

RANDOM ATTRACTOR FOR THE REACTION-DIFFUSION EQUATION PERTURBED BY A STOCHASTIC CÀDLÀG PROCESS

UDC 519.21

O. V. KAPUSTYAN, J. VALERO, AND O. V. PEREGUDA

ABSTRACT. We study a stochastically perturbed reaction-diffusion equation by using the methods of the theory of stochastic attractors. It is proved that solutions of the equation form a multivalued random dynamic system for which there exists a random attractor in the phase space.

INTRODUCTION

The theory of random dynamic systems [1] provides convenient tools to study many random and stochastic equations. Evolution systems described by these equations have, as a rule, unbounded trajectories in ordinary time. The theory of random attractors (i.e., measurable invariant sets attracting trajectories in inverse time) is developed in the papers [5]–[7] to obtain a qualitative description of such systems and to study finite-dimensional systems of stochastically perturbed differential equations that have unique solutions.

A generalization of the theory of random attractors is proposed in the paper [8] for the case of infinite-dimensional systems that do not necessarily have unique solutions. However an extra assumption is made in [8] that the corresponding multivalued random dynamic system is dissipative with probability one.

In this paper, we develop the theory of random attractors for a system that is dissipative in probability. Based on this extension we study the qualitative behavior of solutions of the reaction-diffusion equation perturbed by a càdlàg process.

THE SETTING OF THE PROBLEM

Consider the problem

$$(1) \quad \begin{cases} \frac{\partial u(t,x)}{\partial t} = a\Delta u(t,x) - f(u(t,x)) + h(x) + g(u(t,x))\xi(t,w), \\ u|_{\partial Q} = 0, \quad u|_{t=0} = u_0(x), \end{cases}$$

where $a > 0$, $Q \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary, $f, g \in C(\mathbb{R})$, $h \in L_2(Q)$, $uf(u) \geq \alpha|u|^p - C$, $|f(u)| \leq C(1 + |u|^{p-1})$, $p \geq 2$, $\alpha > 0$, and

$$|g(u)| \leq C_1|u| + C_2.$$

We assume that $\xi(t, w)$ is a stochastic càdlàg process; that is, its trajectories have no discontinuities of the second kind. In view of the canonical representation of such processes

2000 *Mathematics Subject Classification.* Primary 34F05, 60H10.

described in [1, 2], we assume in what follows that $\xi(t, w): \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}$, where

$$\begin{aligned} \Omega &= D(\mathbb{R}) \\ &= \left\{ w(\cdot): \mathbb{R} \mapsto \mathbb{R} \mid \text{for all } t \in \mathbb{R} \text{ there exist } \lim_{s \rightarrow t-} w(s) =: w(t-) \right. \\ &\qquad \qquad \qquad \left. \text{and } \lim_{s \rightarrow t+} w(s) = w(t) \right\} \end{aligned}$$

is the Skorokhod metric space [2] with the Borel σ -algebra Φ , the transformation

$$\theta_t w(\cdot) = w(t + \cdot),$$

and with a θ_t -invariant probabilistic measure \mathbf{P} . In this case $\xi(t, w) = w(t) = \pi(\theta_t w)$, where $\pi: \Omega \mapsto \mathbb{R}$ and $\pi(w) = w(0)$. The metric ρ is defined by

$$\rho(w_1, w_2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\rho_i(w_1, w_2)}{1 + \rho_i(w_1, w_2)},$$

where

$$\rho_i(w_1, w_2) = \inf_{\lambda \in \Lambda} \left(\sup_{t \in [-i, i]} |w_1(t) - w_2(\lambda(t))| + \sup_{t \in [-i, i]} |t - \lambda(t)| \right)$$

and Λ is the set of continuous increasing functions

$$\lambda(\cdot): [-i, i] \rightarrow [-i, i]$$

such that $\lambda(-i) = -i$ and $\lambda(i) = i$.

