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PROKHOROV–LOÈVE STRONG LAW OF LARGE NUMBERS FOR MARTINGALES NORMALIZED BY OPERATORS

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Abstract. We study strong laws of large numbers for multivariate martingales normalized by linear operators in a finite-dimensional Euclidean space. Corollaries of the general results are considered for martingales under moment restrictions.

1. Introduction

Let $(X_n, n \geq 1)$ be a sequence of independent symmetric random vectors in the Euclidean space $\mathbf{R}^m$ $(m \geq 1)$; $(A_n, n \geq 1)$ a sequence of nonrandom linear operators acting from $\mathbf{R}^m$ to $\mathbf{R}^d$ $(d \geq 1)$; $\|x\|$ the Euclidean norm of a vector x; and $\mathfrak{N}$ the set of all nondecreasing unbounded sequences of positive integers.

Put $S_n = \sum_{i=1}^n X_i$, $n \geq 1$. The Prokhorov–Loève strong law of large numbers is an assertion about the equivalence between the almost sure convergence to zero of the sequence $(A_n S_n, n \geq 1)$, that is,

$$\text{(a)} \qquad \|A_n S_n\| \to 0 \quad \text{a.s.} \qquad (n \to \infty),$$

and the almost sure convergence to zero of the sequence of independent random vectors $(A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j}), j \geq 1)$, that is,

$$\text{(b)} \qquad \left\| A_{n_{j+1}} \left(S_{n_{j+1}} - S_{n_j} \right) \right\| \to 0 \quad \text{a.s.} \qquad (j \to \infty),$$

where the sequences $(n_j, j \geq 1)$ belong to a specified subset $\tilde{\mathfrak{N}}$ of the set $\mathfrak{N}$. The principal problem is to determine the set $\tilde{\mathfrak{N}}$ and find conditions for the equivalence of assertions (a) and (b). In what follows, the set $\tilde{\mathfrak{N}}$ is denoted by $\mathfrak{N}((A_n))$ and called the *test class* for the sequence $(A_n, n \geq 1)$.

The first results concerning this form of the strong law of large numbers are obtained in Prokhorov [1] and Loève [2] in the case of $m = d = 1$. The most advanced results in this direction for $m = d = 1$ are obtained in [3, 4]. The test class $\mathfrak{N}((A_n))$ is simple in this case; namely if $(a_n, n \geq 1)$ is a normalizing sequence of positive numbers such that $a_n \to 0$ as $n \to \infty$, then $\mathfrak{N}((a_n))$ consists of the single sequence $(n_j, j \geq 1)$ defined as $n_j = \max\{n \colon a_n \geq \lambda^{-j}\}$, $j \geq 1$, where $\lambda > 1$.

The multidimensional case is not so easy. The Prokhorov–Loève strong law of large numbers for sums of independent symmetric random vectors normalized by operators is obtained in [5]. The test class $\mathfrak{N}((A_n))$ for this case depends on the normalizing sequence of operators $(A_n, n \geq 1)$ and contains a finite number of sequences. The construction of the test class is used in [5] in the proof of the equivalence of assertions (a) and (b)

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for sums of independent symmetric random vectors. It is worth mentioning that this construction cannot be applied for more complicated cases.

The procedure for the construction of the test class is given in Section 2 of this paper. General results on the almost sure convergence to zero and on the almost sure boundedness of sequences of random vectors normalized by operators are also given in Section 2. These results are applied to partial sums of orthogonal random vectors in Section 3. Martingales normalized by operators are considered in Sections 4 and 5. Note that the results of Sections 3–5 are discussed in [6, 7]. However, the test class is not well defined there for the general case.

We use the following notation: N means the set of positive integers; $\langle x, y\rangle$ is the scalar product of vectors x and y; $\|A\| = \sup_{\|x\|=1} \|Ax\|$ is the norm of an operator A; $\xrightarrow{\mathsf{P}}$ stands for the convergence in probability; and E means the mathematical expectation. For two operators, A and B, the notation $A \geq B$ means that the operator $A - B$ is positive semidefinite. For the sake of definiteness, we put $\max_{\varnothing}(\cdot) = 0$ and $\sum_{i=m}^{n}(\cdot) = 0$ for $m > n$.

2. General conditions for the almost sure convergence to zero and almost sure boundedness of sequences of random vectors normalized by operators

Below we obtain some Prokhorov–Loève type sufficient conditions for the almost sure convergence to zero and the almost sure boundedness of arbitrary sequences of random vectors normalized by operators. First we consider the procedure for the construction of the test class $\mathfrak{N}((A_n))$ for a given sequence $(A_n, n \geq 1)$.

Construction of the test class $\mathfrak{N}((A_n))$. The procedure for the construction of the test class is rather complicated. Thus we split it into several steps. Without loss of generality we assume that $(A_n, n \geq 1)$ are $d \times m$ matrices.

Case 1: $d = 1$, $m = 1$, and $\|A_n\| \to 0$ as $n \to \infty$. In this case,

$$(A_n, n \geq 1) = (a_n, n \geq 1)$$

is a sequence of real numbers. Fix an arbitrary number $\lambda > 1$ and define the sequence of positive integers $(n_j, j \geq 1)$ as follows:

$$n_j = \max\left\{n \colon |a_n| \geq \lambda^{-j}\right\}, \qquad j \geq 1. \tag{1}$$

If there are infinitely many nonzero terms in the sequence $(a_n, n \geq 1)$, then we put

$$\mathfrak{N}((A_n)) = \mathfrak{N}((a_n), \lambda) = \{(n_j, j \geq 1)\}. \tag{2}$$

Otherwise, i.e., if all the terms of the sequence starting with some term are zero, we put

$$\mathfrak{N}((A_n)) = \varnothing. \tag{3}$$

Case 2: $d = 1$, $m \geq 1$, and $\|A_n\| \to 0$ as $n \to \infty$. In this case,

$$(A_n, n \geq 1) = (T_n, n \geq 1)$$

is a sequence of row vectors of dimension m and the class $\mathfrak{N}((A_n))$ is defined by induction on m. The class is already defined for $m = 1$ (see Case 1). Assume that the class is defined for some $m = s \geq 1$ and define it for $m = s + 1$. The last coordinate of the vector $T_n \in \mathbf{R}^{s+1}$ is denoted by t_n.

a) First we consider the case of $t_n \neq 0$, $n \geq 1$. Fix an arbitrary number $\lambda > 1$ and put

$$\hat{n}_j = \max\left\{n \colon |t_n| \geq \lambda^{-j}\right\}, \qquad j \geq 1.$$

Since $\|T_n\| \to 0$ as $n \to \infty$, we have $|t_n| \to 0$ as $n \to \infty$. Thus the sequence $(\hat{n}_j, j \geq 1)$ is well defined. By $\tilde{T}_n$ we denote the vector formed by the first s coordinates of the vector T_n, that is, $T_n = (\tilde{T}_n, t_n)$. For each positive integer $n \geq 1$, put

$$B_n = \theta_n \left(t_n^{-1} \tilde{T}_n - t_{\hat{n}_{j+1}}^{-1} \tilde{T}_{\hat{n}_{j+1}} \right), \qquad \hat{n}_j \leq n < \hat{n}_{j+1}, \tag{4}$$

where $\theta_n = \min\left\{|t_n|, |t_{\hat{n}_{j+1}}|\right\}$. If $\hat{n}_j \leq n < \hat{n}_{j+1}$, then $\|B_n\| \leq \|\tilde{T}_n\| + \|\tilde{T}_{\hat{n}_{j+1}}\|$. This implies that

$$\|B_n\| \to 0 \qquad (n \to \infty), \tag{5}$$

since $\|\tilde{T}_n\| \to 0$ as $n \to \infty$. The latter relation follows from $\|T_n\| \to 0$ as $n \to \infty$. Since $B_n \in \mathbf{R}^s$, $n \geq 1$, and relation (5) holds, the test class $\mathfrak{N}((B_n))$ is defined for the sequence $(B_n, n \geq 1)$ by the induction assumption.

