}^{\alpha s}(y-u)^{2H-1}u^{1-2H}\,du&\le(y-\alpha y)^{2H-1}(\alpha y)^{1-2H}(\alpha s-\alpha y)\\&\le(1-\alpha)^{2H-1}(s-y),\end{aligned}$$

$$\begin{aligned}&\int_{\alpha s}^{y}\left((s-u)^{H-1/2}-(y-u)^{H-1/2}\right)^2u^{1-2H}\,du\\&\quad\le(\alpha s)^{1-2H}\int_{\alpha s}^{y}\left(\left(H-\frac{1}{2}\right)\int_y^s(v-u)^{H-3/2}\,dv\right)^2du\\&\quad\le C(H)s^{1-2H}(s-y)\int_{\alpha s}^{y}\int_y^s(v-u)^{2H-3}\,dv\,du\\&\quad=C(H)s^{1-2H}(s-y)\left((y-\alpha s)^{2H-1}-(s-\alpha s)^{2H-1}+(s-y)^{2H-1}\right)\\&\quad\le C(H)(1-\alpha)^{2H-1}(s-y)\quad\text{as }\alpha\in(1/2,1),\end{aligned}$$

$$\begin{aligned}\int_y^s(s-u)^{2H-1}u^{1-2H}\,du&\le(s-\alpha s)^{2H-1}(\alpha s)^{1-2H}(s-y)\\&\le 2^{2H-1}(1-\alpha)^{2H-1}(s-y)\quad\text{as }\alpha\in(1/2,1).\end{aligned}$$

Applying (3.4) we estimate the term $J_2(\alpha)$:

$$J_2(\alpha) \le C(H)(1-\alpha)^{H-1/2} \int_a^b s^{H-1/2} \int_{(\alpha s)\vee a}^s (s-y)^{-\lambda-1/2}\, dy\, ds$$
$$\le C(H)(1-\alpha)^{H-1/2} \int_a^b s^{H-\lambda}\, ds \int_\alpha^1 (1-y)^{-\lambda-1/2}\, dy \le C(H,\lambda,T)(1-\alpha)^{H-\lambda}.$$

Note again that the bound $\lambda < \frac{1}{2}$ is essential for the latter relation.

This completes the proof of the theorem. □

4. Convergence of integrals

Consider the stochastic integral with respect to the fractional Brownian motion

$$\int_a^b f(u)\, dB_u^H.$$

Proposition 4.1. *Let*

$$f(\cdot,\omega) \in \bigcup_{\lambda > 1-H} C^\lambda[a,b] \tag{4.1}$$

for almost all $\omega \in \Omega$. *Then the integral*

$$\int_a^b f(u,\omega)\, dB_u^H(\omega) = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} f(u_i^*,\omega) \left(B_{u_{i+1}}^H(\omega) - B_{u_i}^H(\omega) \right)$$

exists almost surely as the Riemann–Stieltjes integral, where the convergence is uniform with respect to all finite partitions

$$\pi := \left\{ a = u_0 \le u_0^* \le u_1 \le u_1^* \le \cdots \le u_{n-1} \le u_{n-1}^* \le u_n = b \right\}$$

with $|\pi| = \max_i |u_{i+1} - u_i|$.

Proof. The Riemann–Stieltjes integral $\int_a^b f\, dg$ exists in the deterministic case for $f \in C^\lambda$ and $g \in C^\mu$ if $\lambda + \mu > 1$ (see Theorem 4.2.1 in [6]). Thus Proposition 4.1 follows, since the trajectories of the process B_t^H belong to $C^{H-}[a,b] = \bigcap_{\lambda<H} C^\lambda[a,b]$ with probability one. □

Therefore one can apply the integration by parts formula to both integrals $\int_a^b f(u)\, dB_u^H$ and $\int_a^b f(u)\, dB_u^{H,\alpha}$ if condition (4.1) holds. If one assumes that

$$f \in C^{1/2+\varepsilon}[a,b] \subset W^{(1/2+\varepsilon/2),\infty}[a,b]$$

with probability one, then estimate (3.1) and the integration by parts formula imply

$$\left| \int_a^b f(u)\, d\left(B_u^H - B_u^{H,\alpha}\right) \right| \le \left| f(a) \left(B_a^H - B_a^{H,\alpha}\right) \right| + \left| f(b) \left(B_b^H - B_b^{H,\alpha}\right) \right|$$
$$+ C \left\| B^H - B^{H,\alpha} \right\|_{(1/2-\varepsilon/2),1} \|f\|_{(1/2+\varepsilon/2),\infty}.$$

Using the latter inequality, estimate (2.3), and Theorem 3.1 we obtain the following results.