Note that an arbitrary function $w(\cdot) \in \Omega$ is measurable, bounded on finite intervals, and has an at most countable set of points of discontinuity of the first kind. Moreover, if $w_n \rightarrow w_0$ in Ω , then

$$w_n(t) \rightarrow w_0(t)$$

almost everywhere on $[a, b]$ for arbitrary $[a, b] \subset \mathbb{R}$; and

$$\sup_{t \in [a, b]} |w_n(t)| \leq A + B \sup_{t \in [a, b]} |w_0(t)|,$$

where the nonnegative constants A and B do not depend on n . Therefore $w_n \rightarrow w_0$ in $L^q(a, b)$ for all $1 \leq q < \infty$ by the Lebesgue theorem.

It follows from [1] that $\{\theta_t: \Omega \mapsto \Omega\}_{t \in \mathbb{R}}$ is a metric dynamic system; that is, the mapping $(w, t) \mapsto \theta_t w$ is measurable, $\theta_0 w = w$, $\theta_{t+s} w = \theta_t \theta_s w$, and $\theta_t \mathbf{P} = \mathbf{P}$.

Our main aim is to prove that solutions of (1) form a (multivalued, generally speaking) random dynamic system whose phase space contains a minimal measurable attracting set called a random attractor.

EXISTENCE OF SOLUTIONS OF PROBLEM (1) AND PRIOR ESTIMATES

Let $H = L_2(Q)$ be equipped with a norm $\|\cdot\|$ and scalar product (\cdot, \cdot) . Applying Galerkin's approximations [3, 4] in a standard way we prove that, for all $w \in \Omega$, the class

$$W = L_p^{loc}(0, +\infty; L_p(Q)) \cap L_2^{loc}(0, +\infty; H_0^1(Q)) \cap C([0, +\infty); H)$$

contains at least one solution $u = u(t, w)u_0$ of problem (1) and, moreover,

$$\begin{aligned} (2) \quad & \frac{d}{dt} \|u(t)\|^2 + 2\alpha \|u(t)\|_{H_0^1(Q)}^2 + 2\alpha \|u(t)\|_{L_p(Q)}^p \\ & \leq 2C + 2(h, u(t)) + C_\gamma |w(t)| + (2C_1 + \gamma) |w(t)| \|u(t)\|^2 \end{aligned}$$

for any solution $u \in W$ of problem (1) and almost all $t > 0$, where $\gamma > 0$ is an arbitrary number and $C_\gamma = C_2^2|Q|/\gamma$. It follows from (2) that

$$(3) \quad \begin{aligned} \|u(t)\|^2 &\leq \|u(s)\|^2 + \left(2C + \frac{1}{a\lambda_1}\|h\|^2\right)(t-s) + C_\gamma \int_s^t |w(p)| dp \\ &\quad + \int_s^t ((2C_1 + \gamma)|w(p)| - \lambda_1 a) \|u(p)\|^2 dp \end{aligned}$$

for all $t \geq s \geq 0$, where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(Q)$. Using the Gronwall lemma we obtain

$$(4) \quad \begin{aligned} \|u(t)\|^2 &\leq \|u_0\|^2 \exp\left(\int_0^t ((2C_1 + \gamma)|w(p)| - \lambda_1 a) dp\right) \\ &\quad + \int_0^t \exp\left(\int_s^t ((2C_1 + \gamma)|w(p)| - a\lambda_1) dp\right) \\ &\quad \quad \quad \times \left(2C + \frac{1}{a\lambda_1}\|h\|^2 + C_\gamma|w(s)|\right) ds \end{aligned}$$

for all $t \geq 0$.

In what follows we need the following auxiliary result.

Lemma 1. *Let $\{u_n = u_n(t, w_n)u_n^0\} \subset W$ be an arbitrary sequence of solutions of problem (1), where $w_n \rightarrow w_0$ in Ω , $u_n^0 \rightarrow u_0$ weakly in H , and $t_n \rightarrow t_0 > 0$.*

Then

$$u_n(t_n, w_n)u_n^0 \rightarrow u(t_0, w_0)u_0$$

in H at least along some subsequence, where $u = u(t, w_0)u_0 \in W$ is a solution of problem (1).