Fix an arbitrary sequence $(n_j, j \geq 1) \in \mathfrak{N}((B_n))$. Using this sequence and the sequences $(\hat{n}_j, j \geq 1)$ we define the sequence of intervals $(I_n, n \geq 1)$ in the set N of positive integers as follows. Let $I_1 = [\hat{n}_j, \hat{n}_{j+1})$ if $n_1 \in [\hat{n}_j, \hat{n}_{j+1})$. If an interval I_i, $i \geq 1$, is already defined, then I_{i+1} is the interval that is different from I_i, is the closest to I_i from the right among the intervals of the sequence $([\hat{n}_j, \hat{n}_{j+1}), j \geq 1)$, and that contains the terms of the sequence $(n_j, j \geq 1)$. For $i \geq 1$, put $\bar{j}(i) = \min\{p\colon n_p \in I_i\}$, $j(i) = \max\{p\colon n_p \in I_i\}$, and $m_i = \max\{n\colon n \in I_i\} + 1$. Consider the two sequences

$$\left(n_{j1}, m_2, n_{j(3)}, m_4, \ldots, n_{j(2p-1)}, m_{2p}, \ldots\right),$$
$$\left(m_1, n_{j2}, m_3, n_{j(4)}, \ldots, m_{2p-1}, n_{j(2p)}, \ldots\right).$$

Since

$$n_{j(i)} < m_i \leq n_{\bar{j}(i+1)} \leq n_{j(i+1)} < m_{i+1}, \qquad i \geq 1,$$

each of the two sequences belongs to the set $\mathfrak{N}$. The set containing these two sequences and the sequence $(n_j, j \geq 1)$ is denoted by $\mathfrak{N}_{(n_j)}$. Then we put

$$\mathfrak{N}((A_n)) = \mathfrak{N}\left((T_n)\right) = \bigcup_{(n_j, j \geq 1) \in \mathfrak{N}((B_n))} \mathfrak{N}_{(n_j)} \cup \{(\hat{n}_j, j \geq 1)\}. \tag{6}$$

b) Now consider a general sequence $(t_n, n \geq 1)$. We split the set N of positive integers into two subsets N_1 and N_2, where $N_1 = \{n \in N\colon t_n \neq 0\}$ and $N_2 = \{n \in N\colon t_n = 0\}$. Then the sequence $(T_n, n \geq 1)$ is split into two subsequences $T^1 = (T_n^1, n \geq 1)$ and $T^{(2)} = (T_n^2, n \geq 1)$ corresponding to the subsets N_1 and N_2. Note that T^2 is a sequence of s-dimensional vectors; thus the class $\mathfrak{N}(T^2)$ is defined for it by the induction assumption. If the set N_1 contains only a finite number of elements, then we let $\mathfrak{N}((T_n)) = \mathfrak{N}(T^2)$. Otherwise, that is, if the set N_1 is infinite, the class $\mathfrak{N}(T^1)$ is defined for T^1 according to a). If both sets N_1 and N_2 are infinite, we let

$$\mathfrak{N}((A_n)) = \mathfrak{N}((T_n)) = \mathfrak{N}(T^1) \cup \mathfrak{N}(T^2).$$

Case 3: $d = 1$, $m \geq 1$, and the sequence $(A_n, n \geq 1) = (T_n, n \geq 1)$ converges. Let $T = \lim_{n\to\infty} T_n$ and put $\breve{T}_n = T_n - T$, $n \geq 1$. Thus $\|\breve{T}_n\| \to 0$ as $n \to \infty$ and the class $\mathfrak{N}((\breve{T}_n))$ is defined according to Case 2. Then we put

$$\mathfrak{N}((A_n)) = \mathfrak{N}((T_n)) = \mathfrak{N}\big((\breve{T}_n)\big).$$

Case 4: $d = 1$, $m \geq 1$, and the sequence $(A_n, n \geq 1) = (T_n, n \geq 1)$ does not converge. With the sequence $(T_n, n \geq 1)$ we associate the set $\mathfrak{U}$ of sequences of pairs of positive integers such that

$$\mathfrak{U} = \left\{ \left((n_j, k_j), j \geq 1\right) \subset \mathfrak{N} \times N^N\colon \inf_{j \geq 1} \left\|T_{n_j + k_j} - T_{n_j}\right\| > 0 \right\}.$$

Now we introduce the set of limit points of normalized differences between the consecutive elements of the sequence $(T_n, n \geq 1)$; namely let $\Delta_j = \left(T_{n_j+k_j} - T_{n_j}\right) / \left\|T_{n_j+k_j} - T_{n_j}\right\|$, $j \geq 1$, and

$$Q = \bigcup_{(n_j,k_j)\in\mathfrak{U}} \left\{x \in S_1(0) : x = \lim_{j\to\infty} \Delta_j\right\},$$

where $S_1(0) = \{x \in \mathbf{R}^m \colon \|x\| = 1\}$. We denote by W the linear closure of the set Q and let $W^\perp$ be the orthogonal complement of W. Denote by $\mathrm{Pr}_{W^\perp}$ the orthogonal projection to the subspace $W^\perp$. Since the sequence $(T_n, n \geq 1)$ does not converge,

$$\varlimsup_{n\to\infty} \sup_{k\geq 1} \|T_{n+k} - T_n\| > 0.$$

According to [8, Lemma 3.2.2], the sequence $(T_n', n \geq 1) = (\mathrm{Pr}_{W^\perp} T_n, n \geq 1)$ converges. Thus we put

$$\mathfrak{N}((A_n)) = \mathfrak{N}((T_n)) = \mathfrak{N}\left((T_n')\right),$$

where $\mathfrak{N}\left((T_n')\right)$ is defined according to Case 3.

Case 5: $d \geq 1$, $m \geq 1$, and the sequence $(A_n, n \geq 1)$ is arbitrary. Let $A_n^{(k)}$ be the row k in the matrix A_n, $k = 1, 2, \ldots, d$. Consider the sequences of m-dimensional vectors $A^{(k)} = \left(A_n^{(k)}, n \geq 1\right)$, $k = 1, \ldots, d$. For each of these sequences, the corresponding class $\mathfrak{N}\left(A^{(k)}\right)$ is defined according to the cases considered above. Then we put

$$\mathfrak{N}\left((A_n)\right) = \bigcup_{k=1}^{d} \mathfrak{N}\left(A^{(k)}\right).$$

Therefore the test class $\mathfrak{N}((A_n))$ is constructed for an arbitrary sequence of $d \times m$ matrices $(A_n, n \geq 1)$.