Theorem 4.1. *Let* $f \in C^{1/2+\varepsilon}[a,b]$ *with probability one for some* $\varepsilon > 0$. *Then*

$$\int_a^b f(u)\, dB_u^{H,\alpha} \to \int_a^b f(u)\, dB_u^H \quad \textit{in probability} \quad \textit{as } \alpha \to 1-.$$

Theorem 4.2. *Suppose that*

(1) *for some $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\|f\|_{C^{1/2+\varepsilon}[a,b]} < C$$

with probability one;

(2) *the random variables $f(a)$ and $f(b)$ have finite second moments.*

Then

$$\mathsf{E}\left|\int_a^b f(u)\,dB_u^H - \int_a^b f(u)\,dB_u^{H,\alpha}\right| \to 0 \quad \text{as } \alpha \to 1-.$$

5. Estimation of integral convergence in terms of Riemannian sums

Theorem 5.1. *Let $f\colon [a,b]\times\Omega \to \mathbb{R}$, $0 \le a < b$, satisfy*

$$|f(x,\omega) - f(y,\omega)| \le K(\omega)\ |x-y|^{2(1-H)+\varepsilon} \tag{5.1}$$

for some $\varepsilon > 0$, P-almost all $\omega \in \Omega$, and all $a \le x < y \le b$, where $K(\omega)$ is a finite random variable. Then

$$\int_a^b f(u)\,dB_u^{H,\alpha} \to \int_a^b f(u)\,dB_u^H \quad \text{in probability} \quad \text{as } \alpha \to 1-. \tag{5.2}$$

Proof. To show, simultaneously with the proof of Theorem 5.1, that its result cannot in a certain sense be improved, we assume that

$$|f(x,\omega) - f(y,\omega)| \le K(\omega)\ |x-y|^{\lambda+\varepsilon}. \tag{5.3}$$

For $\Delta > 0$ introduce the function

$$f_\Delta(u) = \sum_{k=0}^{n} f(u_k)\,\mathbb{1}[u_k, u_{k+1})(u), \qquad u \in [a,b), \qquad f_\Delta(b) = f(u_n),$$

where

$$n = n(\Delta) = \left[\frac{b-a}{\Delta}\right],$$
$$u_k = u_k(\Delta) = a + k\Delta, \quad k = 0,\dots,n, \qquad u_{n+1} = b.$$

For all $\Delta > 0$, we have

$$\begin{aligned}
&\left|\int_a^b f(u)\,dB_u^{H,\alpha} - \int_a^b f(u)\,dB_u^H\right| \\
&\quad\le \left|\int_a^b \big(f(u) - f_\Delta(u)\big)\,dB_u^{H,\alpha}\right| + \left|\int_a^b f_\Delta(u)\,d\big(B_u^{H,\alpha} - B_u^H\big)\right| \\
&\qquad + \left|\int_a^b (f_\Delta(u) - f(u))\,dB_u^H\right| \\
&\quad =: I_1(\Delta,\alpha) + I_2(\Delta,\alpha) + I_3(\Delta).
\end{aligned} \tag{5.4}$$

Put $\Delta = \Delta(\alpha) = (1-\alpha)^r$ and choose $r > 0$ such that the first and second terms in (5.4) converge to zero as $\alpha \to 1-$ with the minimal λ in (5.3). The optimal (minimal) λ coincides with the Hölder exponent in (5.1). Since $2(1-H) > 1-H$, the third term in (5.4) converges to zero in probability in view of Proposition 4.1 for the specified $r > 0$.