Proof. Let $T > 0$. It follows from estimates (2) and (4) that the sequence $\{u_n\}$ is bounded in

$$L_p(0, T; L_p(Q)) \cap L_2(0, T; H_0^1(Q)) \cap L_\infty(0, T; H).$$

Since

$$H_0^s(Q) \subset L_p(Q) \cap H_0^1(Q)$$

for $s \geq \max\{1; (1/2 - 1/p)n\}$, we have $(H_0^1(Q))^* \subset (H_0^s(Q))^*$ and $L_q(Q) \subset (H_0^s(Q))^*$, where $q = p/(p-1)$. By the assumptions imposed on the function f ,

$$u_{n_i} \in L_2(0, T; (H_0^1(Q))^*) + L_q(0, T; L_q(Q));$$

thus the sequence $\{u_{n_i}\}$ is bounded in $L_q(0, T; (H_0^s(Q))^*)$. The lemma on compactness [3] implies that (at least along a subsequence) $u_n(\cdot, w_n)u_n^0 \rightarrow u(\cdot)$ in $L_2(0, T; H)$, $u_n(t, w_n)u_n^0 \rightarrow u(t)$ strongly in H for almost all $t \in (0, T)$ and weakly in H uniformly in $t \in [0, T]$.

Passing to the limit in (1) we prove that $u = u(t, w_0)u_0 \in W$ is a solution of (1). Let $K := 2C + \|h\|^2/(a\lambda_1)$ and

$$L_n(p) := (2C_1 + \gamma)|w_n(p)| - \lambda_1 a, \quad L_0(p) := (2C_1 + \gamma)|w_0(p)| - \lambda_1 a.$$

Then it follows from (3) that

$$J(t, w_0) \leq J(s, w_0), \quad J_n(t, w_n) \leq J_n(s, w_n)$$

for all $t \geq s$, $t, s \in [0, T]$, and $J_n(t, w_n) \rightarrow J(t, w_0)$ for almost all $t \in (0, T)$ where the functions

$$J_n(t, w_n) := \|u_n(t)\|^2 - Kt - C_\gamma \int_0^t |w_n(p)| dp - \int_0^t L_n(p) \|u_n(p)\|^2 dp,$$

$$J(t, w_0) := \|u(t)\|^2 - Kt - C_\gamma \int_0^t |w_0(p)| dp - \int_0^t L_0(p) \|u(p)\|^2 dp$$

belong to $C([0, T])$. Now it is easy to show that $J_n(t, w_n) \rightarrow J(t, w_0)$ uniformly on an arbitrary interval $[a, b] \subset (0, T)$.

The convergence $w_n \rightarrow w_0$ implies that $L_n(\cdot) \rightarrow L_0(\cdot)$ in $L^1(0, t)$ and that $L_n(\cdot)$, $n \geq 1$, are bounded in $L^\infty(0, t)$. Then

$$\int_0^t L_n(p) \|u_n(p)\|^2 dp \rightarrow \int_0^t L_0(p) \|u(p)\|^2 dp.$$

Thus

$$\begin{aligned} \liminf_{n \rightarrow \infty} J_n(t_n, w_n) &= J(t_0, w_0) \\ &\geq \liminf_{n \rightarrow \infty} \|u_n(t_n)\|^2 - Kt_0 - C_\gamma \int_0^{t_0} |w_0(p)| dp - \int_0^{t_0} L_0(p) \|u(p)\|^2 dp \end{aligned}$$

for all $t_n \rightarrow t_0 > 0$.

Hence

$$\|u(t_0)\| \geq \liminf_{n \rightarrow \infty} \|u_n(t_n)\|.$$

Since $u_n(t_n) \rightarrow u(t_0)$ weakly in H , the converse inequality also holds and therefore

$$\liminf_{n \rightarrow \infty} \|u_n(t_n)\| \leq \|u(t_0)\| \leq \liminf_{n \rightarrow \infty} \|u_n(t_n)\|.$$

Moreover $u_n(t_n)$ converges (at least along a subsequence) strongly in H to $u(t_0)$. The lemma is proved. \square

Consider the mapping $G: \mathbb{R}_+ \times \Omega \times H \mapsto 2^H$ defined by

$$(5) \quad G(t, w)u_0 := \{u(t, w)u_0 \mid u(t, w)u_0 \in W \text{ is a solution of (1), } u(0, w)u_0 = u_0\}.$$

Mapping (5) describes the dynamics of the solutions of problem (1), and our aim is to study the behavior of (5) as $t \rightarrow +\infty$. In doing so we apply some methods of the theory of random attractors of dynamic systems.