The convergence to zero of sequences of random vectors normalized by operators. Now we consider the almost sure convergence to zero of arbitrary sequences of random vectors normalized by operators. Let $(S_n, n \geq 0)$, $S_0 = 0$, be an arbitrary sequence of random vectors in $\mathbf{R}^m$ defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$.

Below we make use of the following condition: for all $i \geq 1$,

$$\|A_n (S_i - S_{i-1})\| \xrightarrow{\mathsf{P}} 0 \qquad (n \to \infty). \tag{7}$$

Condition (7) holds, in particular, if $\|A_n\| \to 0$ as $n \to \infty$.

Theorem 2.1. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$. If condition (7) holds and*

$$\max_{n_j < n \leq n_{j+1}} \left\|A_{n_{j+1}} \left(S_n - S_{n_j}\right)\right\| \to 0 \quad a.s. \qquad (j \to \infty) \tag{8}$$

for every sequence $(n_j, j \geq 1) \in \mathfrak{N}\left((A_n)\right)$, then

$$\|A_n S_n\| \to 0 \quad a.s. \qquad (n \to \infty). \tag{9}$$

The proof of Theorem 2.1 is given after two corollaries useful for several applications in this paper.

Theorem 2.2. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$. If condition (7) holds and*

$$\sum_{j=1}^{\infty} \mathsf{P}\left(\max_{n_j < n \leq n_{j+1}} \left\|A_{n_{j+1}} \left(S_n - S_{n_j}\right)\right\| > \varepsilon\right) < \infty \tag{10}$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$ and all $\varepsilon > 0$, then relation (9) holds.

Theorem 2.2 follows from Theorem 2.1 and the Borel–Cantelli lemma.

The construction of the class $\mathfrak{N}((A_n))$ is complicated and the assumptions of Theorem 2.2 are not easy to check in the general case. Thus we provide another corollary of Theorem 2.2, useful for several applications.

Theorem 2.3. *Let condition* (7) *hold. If condition* (10) *holds for every sequence*

$$(n_j, j \geq 1) \in \mathfrak{N}$$

and all $\varepsilon > 0$*, then relation* (9) *holds.*

In proving Theorem 2.1, we follow the idea of [5] and use the induction on the dimension of the space $\mathbf{R}^m$. Since the proof is rather complicated, we split it into three steps.

Lemma 2.1. *Let a sequence of vectors* $(T_n, n \geq 1) \subset \mathbf{R}^m$*,* $m \geq 1$*, satisfy*

$$\|T_n\| \to 0 \qquad (n \to \infty) \tag{11}$$

and let $\mathfrak{N}((T_n))$ *be the test class for this sequence. If*

$$\max_{n_j < n \leq n_{j+1}} \left|\left\langle T_{n_{j+1}}, S_n - S_{n_j}\right\rangle\right| \to 0 \quad a.s. \qquad (j \to \infty) \tag{12}$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((T_n))$*, then*

$$\langle T_n, S_n \rangle \to 0 \quad a.s. \qquad (n \to \infty).$$

Proof. Our further argument applies to an arbitrary elementary random event $\omega \in \Omega$, where condition (12) holds for all $(n_j, j \geq 1) \in \mathfrak{N}((T_n))$. Thus we do not write "almost surely". Now we start the induction on the dimension of the space $\mathbf{R}^m$.

First we prove the lemma for $m = 1$. Let $(T_n, n \geq 1) = (a_n, n \geq 1)$ be a sequence of real numbers such that $a_n \to 0$ as $n \to \infty$. Then the test class is defined according to (1) and (2), (3). The case of (3) is trivial; thus consider the case of (2). Put

$$Z_j = \max_{n_j < n \leq n_{j+1}} \left| a_{n_{j+1}} \left(S_n - S_{n_j}\right)\right|, \qquad j \geq 0,$$

where $n_0 = 0$ and $a_0 = 1$. By the assumption of the lemma

$$Z_j \to 0 \qquad (j \to \infty). \tag{13}$$

If $n_j < n \leq n_{j+1}$, then

$$|a_n S_n| \leq \lambda^{1-j} \sum_{k=0}^{j} \lambda^k Z_k.$$

Now the Toeplitz lemma and relation (13) imply that $a_n S_n \to 0$ as $n \to \infty$. This completes the proof of Lemma 2.1 in the case $m = 1$.

Next we assume that the lemma is true for some $m = s \geq 1$ and prove it for $m = s+1$.

We denote the last coordinate of the vector $T_n \in \mathbf{R}^{s+1}$ by t_n. Relation (11) implies that $t_n \to 0$ as $n \to \infty$. Without loss of generality we assume that $t_n \neq 0$, $n \geq 1$, in Case 2b) of the construction of the test class. Fix an arbitrary number $\lambda > 1$ and put

$$\hat{n}_j = \max\left\{n \colon |t_n| \geq \lambda^{-j}\right\}, \qquad j \geq 1.$$

It is clear that

$$\begin{aligned} t_{\hat{n}_{j+1}} \langle T_n, S_n \rangle &= \left\langle t_{\hat{n}_{j+1}} T_n - t_n T_{\hat{n}_{j+1}}, S_n \right\rangle \\ &\quad + t_n \left(\left\langle T_{\hat{n}_{j+1}}, S_n - S_{\hat{n}_j}\right\rangle + \left\langle T_{\hat{n}_{j+1}}, S_{\hat{n}_{j+1}}\right\rangle - \left\langle T_{\hat{n}_{j+1}}, S_{\hat{n}_{j+1}} - S_{\hat{n}_j}\right\rangle\right). \end{aligned} \tag{14}$$

The last coordinate of the vector $t_{\hat{n}_{j+1}} T_n - t_n T_{\hat{n}_{j+1}}$ is equal to zero. We denote the vector formed by the first s coordinates of the vector T_n or S_n by $\tilde{T}_n$ or $\tilde{S}_n$, respectively. Now we introduce the sequence of vectors $(B_n, n \geq 1)$ in $\mathbf{R}^s$ according to (4). Then the test class $\mathfrak{N}((T_n))$ for the sequence $(T_n, n \geq 1)$ is defined by (6).

If $\hat{n}_j < n \leq \hat{n}_{j+1}$, then it follows from (14) that

$$
\begin{aligned}
|\langle T_n, S_n\rangle| \leq \lambda \max_{\hat{n}_j < n < \hat{n}_{j+1}} \left|\left\langle B_n, \tilde{S}_n\right\rangle\right| \\
+ 2\lambda \max_{\hat{n}_j < n \leq \hat{n}_{j+1}} \left|\left\langle T_{\hat{n}_{j+1}}, S_n - S_{\hat{n}_j}\right\rangle\right| + \lambda \left|\left\langle T_{\hat{n}_{j+1}}, S_{\hat{n}_{j+1}}\right\rangle\right| .
\end{aligned} \tag{15}
$$

Indeed, inequality (15) is obvious for $n = \hat{n}_{j+1}$. Let $\hat{n}_j < n < \hat{n}_{j+1}$. If n is such that $|t_n| \geq \left|t_{\hat{n}_{j+1}}\right|$, then $|t_n|/\left|t_{\hat{n}_{j+1}}\right| < \lambda$ and $\theta_n = \left|t_{\hat{n}_{j+1}}\right|$. Thus we obtain (15) from (14). If n is such that $|t_n| < \left|t_{\hat{n}_{j+1}}\right|$, then we obtain from (14) that

$$
\begin{aligned}
|\langle T_n, S_n\rangle| \leq \max_{\hat{n}_j < n < \hat{n}_{j+1}} \left|\left\langle B_n, \tilde{S}_n\right\rangle\right| \\
+ 2 \max_{\hat{n}_j < n \leq \hat{n}_{j+1}} \left|\left\langle T_{\hat{n}_{j+1}}, S_n - S_{\hat{n}_j}\right\rangle\right| + \left|\left\langle T_{\hat{n}_{j+1}}, S_{\hat{n}_{j+1}}\right\rangle\right| .
\end{aligned}
$$

This implies inequality (15), since $\lambda > 1$.