Consider $I_1(\Delta,\alpha)$. It follows from condition (5.3) and definition (2.2) of the process $B_t^{H,\alpha}$ that

$$\begin{aligned} I_1(\Delta,\alpha) &= \left|\int_a^b \big(f(u)-f_\Delta(u)\big)\, dB_u^{H,\alpha}\right| \\ &= C\left|\sum_{k=0}^n \int_{u_k}^{u_{k+1}} \big(f(u)-f(u_k)\big)\cdot\left(u^{H-1/2}\int_0^{\alpha u}(u-y)^{H-3/2}y^{1/2-H}\, dW_y\right) du\right| \\ &\le CK\sum_{k=0}^n (u_{k+1}-u_k)^{\lambda+\varepsilon}\int_{u_k}^{u_{k+1}} u^{H-1/2}\left|\int_0^{\alpha u}(u-y)^{H-3/2}y^{1/2-H}\, dW_y\right| du \\ &=: CK\cdot\zeta_1(\Delta,\alpha). \end{aligned}$$

The expectation of $\zeta_1(\Delta,\alpha)$ is estimated as follows:

$$\begin{aligned} \mathsf{E}\,\zeta_1(\Delta,\alpha) &\le \Delta^{\lambda+\varepsilon}\sum_{k=0}^n\int_{u_k}^{u_{k+1}} u^{H-1/2}\,\mathsf{E}\left|\int_0^{\alpha u}(u-y)^{H-3/2}y^{1/2-H}\, dW_y\right| du \\ &\le \Delta^{\lambda+\varepsilon}\sum_{k=0}^n\int_{u_k}^{u_{k+1}} u^{H-1/2}\sqrt{\int_0^{\alpha u}(u-y)^{2H-3}y^{1-2H}\, dy}\, du \\ &= \Delta^{\lambda+\varepsilon}\sqrt{\int_0^{\alpha}(1-y)^{2H-3}y^{1-2H}\, dy}\sum_{k=0}^n\int_{u_k}^{u_{k+1}} u^{H-1}\, du \\ &\le C\Delta^{\lambda+\varepsilon}\left(1+(1-\alpha)^{2H-2}\right)^{1/2}. \end{aligned} \tag{5.5}$$

Substituting $\Delta(\alpha)=(1-\alpha)^r$ we get

$$\mathsf{E}\,\zeta_1\big(\Delta(\alpha),\alpha\big) = O\left((1-\alpha)^{r(\lambda+\varepsilon)+(H-1)}\right), \qquad \alpha\to 1-. \tag{5.6}$$

Now we consider $I_2(\Delta,\alpha)$:

$$\begin{aligned} I_2(\Delta,\alpha) &= \left|\sum_{k=0}^n f(u_k)\left(\big(B_{u_{k+1}}^{H,\alpha}-B_{u_{k+1}}^H\big)-\big(B_{u_k}^{H,\alpha}-B_{u_k}^H\big)\right)\right| \\ &\le \left|f(a)\big(B_a^{H,\alpha}-B_a^H\big)\right| + \sum_{k=0}^n \left|f(u_{k+1})-f(u_k)\right|\cdot\left|B_{u_{k+1}}^{H,\alpha}-B_{u_{k+1}}^H\right| \\ &\quad + \left|f(b)\big(B_b^{H,\alpha}-B_b^H\big)\right| \\ &=: \xi_1(\alpha)+\xi_2(\Delta,\alpha)+\xi_3(\alpha). \end{aligned} \tag{5.7}$$

Since $B_t^{H,\alpha}-B_t^H\to 0$ in probability for $t\ge 0$, the first and third terms in (5.7) converge in probability to zero. Estimating $\xi_2(\Delta,\alpha)$ we obtain from (5.3) that

$$\xi_2(\Delta,\alpha) \le K\sum_{k=0}^n \big(u_{k+1}-u_k\big)^{\lambda+\varepsilon}\cdot\left|B_{u_{k+1}}^{H,\alpha}-B_{u_{k+1}}^H\right| =: K\cdot\zeta_2(\Delta,\alpha).$$

To estimate the expectation of $\zeta_2(\Delta,\alpha)$ we obtain from (2.3) that

$$\mathsf{E}\,\zeta_2(\Delta,\alpha) \le \Delta^{\lambda+\varepsilon}\sum_{k=0}^n\sqrt{\mathsf{E}\left(B_{u_{k+1}}^{H,\alpha}-B_{u_{k+1}}^H\right)^2} \le C\Delta^{\lambda+\varepsilon}(1-\alpha)^{H-1/2}\left(\left[\frac{b-a}{\Delta}\right]+1\right).$$