A GENERAL THEORY OF MULTIVALUED STOCHASTIC DYNAMIC SYSTEMS

Let $(X, \|\cdot\|)$ be a separable Banach space equipped with the Borel σ -algebra $\sigma(X)$ and let $C(X)$ ($\beta(X)$) be the family of all nonempty closed (nonempty and bounded) subsets of X . For $A, B \subset X$ we put $\|A\| := \sup_{a \in A} \|a\|$,

$$\text{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|,$$

$B_\delta(A) = \{x \in X \mid \text{dist}(x, A) < \delta\}$, and $B_r = \{x \in X \mid \|x\| \leq r\}$, where \bar{A} is the closure of A in X . Let $(\Omega, \Phi, \mathbb{P})$ be a probability space, $\{\theta_t: \Omega \mapsto \Omega\}_{t \in \mathbb{R}}$ be a metric dynamic system, and let $\bar{\Phi}$ be the Lebesgue completion of Φ with respect to the measure \mathbb{P} . The definitions and properties of multivalued mappings given below are taken from [9].

A mapping $F: \Omega \mapsto C(X)$ is called measurable if $\{w \in \Omega \mid F(w) \cap O \neq \emptyset\} \in \bar{\Phi}$ for any open set $O \subset X$. The following conditions are equivalent:

- (1) F is measurable;
- (2) $w \mapsto \text{dist}(x, F(w))$ is measurable for an arbitrary $x \in X$;
- (3) there exist measurable functions $\{f_n(w)\}_{n=1}^\infty$ such that $F(w) = \overline{\bigcup_{n=1}^\infty f_n(w)}$.

It is also known that if F is measurable, then $w \mapsto \|F(w)\|$ is $\overline{\Phi}$ -measurable, and if $G \in \overline{\Phi} \times \sigma(X)$, then $\pi_\Omega G := \{w \in \Omega \mid \text{there exists } x \in X, (w, x) \in G\} \in \overline{\Phi}$.

Definition 1. A multivalued mapping $G: \mathbb{R}_+ \times \Omega \times X \mapsto C(X)$ is called a multivalued random dynamic system if

- 1) the mapping $(t, w) \mapsto G(t, w)x$ is measurable for arbitrary $x \in X$;
- 2) $G(0, w)x = x$ and $G(t + s, w)x \subset G(t, \theta_s w)G(s, w)x$ for all $t, s \in \mathbb{R}_+$, $x \in X$, $w \in \Omega$.

Having in mind some specific applications, it is worth mentioning that an invariant set $\{\theta_t\}_{t \in \mathbb{R}}$ of full measure can replace the set Ω in the definition of a multivalued random dynamic system.

A measurable mapping $F: \Omega \mapsto C(X)$ is called a measurable set $F(w)$.

Definition 2. A measurable set $A(w)$ is called a random attractor for a multivalued random dynamic system G if for \mathbf{P} -almost all $w \in \Omega$,

- 1) $A(\theta_t w) \subset G(t, w)A(w)$ for all $t \in \mathbb{R}_+$ (semi-invariance);
- 2) for all $B \in \beta(X)$,

$$\text{dist}(G(t, \theta_{-t} w)B, A(w)) \rightarrow 0, \quad t \rightarrow +\infty$$

(attraction);

- 3) $A(w)$ is a compact set in X .

Below are some assumptions concerning the multivalued random dynamic system G .

- (G1) The mapping $(t, w) \mapsto \overline{G(t, w)B}$ is measurable for all $B \in \beta(X)$;
- (G2) for all $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that for all $B \in \beta(X)$ there is $T = T(B, R, \varepsilon)$ for which

$$\mathbf{P} \left\{ \sup_{t \geq T} \|G(t, \theta_{-t} w)B\| > R \right\} < \varepsilon.$$

Assumption (G1) implies the measurability of the mapping $(t, w) \mapsto \overline{G(t, \theta_{-t} w)B}$. Indeed, the set $K_1 = \{(t, \xi) \mid \text{dist}(x, G(t, \xi)B) < a\}$ is measurable for all $a \in \mathbb{R}$. Since the mapping $v: (t, w) \mapsto (t, \theta_{-t} w)$ is measurable, the set

$$K_2 = v^{-1}(K_1) = \{(t, w) \mid \xi = \theta_{-t} w, (t, \xi) \in K_1\}$$

is measurable too. We also have $K_2 = \{(t, w) \mid \text{dist}(x, G(t, \theta_{-t} w)B) < a\}$, whence the required statement follows.