The second term on the right hand side of (15) tends to zero as $j \to \infty$ in view of condition (12). Now we show that also the first term tends to zero. Moreover we show that

$$
\left\langle B_n, \tilde{S}_n\right\rangle \to 0 \qquad (n \to \infty) . \tag{16}
$$

By the induction assumption, if

$$
\max_{n_{j-1} < n \leq n_j} \left|\left\langle B_{n_j}, \tilde{S}_n - \tilde{S}_{n_{j-1}}\right\rangle\right| \to 0 \qquad (j \to \infty) \tag{17}
$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((B_n))$, then relation (16) holds. Now we prove that condition (12) implies relation (17). Fix an arbitrary sequence $(n_j, j \geq 1) \in \mathfrak{N}((B_n))$. Since condition (12) holds for every sequence of the set $\mathfrak{N}_{(n_j)}$ (see Case 2 in the construction of the test class), we have

$$
\begin{aligned}
\max_{n_{j(2p-1)} < n \leq m_{2p}} \left|\left\langle T_{m_{2p}}, S_n - S_{n_{j(2p-1)}}\right\rangle\right| \to 0 \qquad (p \to \infty) , \\
\max_{n_{j(2p)} < n \leq m_{2p+1}} \left|\left\langle T_{m_{2p+1}}, S_n - S_{n_{j(2p)}}\right\rangle\right| \to 0 \qquad (p \to \infty) , \\
L_j = \max_{n_{j-1} < n \leq n_j} \left|\left\langle T_{n_j}, S_n - S_{n_{j-1}}\right\rangle\right| \to 0 \qquad (j \to \infty).
\end{aligned} \tag{18}
$$

Here and in what follows we use the notation introduced in Case 2 of the construction of the test class. Since $n_{j(2p)} < m_{2p}$ and $n_{j(2p+1)} < m_{2p+1}$,

$$
\begin{aligned}
\max_{n_{j(2p-1)} < n \leq n_{j(2p)}} \left|\left\langle T_{m_{2p}}, S_n - S_{n_{j(2p-1)}}\right\rangle\right| \to 0 \qquad (p \to \infty) , \\
\max_{n_{j(2p)} < n \leq n_{j(2p+1)}} \left|\left\langle T_{m_{2p+1}}, S_n - S_{n_{j(2p)}}\right\rangle\right| \to 0 \qquad (p \to \infty) .
\end{aligned}
$$

Combining these two relations, we obtain

$$
\max_{n_{j(i)} < n \leq n_{j(i+1)}} \left|\left\langle T_{m_{i+1}}, S_n - S_{n_{j(i)}}\right\rangle\right| \to 0 \qquad (i \to \infty). \tag{19}
$$

Note that $N = \bigcup_{i=1}^{\infty} [\bar{j}(i), j(i)]$. Furthermore, let $j \in [\bar{j}(i+1), j(i+1)]$. Then

$$
n_j \in I_{i+1} = [\hat{n}_p, \hat{n}_{p+1})
$$

and

$$
\left\langle B_{n_j}, \tilde{S}_n - \tilde{S}_{n_{j-1}}\right\rangle = \theta_{n_j} t_{n_j}^{-1} \left\langle T_{n_j}, S_n - S_{n_{j-1}}\right\rangle - \theta_{n_j} t_{\hat{n}_{p+1}}^{-1} \left\langle T_{\hat{n}_{p+1}}, S_n - S_{n_{j-1}}\right\rangle
$$

for all n and j. Indeed, after simple algebra in the right hand side of the latter equality we get

$$\theta_{n_j}\left\langle t_{n_j}^{-1}T_{n_j}-t_{\hat n_{p+1}}^{-1}T_{\hat n_{p+1}}, S_n-S_{n_{j-1}}\right\rangle=\theta_{n_j}\left\langle t_{n_j}^{-1}\tilde T_{n_j}-t_{\hat n_{p+1}}^{-1}\tilde T_{\hat n_{p+1}}, \tilde S_n-\tilde S_{n_{j-1}}\right\rangle$$
$$=\left\langle \theta_{n_j}\left(t_{n_j}^{-1}\tilde T_{n_j}-t_{\hat n_{p+1}}^{-1}\tilde T_{\hat n_{p+1}}\right), \tilde S_n-\tilde S_{n_{j-1}}\right\rangle=\left\langle B_{n_j}, \tilde S_n-\tilde S_{n_{j-1}}\right\rangle.$$

Since $\hat n_{p+1}=m_{i+1}$,

$$\begin{aligned}\max_{n_{j-1}<n\le n_j}\left|\left\langle B_{n_j},\tilde S_n-\tilde S_{n_{j-1}}\right\rangle\right| &\le \max_{n_{j-1}<n\le n_j}\left|\left\langle T_{n_j}, S_n-S_{n_{j-1}}\right\rangle\right| \\ &\quad+\max_{n_{j-1}<n\le n_j}\left|\left\langle T_{m_{i+1}}, S_n-S_{n_{j-1}}\right\rangle\right| \\ &= L_j+Q_j,\end{aligned}\tag{20}$$

where

$$Q_j=\max_{n_{j-1}<n\le n_j}\left|\left\langle T_{m_{i+1}}, S_n-S_{n_{j-1}}\right\rangle\right|.$$

Since $n_{j(i)}\le n_{j-1}\le n_j\le n_{j(i+1)}$, we have

$$\begin{aligned}Q_j&\le \max_{n_{j-1}<n\le n_j}\left|\left\langle T_{m_{i+1}}, S_n-S_{n_{j(i)}}\right\rangle\right|+\left|\left\langle T_{m_{i+1}}, S_{n_{j-1}}-S_{n_{j(i)}}\right\rangle\right| \\ &\le 2\max_{n_{j-1}\le n\le n_j}\left|\left\langle T_{m_{i+1}}, S_n-S_{n_{j(i)}}\right\rangle\right| \\ &\le 2\max_{n_{j(i)}<n\le n_{j(i+1)}}\left|\left\langle T_{m_{i+1}}, S_n-S_{n_{j(i)}}\right\rangle\right|, \qquad j\in\left[\bar j\,(i+1)\,, j(i+1)\right].\end{aligned}$$

This together with relation (19) implies that $\max_{\bar j(i+1)\le j\le j(i+1)} Q_j\to 0$ as $i\to\infty$. Hence $Q_j\to 0$ as $j\to\infty$. Using the latter result we deduce (17) from (18) and (20).