Substituting $\Delta(\alpha)=(1-\alpha)^r$ we get

$$\mathsf{E}\,\zeta_2\big(\Delta(\alpha),\alpha\big) = O\left((1-\alpha)^{r(\lambda-1+\varepsilon)+(H-1/2)}\right), \qquad \alpha\to 1-. \tag{5.8}$$

Now we find restrictions on r and λ such that the random variables $\zeta_1\big(\Delta(\alpha),\alpha\big)$ and $\zeta_2\big(\Delta(\alpha),\alpha\big)$ converge to zero in L_1 which implies the convergence in probability to zero of the first two terms in (5.4). One can derive these restrictions from relations (5.6) and (5.8):

$$\begin{cases} r\lambda + (H-1) \geq 0, \\ r(\lambda-1) + \big(H-\frac{1}{2}\big) \geq 0. \end{cases} \tag{5.9}$$

Solving (5.9) as an extremal problem $\lambda \to \min$ for $r \in (0,1)$ we find that λ attains its minimal value if $r = \frac{1}{2}$ and that this minimal value coincides with $2(H-1)$. Thus all the terms in (5.4) converge in probability to zero if condition (5.1) holds and

$$\Delta(\alpha) = (1-\alpha)^{1/2}.$$

This completes the proof of the theorem. □

Corollary 5.1. *Combining Theorems* 4.1 *and* 5.1 *we prove that the convergence*

$$\int_a^b f(u)\, dB_u^{H,\alpha} \to \int_a^b f(u)\, dB_u^H \quad \textit{in probability} \quad \textit{as } \alpha \to 1-$$

if, for some $\varepsilon > 0$, the trajectories of the process f belong to the class $C^{\lambda_0+\varepsilon}[a,b]$ with probability one, where $\lambda_0 = \min\big\{\frac{1}{2}, 2(1-H)\big\}$.

It also turns out that Theorem 4.1 *is stronger for $H < \frac{3}{4}$, while Theorem* 5.1 *is stronger for $H > \frac{3}{4}$.*

Remark 5.1. One can study other approximations of the fractional Brownian motion $B^{H,\alpha}$, for which (5.2) holds under other assumptions. For example, one can prove that

$$\mathsf{E}\left(B_t^H - \tilde{B}_t^{H,\alpha}\right)^2 \to 0 \quad \text{as } \alpha \to 0+$$

for the process

$$\tilde{B}_t^{H,\alpha} = \int_0^t s^{H-1/2}\, d\tilde{Y}_s^\alpha,$$

where

$$\tilde{Y}_t^\alpha = C_0\left(H-\frac{1}{2}\right)\int_0^t \left[\int_0^{(s-\alpha)_+} (s-u)^{H-3/2} u^{1/2-H}\, dW_u\right] ds.$$

Here $(x)_+ = \max(x,0)$.

Note also that the process $\tilde{B}_t^{H,\alpha}$ has continuous derivative on $[0,\infty)$ for a fixed α. This, of course, means that its variation on $[0,T]$ is finite for any $T > 0$.

An analog of Theorem 5.1 can be proved for the process $\tilde{B}^{H,\alpha}$. Namely,

$$\int_a^b f(u)\, d\tilde{B}_u^{H,\alpha} \to \int_a^b f(u)\, dB_u^H \quad \text{in probability} \quad \text{as } \alpha \to 0+$$

if $f \in C^{2(1-H)+\varepsilon}[a,b]$ for some $\varepsilon > 0$.

6. Concluding remarks

We considered an absolutely continuous process $B_t^{H,\alpha}$ converging to the fractional Brownian motion B_t^H in L_1. The derivative of $B_t^{H,\alpha}$ is continuous on $(0,\infty)$. We proved that $B_t^{H,\alpha} \to B_t^H$ in the mean in the Besov space $W^{\lambda,1}[a,b]$ as $\alpha \to 1-$. We also obtained sufficient conditions for the convergence of integrals $\int_a^b f(u)\, dB_u^{H,\alpha} \to \int_a^b f(u)\, dB_u^H$ as $\alpha \to 1-$ in L_1 and in probability.

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