To complete the proof of the lemma, it remains to show that the third term in (15) approaches zero, that is,

$$V_j=\left|\left\langle T_{\hat n_j}, S_{\hat n_j}\right\rangle\right|\to 0 \qquad (j\to\infty).\tag{21}$$

Note that if $\hat n_j=\hat n_{j+1}=\dots=\hat n_{j+l}$, then $V_j=V_{j+1}=\dots=V_{j+l}$ by construction. This allows one to assume (without any loss of generality) that the sequence $(\hat n_j, j\ge 1)$ is increasing.

We have

$$\begin{aligned}t_{\hat n_j}\left\langle T_{\hat n_{j+1}}, S_{\hat n_{j+1}}\right\rangle &= t_{\hat n_j}\left\langle T_{\hat n_{j+1}}, S_{\hat n_{j+1}}-S_{\hat n_j}\right\rangle+t_{\hat n_{j+1}}\left\langle T_{\hat n_j}, S_{\hat n_j}\right\rangle \\ &\quad-\left\langle t_{\hat n_{j+1}}T_{\hat n_j}-t_{\hat n_j}T_{\hat n_{j+1}}, S_{\hat n_j}\right\rangle.\end{aligned}$$

Thus

$$V_{j+1}\le\left|t_{\hat n_{j+1}}t_{\hat n_j}^{-1}\right|\cdot V_j+\delta_j, \qquad j\ge 1,\tag{22}$$

where

$$\delta_j=\left|\left\langle T_{\hat n_{j+1}}, S_{\hat n_{j+1}}-S_{\hat n_j}\right\rangle\right|+\left|\left\langle B_{\hat n_j}, \tilde S_{\hat n_j}\right\rangle\right|.$$

It follows from (12) and (16) that

$$\delta_j\to 0 \qquad (j\to\infty).\tag{23}$$

In turn, inequality (22) implies that

$$V_{j+2}\le\left|t_{\hat n_{j+2}}t_{\hat n_j}^{-1}\right|\cdot V_{j+1}+\left|t_{\hat n_{j+2}}t_{\hat n_{j+1}}^{-1}\right|\cdot\delta_j+\delta_{j+1}, \qquad j\ge 1.\tag{24}$$

Since $|t_{\hat n_{j+2}}t_{\hat n_j}^{-1}|\le\lambda^{-1}<1$ and $|t_{\hat n_{j+2}}t_{\hat n_{j+1}}^{-1}|\le 1$, we obtain relation (21) from (23) and (24) by using the Toeplitz lemma. Lemma 2.1 is proved. □

Now we replace (11) with a weaker assumption in the following result.

Lemma 2.2. *Let a sequence of vectors $(T_n, n \geq 1)$ in $\mathbf{R}^m$ be such that*

$$\langle T_n, S_i - S_{i-1}\rangle \xrightarrow{\mathsf{P}} 0 \qquad (n \to \infty) \tag{25}$$

for all $i \geq 1$. Let $\mathfrak{N}((T_n))$ be the test class for this sequence. If condition (12) *holds for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((T_n))$, then*

$$\langle T_n, S_n\rangle \to 0 \quad a.s. \qquad (n \to \infty).$$

Proof. First we assume that the sequence $(T_n, n \geq 1)$ is such that

$$\langle T_n, S_i - S_{i-1}\rangle \to 0 \quad \text{a.s.} \qquad (n \to \infty) \tag{26}$$

for all $i \geq 1$. We distinguish between the two cases in the proof, namely a) the sequence $(T_n, n \geq 1)$ converges; and b) the sequence $(T_n, n \geq 1)$ does not converge.

a) Suppose the sequence $(T_n, n \geq 1)$ converges. Put

$$T = \lim_{n\to\infty} T_n, \qquad \breve{T}_n = T_n - T, \quad n \geq 1.$$

Then $\mathfrak{N}((T_n))$ is defined according to Case 3 of the construction of the test class. Condition (26) implies that $\langle T, S_i - S_{i-1}\rangle = 0$ almost surely for all $i \geq 1$. This means that

$$\max_{n_j < n \leq n_{j+1}} \left|\left\langle T_{n_{j+1}}, S_n - S_{n_j}\right\rangle\right| = \max_{n_j < n \leq n_{j+1}} \left|\left\langle \breve{T}_{n_{j+1}}, S_n - S_{n_j}\right\rangle\right| \quad \text{a.s.}$$

for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$, and $\langle T_n, S_n\rangle = \langle \breve{T}_n, S_n\rangle$ almost surely for all $n \geq 1$. Now the lemma follows from Lemma 2.1.

b) Suppose the sequence $(T_n, n \geq 1)$ does not converge. Then $\mathfrak{N}((T_n))$ is defined according to Case 4 of the construction of the test class. Recall that the terms of the sequence $(T'_n, n \geq 1)$ satisfy $T'_n = \mathrm{Pr}_{W^\perp} T_n$, $n \geq 1$. Condition (26) and [8, Lemma 3.2.3] imply that

$$\langle T_n, S_i - S_{i-1}\rangle = \langle T'_n, S_i - S_{i-1}\rangle \quad \text{a.s.}$$

for all n and $i \geq 1$, and

$$\langle T'_n, S_i - S_{i-1}\rangle \to 0 \quad \text{a.s.} \qquad (n \to \infty)$$

for all $i \geq 1$. Since

$$\max_{n_j < n \leq n_{j+1}} \left|\left\langle T_{n_{j+1}}, S_n - S_{n_j}\right\rangle\right| = \max_{n_j < n \leq n_{j+1}} \left|\left\langle T'_{n_{j+1}}, S_n - S_{n_j}\right\rangle\right| \quad \text{a.s.}$$

for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$, and $\langle T_n, S_n\rangle = \langle T'_n, S_n\rangle$ almost surely for all $n \geq 1$, the general case of the lemma follows from case a).

One can reduce condition (26) to condition (25) in view of [8, Lemma 3.2.5]. Lemma 2.2 is proved. □

Proof of Theorem 2.1. Let $(e_1, \ldots, e_d)$ be an orthonormal basis in $\mathbf{R}^d$, and let A_n^* be the conjugate operator to A_n. Since

$$A_n S_n = \sum_{k=1}^{d} \langle A_n S_n, e_k\rangle e_k = \sum_{k=1}^{d} \langle A_n^* e_k, S_n\rangle\, e_k,$$

relation (9) holds if and only if

$$\left\langle A_n^{(k)}, S_n\right\rangle \to 0 \quad \text{a.s.} \qquad (n \to \infty) \tag{27}$$

for all $k = 1, \ldots, d$, where $A_n^{(k)} = A_n^* e_k$. The set $\mathfrak{N}((A_n))$ is defined according to Case 5 of the construction of the test class. It follows from (7) that

$$\left\langle A_n^{(k)}, S_i - S_{i-1}\right\rangle \xrightarrow{\mathsf{P}} 0 \qquad (n \to \infty) \tag{28}$$

for all $k = 1, \dots, d$ and for all $i \geq 1$. Furthermore, condition (8) implies that

$$\max_{n_j < n \leq n_{j+1}} \left| \left\langle A^{(k)}_{n_{j+1}}, S_n - S_{n_j} \right\rangle \right| \to 0 \quad \text{a.s.} \qquad (j \to \infty) \tag{29}$$

for all $k = 1, \dots, d$ and all sequences $(n_j, j \geq 1) \in \mathfrak{N}(A^{(k)})$. In view of (28) and (29) Lemma 2.2 implies relation (27), whence (9) follows. Theorem 2.1 is proved. □

If $(S_n, n \geq 1)$ is a sequence of partial sums of independent symmetric random vectors, then conditions (7) and (10) are necessary for relation (9). This follows from Theorem 3.2.3 in [8]. Indeed, condition (7) is obvious in view of Theorem 3.2.3 in [8]. Since the series in (10) does not exceed the series

$$2 \sum_{j=1}^{\infty} \mathsf{P} \left(\left\| A_{n_{j+1}} \left(S_{n_{j+1}} - S_{n_j} \right) \right\| > \varepsilon \right)$$

by the Lévy inequality, condition (10) follows. The convergence of the latter series is implied by the Borel–Cantelli lemma, by the relation $\|A_{n_{j+1}}(S_{n_{j+1}} - S_{n_j})\| \to 0$ a.s. as $j \to \infty$, and by the independence of the vectors $(S_{n_{j+1}} - S_{n_j}, j \geq 1)$.

The boundedness of sequences of random vectors normalized by operators. Consider conditions for the almost sure boundedness of arbitrary sequences of random vectors normalized by operators. The following result is proved similarly to Theorem 2.1.

Theorem 2.4. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$. If condition* (7) *holds and*

$$\sup_{j \geq 1} \max_{n_j < n \leq n_{j+1}} \left\| A_{n_{j+1}} \left(S_n - S_{n_j} \right) \right\| < \infty \quad \textit{a.s.}$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$, then

$$\sup_{n \geq 1} \|A_n S_n\| < \infty \quad \textit{a.s.} \tag{30}$$

Theorem 2.4 and the Borel–Cantelli lemma imply the following result.

Theorem 2.5. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$. If condition* (7) *holds and for any sequence $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$ there exists a number $\varepsilon > 0$ (possibly depending on the sequence $(n_j, j \geq 1)$) such that condition* (10) *holds, that is,*

$$\sum_{j=1}^{\infty} \mathsf{P} \left(\max_{n_j < n \leq n_{j+1}} \left\| A_{n_{j+1}} \left(S_n - S_{n_j} \right) \right\| > \varepsilon \right) < \infty, \tag{31}$$

then inequality (30) *holds.*

The following result also follows from Theorem 2.5.

Theorem 2.6. *Let condition* (7) *hold. If for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$ there exists a number $\varepsilon > 0$ (possibly depending on $(n_j, j \geq 1)$) such that condition* (31) *holds, then inequality* (30) *holds.*

The following result shows that condition (7) can be weakened.

Theorem 2.7. *Let*

$$\sup_{n \geq 1} \|A_n\| < \infty.$$

If for every sequence $(n_j, j \geq 1) \in \mathfrak{N}$ there exists a number $\varepsilon > 0$ (possibly depending on the sequence $(n_j, j \geq 1)$) such that condition (31) *holds, then inequality* (30) *holds.*

The proof of Theorem 2.7 follows from Theorem 2.1 and Lemma 3 in [9].

If $(S_n, n \geq 1)$ is the sequence of partial sums of independent symmetric random vectors, then condition (31) for all sequences $(n_j, j \geq 1) \in \mathfrak{N}$ and some number $\varepsilon > 0$ (possibly depending on the sequence $(n_j, j \geq 1)$) is necessary for inequality (30). This follows from [10, Theorem 3.4.1] and Lévy's inequality.

3. Strong law of large numbers for sums of orthogonal random vectors normalized by operators

A sequence of random vectors $(X_i, i \geq 1)$ in $\mathbf{R}^m$ is called orthogonal if $\mathsf{E}\,\|X_i\|^2 < \infty$, $i \geq 1$, and $\mathsf{E}\left(\langle T_1, X_i\rangle \langle T_2, X_j\rangle\right) = 0$ for all vectors $T_1, T_2 \in \mathbf{R}^m$ and all $i \neq j$. Put $S_n = \sum_{i=1}^n X_i$, $n \geq 1$, and $\log_+ t = \ln \max\{t, e\}$, $t \geq 0$.

Theorem 3.1. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$ and let*

$$(X_i, i \geq 1)$$

be an orthogonal sequence in $\mathbf{R}^m$. If

$$\|A_n X_i\| \xrightarrow{\mathsf{P}} 0 \qquad (n \to \infty) \tag{32}$$

for all $i \geq 1$, and

$$\sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \mathsf{E}\left\|A_{n_{j+1}} X_i\right\|^2 \log_+^2 (n_{j+1} - n_j) < \infty \tag{33}$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$, then

$$\|A_n S_n\| \to 0 \quad a.s. \qquad (n \to \infty)\,. \tag{34}$$

Proof. We check assumptions of Theorem 2.2. Condition (7) holds in view of relation (32). Using Chebyshev's and Rademacher–Men'shov's inequalities we obtain

$$\begin{aligned} \mathsf{P}\left(\max_{n_j < n \leq n_{j+1}} \left\|A_{n_{j+1}}\left(S_n - S_{n_j}\right)\right\| > \varepsilon\right) &\leq \varepsilon^{-2}\,\mathsf{E}\left(\max_{n_j < n \leq n_{j+1}} \left\|\sum_{i=n_j+1}^{n} A_{n_{j+1}} X_i\right\|^2\right) \\ &\leq (\varepsilon \ln 2)^{-2} \ln^2\left[4\left(n_{j+1} - n_j\right)\right] \sum_{i=n_j+1}^{n_{j+1}} \mathsf{E}\left\|A_{n_{j+1}} X_i\right\|^2 \end{aligned}$$

for all $\varepsilon > 0$.

Thus condition (33) implies (10), and this completes the proof of Theorem 3.1. □

Theorem 3.1 yields the following result.

Corollary 3.1. *Let $(X_i, i \geq 1)$ be an orthogonal sequence in $\mathbf{R}^m$. Assume that condition (32) holds. If*

$$\sum_{i=1}^{\infty} \sup_{n \geq i} \left(\mathsf{E}\,\|A_n X_i\|^2 \ln^2 n\right) < \infty, \tag{35}$$

then relation (34) holds.

If $A_n = a_n I$, $n \geq 1$, in Corollary 3.1, where I is the identity operator, and if the sequence of numbers $(a_n \ln n, n \geq n_0 > 1)$ is decreasing to zero, then condition (35) becomes the classical Rademacher–Men'shov condition

$$\sum_{i=2}^{\infty} \mathsf{E}\,\|X_i\|^2 a_i^2 \ln^2 i < \infty$$

(see, for example, [2]).

4. Convergence to zero and boundedness of martingales normalized by operators

Given a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with filtration $(\mathcal{F}_n, n \geq 0)$ let $(S_n, n \geq 0)$ be a martingale with respect to $(\mathcal{F}_n, n \geq 0)$. Put $S_0 = 0$ and denote by $(X_n, n \geq 1)$ the corresponding martingale difference, that is,

$$X_n = S_n - S_{n-1}, \qquad n \geq 1.$$

In what follows we use the following form of condition (7): for all $i \geq 1$,

$$\|A_n X_i\| \xrightarrow{\mathsf{P}} 0 \qquad (n \to \infty). \tag{36}$$

For the sake of brevity we use the notation

$$U_{n_j} = A_{n_{j+1}} \left(S_{n_{j+1}} - S_{n_j} \right), \qquad j \geq 1,\ (n_j, j \geq 1) \in \mathfrak{N}. \tag{37}$$

The indicator of a random event is denoted by $I(\cdot)$.

Theorem 4.1. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$. Assume that condition* (36) *holds. If*

$$\sum_{j=1}^{\infty} \mathsf{E}\left[\|U_{n_j}\| I(\|U_{n_j}\| > \varepsilon) \right] < \infty \tag{38}$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$ and all numbers $\varepsilon > 0$, or, equivalently, if

$$\sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \mathsf{P}\left(\|U_{n_j}\| > t \right) dt < \infty \tag{39}$$

for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$, then

$$\|A_n S_n\| \to 0 \quad a.s. \qquad (n \to \infty). \tag{40}$$

Proof. The proof of the theorem is based on Theorem 2.2. Condition (7) is equivalent to condition (36) in this case. Now we show that condition (10) follows from (38). Note that

$$\left(A_{n_{j+1}} \left(S_n - S_{n_j} \right), \mathcal{F}_n, n_j \leq n \leq n_{j+1} \right)$$

is a martingale with zero mean for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$. Using the Brown inequality [11], we get

$$\begin{aligned} &\sum_{j=1}^{\infty} \mathsf{P}\left(\max_{n_j < n \leq n_{j+1}} \left\| A_{n_{j+1}} \left(S_n - S_{n_j} \right) \right\| > \varepsilon \right) \\ &\qquad \leq \frac{2d^{3/2}}{\varepsilon} \sum_{j=1}^{\infty} \mathsf{E}\left[\|U_{n_j}\| I\left(\|U_{n_j}\| > \varepsilon / 2d^{1/2} \right) \right] < \infty. \end{aligned}$$

Thus Theorem 2.2 implies relation (40). The equivalence of conditions (38) and (39) is proved by integrating by parts. Theorem 4.1 is proved. □

Remark 4.1. Condition (39) is equivalent to the following two conditions:

(i) for all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$ and all numbers $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} \mathsf{P}\left(\|U_{n_j}\| > \varepsilon \right) < \infty;$$

(ii) for every sequence $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$ there exists a number $\varepsilon_0 > 0$ (possibly depending on the sequence $(n_j, j \geq 1)$) such that

$$\sum_{j=1}^{\infty} \int_{\varepsilon_0}^{\infty} \mathsf{P}\left(\|U_{n_j}\| > t\right) dt < \infty.$$

Now we discuss whether the assumptions of Theorem 4.1 can be weakened in some sense.

1) If there exists a constant $c > 0$ such that $\|A_n S_n\| \leq c$ almost surely for all $n \geq 1$, then condition (i) is sufficient for relation (40). This obviously follows from (38), since

$$\begin{aligned}\|U_{n_j}\| &\leq \left\|A_{n_{j+1}} S_{n_{j+1}}\right\| + \left\|A_{n_{j+1}} S_{n_j}\right\| \leq c + \left\|\mathsf{E}\left(A_{n_{j+1}} S_{n_{j+1}} \;\middle|\; \mathcal{F}_{n_j}\right)\right\| \\ &\leq c + \mathsf{E}\left(\left\|A_{n_{j+1}} S_{n_{j+1}}\right\| \;\middle|\; \mathcal{F}_{n_j}\right) \leq 2c \quad \text{a.s.}\end{aligned}$$

in view of the martingale property.

2) Let $(X_i, i \geq 1)$ be a sequence of independent centered ($\mathsf{E}\, X_i = 0$, $i \geq 1$) random vectors. Consider the martingale

$$S_n = \sum_{i=1}^{n} X_i, \qquad n \geq 1.$$

It follows from [12] that if (36) holds, then (i) and

$$\|A_n S_n\| \xrightarrow{\mathsf{P}} 0 \qquad (n \to \infty) \tag{41}$$

imply relation (40). This form of the strong law of large numbers originates from the paper [13] for the one-dimensional case (see also [14, p. 160]). Condition (39) implies (i) and (41) in view of Remark 4.1 and Theorem 4.1. An example is constructed in [6] demonstrating that conditions (i) and (41) are not sufficient for (40) in the case of general martingales. This also means that condition (39) cannot be weakened in general.

The boundedness of martingales normalized by operators. Consider conditions for the boundedness of martingales normalized by operators.

The next result follows from Theorem 2.5 and Brown's inequality [11].

Theorem 4.2. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \geq 1)$. Assume that condition* (36) *holds. If for any sequence $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$ there exists a number $\varepsilon > 0$ (possibly depending on the sequence $(n_j, j \geq 1)$) such that*

$$\sum_{j=1}^{\infty} \mathsf{E}\left[\left\|U_{n_j}\right\| I\left(\|U_{n_j}\| > \varepsilon\right)\right] < \infty,$$

then

$$\sup_{n \geq 1} \|A_n S_n\| < \infty \quad \textit{a.s.} \tag{42}$$

The following result demonstrates that condition (36) can be weakened.

Theorem 4.3. *Let*

$$\sup_{n \geq 1} \|A_n\| < \infty.$$

If for any sequence $(n_j, j \geq 1) \in \mathfrak{N}$ there exists a number $\varepsilon > 0$ (possibly depending on the sequence $(n_j, j \geq 1)$) such that

$$\sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \mathsf{P}\left(\|U_{n_j}\| > t\right) dt < \infty,$$

then relation (42) *holds.*

The proof of Theorem 4.3 follows from Theorem 4.1 and Lemma 3 in [9].

Comparing Theorem 4.3 with Remark 4.1, we see that condition (ii) (which is true for the set $\mathfrak{N}$) implies relation (42), that is, (ii) implies the almost sure boundedness.

5. Strong law of large numbers for martingales that are normalized by operators and have higher moments

In this section, we consider a number of corollaries of Theorem 4.1 for martingales that have a finite absolute moment of order $\nu \ge 1$. A comprehensive study of the strong law of large numbers for martingales normalized by sequences of real numbers can be found in [14]–[17].

Theorem 5.1. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \ge 1)$. Assume that condition* (36) *holds. If*

$$\sum_{j=1}^{\infty} \mathsf{E}\, \|U_{n_j}\|^{\nu} < \infty \tag{43}$$

for some $\nu \ge 1$ and all sequences $(n_j, j \ge 1) \in \mathfrak{N}((A_n))$, then

$$\|A_n S_n\| \to 0 \quad a.s. \qquad (n \to \infty). \tag{44}$$

Proof. The case of $\nu = 1$ is obvious. Consider the case of $\nu > 1$. Using the Chebyshev inequality we get

$$\sum_{j=1}^{\infty} \int_{\varepsilon}^{\infty} \mathsf{P}\left(\|U_{n_j}\| > t\right)\, dt \le \sum_{j=1}^{\infty} \mathsf{E}\, \|U_{n_j}\|^{\nu} \int_{\varepsilon}^{\infty} \frac{dt}{t^{\nu}} = \frac{\varepsilon^{1-\nu}}{\nu - 1} \sum_{j=1}^{\infty} \mathsf{E}\, \|U_{n_j}\|^{\nu} < \infty.$$

Thus Theorem 4.1 implies relation (44). The theorem is proved. □

Theorem 5.2. *Let $\mathfrak{N}((A_n))$ be the test class for a sequence $(A_n, n \ge 1)$. Assume that condition* (36) *holds. If*

$$\sum_{j=1}^{\infty} \sum_{i=n_j+1}^{n_{j+1}} \mathsf{E}\, \left\|A_{n_{j+1}} X_i\right\|^{\nu} < \infty$$

for some number $\nu \in [1, 2]$ and all sequences $(n_j, j \ge 1) \in \mathfrak{N}\left((A_n)\right)$, then relation (44) *holds.*

The proof of Theorem 5.2 follows from Theorem 5.1 and the inequality (see [18])

$$\mathsf{E}\, \|U_{n_j}\|^{\nu} = \mathsf{E} \left\| \sum_{i=n_j+1}^{n_{j+1}} A_{n_{j+1}} X_i \right\|^{\nu} \le 2^{2-\nu} \sum_{i=n_j+1}^{n_{j+1}} \mathsf{E}\, \left\|A_{n_{j+1}} X_i\right\|^{\nu}$$

(we use notation (37)).

Remark 5.1. An equivalent form of Theorem 5.2 is obtained in [4] for sequences of independent centered random variables $(X_n, n \ge 1)$.

Corollary 5.1. *If condition* (36) *holds and*

$$\sum_{i=1}^{\infty} \sup_{n \ge i} \mathsf{E}\, \|A_n X_i\|^{\nu} < \infty$$

for some number $\nu \in [1, 2]$, then relation (44) *holds.*

Definition 5.1. A sequence of operators $(A_n, n \ge 1)$ is called *monotone* if

$$\|A_n x\| \ge \|A_{n+1} x\| \tag{45}$$

for all $x \in \mathbf{R}^m$ and $n \ge 1$.

Note that condition (45) is equivalent to $A_n^* A_n \geq A_{n+1}^* A_{n+1}$, $n \geq 1$, where A_n^* is the conjugate operator to A_n.

Corollary 5.1 and condition (45) imply the following result originally proved in [18, Theorem 2].

Corollary 5.2. *Let $(A_n, n \geq 1)$ be a monotone sequence of operators and let*

$$\|A_n\| \to 0, \qquad n \to \infty.$$

If

$$\sum_{i=1}^{\infty} \mathsf{E} \|A_i X_i\|^{\nu} < \infty$$

for some number $\nu \in [1,2]$, then relation (44) holds.

Now we consider the case where a martingale has a finite absolute moment of order $\nu \geq 2$.

Theorem 5.3. *Let $\mathfrak{N}((A_n))$ be the test class of a sequence $(A_n, n \geq 1)$. Assume that condition (36) holds. If*

$$\sum_{j=1}^{\infty} (n_{j+1} - n_j)^{(\nu/2)-1} \sum_{i=n_j+1}^{n_{j+1}} \mathsf{E} \left\|A_{n_{j+1}} X_i\right\|^{\nu} < \infty$$

for some number $\nu \in [2, +\infty)$ and all sequences $(n_j, j \geq 1) \in \mathfrak{N}((A_n))$, then relation (44) holds.

The proof of Theorem 5.3 follows from Theorem 5.1 and the Dharmadhikari–Fabian–Jogdeo inequality [19].

Theorem 5.3 implies the following result.

Corollary 5.3. *Suppose condition (36) holds. If*

$$\sum_{i=1}^{n} \sup_{n \geq i} \left(n^{\nu/2-1} \mathsf{E} \|A_n X_i\|^{\nu}\right) < \infty$$

for some number $\nu \in [2, +\infty)$, then relation (44) holds.

Corollary 5.3 contains the following result obtained in [20].

Corollary 5.4. *Let $A_n = a_n I$, $n \geq 1$, where I is the identity operator and $(a_n, n \geq 1)$ is a sequence of positive numbers such that $a_n \to 0$ as $n \to \infty$ and the sequence*

$$\left(a_n n^{(\nu-2)/(2\nu)}, n \geq n_0 \geq 1\right)$$

decreases for some $\nu \in [2, +\infty)$. If

$$\sum_{i=1}^{\infty} i^{\nu/2-1} a_i^{\nu} \mathsf{E} \|X_i\|^{\nu} < \infty,$$

then

$$a_n \|S_n\| \to 0 \quad a.s. \qquad (n \to \infty).$$

The following result is useful for some cases.

Corollary 5.5. *Suppose condition (36) holds. Assume that*

$$\mathsf{E} \|A_n (S_n - S_k)\|^{\nu} \leq c\,(f_k - f_n)$$

for some number $\nu \in [1, +\infty)$ and for all positive integers $k \leq n$, where c is a positive constant (that does not depend on n and k). If $(f_n, n \geq 1)$ is a sequence of real numbers such that $f_n \downarrow 0$ as $n \to \infty$, then relation (44) holds.

The proof of Corollary 5.5 follows from Theorem 5.1, since relation (42) holds for all sequences

$$(n_j, j \geq 1) \in \mathfrak{N}.$$

Indeed,

$$\sum_{j=1}^{\infty} \mathsf{E}\, \|U_{n_j}\|^{\nu} = \sum_{j=1}^{\infty} \mathsf{E}\, \left\|A_{n_{j+1}} \left(S_{n_{j+1}} - S_{n_j}\right)\right\|^{\nu} \leq c \sum_{j=1}^{\infty} \left(f_{n_j} - f_{n_{j+1}}\right) = c f_{n_1} < \infty.$$

Consider an application of Corollary 5.5 to the proof of the strong consistency of the least squares estimator of an unknown parameter for vector linear regression. This problem is studied in [21]. The most general result for this problem is obtained in [22].

Assume that elements of $\mathbf{R}^m$ are represented as vector columns and denote by $\|\cdot\|$ the Euclidean norm of a matrix (or a vector). Consider the model of vector linear regression

$$Y_i = D_i^T V + Z_i, \qquad i \geq 1,$$

where $(Z_i, i \geq 1)$ is a sequence of random vectors in $\mathbf{R}^m$; $(D_i, i \geq 1)$ is a sequence of nonrandom $d \times m$ matrices; V is an unknown parameter; and ^{T} means transposition. Assume that the matrices

$$B_n = \sum_{i=1}^{n} D_i D_i^T, \qquad n \geq 1,$$

are nonsingular. Then the least squares estimator of the unknown parameter V can be written as

$$\hat{V}_n = B_n^{-1} \sum_{i=1}^{n} D_i Y_i, \qquad n \geq 1,$$

where B_n^{-1} is the inverse function to B_n.

It is shown in [7] that Corollary 5.5 implies the following result on the strong consistency of the estimator $(\hat{V}_n, n \geq 1)$ (this result is proved earlier in [22]).

Theorem 5.4. *Let $(Z_i, i \geq 1)$ be a martingale difference and let $\sup_{i\geq 1} \mathsf{E}\, \|Z_i\|^2 < \infty$. If*

$$\left\|B_n^{-1}\right\| \to 0 \qquad (n \to \infty), \tag{46}$$

then

$$\|\hat{V}_n - V\| \to 0 \quad a.s. \qquad (n \to \infty).$$

Note that condition (46) is equivalent to

$$\lambda_{\min}(B_n) \to \infty \qquad (n \to \infty),$$

where $\lambda_{\min}(B_n)$ is the minimal eigenvalue of the matrix B_n.

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