On the other hand, the mapping $w \mapsto \text{dist}(x, \bigcup_{t \geq T} G(t, \theta_{-t} w)B)$ is $\overline{\Phi}$ -measurable, since

$$\text{dist}\left(x, \bigcup_{t \geq T} G(t, \theta_{-t} w)B\right) = \inf_{t \geq T} \text{dist}(x, G(t, \theta_{-t} w)B)$$

and

$$\left\{ w \mid \inf_{t \geq T} \text{dist}(x, G(t, \theta_{-t} w)B) < a \right\} = \pi_\Omega \left\{ (t, w) \mid \text{dist}(x, G(t, \theta_{-t} w)B) < a, t \geq T \right\}.$$

This implies that the mapping $w \mapsto \sup_{t \geq T} \|G(t, \theta_{-t} w)B\|$ is $\overline{\Phi}$ -measurable. Thus all terms in condition (G2) are well defined provided condition (G1) holds.

Condition (G2) can be checked for some classes of systems by using the following result.

Lemma 2. *If condition (G1) holds for a multivalued random dynamic system G and if there exists a bounded, measurable set $B(w)$ such that for \mathbf{P} -almost all $w \in \Omega$ and all $B \in \beta(X)$ there is $T = T(B, w)$ for which $G(t, \theta_{-t} w)B \subset B(w)$ for all $t \geq T$, then condition (G2) holds.*

Proof. Suppose condition (G2) does not hold; that is, there exists $\varepsilon^* > 0$ such that for all R , there exists $B \in \beta(X)$ for which

$$\mathbb{P} \left\{ \sup_{t \geq T} \|G(t, \theta_{-t}w)B\| \geq R \right\} \geq \varepsilon^* \quad \text{for all } T.$$

Since the mapping $w \mapsto \|B(w)\|$ is $\overline{\Phi}$ -measurable and \mathbb{P} -almost everywhere bounded,

$$\mathbb{P}\{\|B(w)\| > R\} \rightarrow 0, \quad R \rightarrow \infty.$$

Then there exists R such that $\mathbb{P}\{\|B(w)\| > R\} < \varepsilon^*/2$. Since

$$K_N := \left\{ w \mid \sup_{t \geq N} \|G(t, \theta_{-t}w)B\| > R \right\},$$

$\mathbb{P}\{K_N\} \geq \varepsilon^*$, and $K_{N+1} \subset K_N$ for all $N \geq 1$, we have $K = \bigcap_{N=1}^{\infty} K_N$,

$$\mathbb{P}\{K\} = \lim_{N \rightarrow \infty} \mathbb{P}\{K_N\} \geq \varepsilon^*,$$

and $K \subset \{w \mid \|B(w)\| > R\}$. This implies that $\mathbb{P}\{K\} < \varepsilon^*/2$, leading to a contradiction. The lemma is proved. \square

For a given $B \in \beta(X)$, consider the sets

$$\Lambda_B(w) := \bigcap_{T>0} \overline{\bigcup_{t \geq T} G(t, \theta_{-t}w)B}, \quad A(w) := \overline{\bigcup_{B \in \beta(X)} \Lambda_B(w)} = \overline{\bigcup_{M=1}^{\infty} \Lambda_{B_M}(w)}.$$

It follows from [8] that $\Lambda_B(w)$ consists of the limits of all convergent sequences $\{\xi_n\}$, where $\xi_n \in G(t_n, \theta_{-t_n}w)B$, $t_n \nearrow \infty$.

Lemma 3. *Let, for all $t \in \mathbb{R}_+$ and $w \in \Omega$, the mapping $x \mapsto G(t, w)x$ be upper semi-continuous and have compact values. Assume that conditions (G1) and (G2) hold for a multivalued random dynamic system G . Let, for all $w \in \Omega$, $t > 0$, and $R > 0$, the set $G(t, w)B_R$ be precompact in X (that is, let the closure of $G(t, w)B_R$ be compact). Then there exists a set $\Omega(B)$ such that $\mathbb{P}(\Omega(B)) = 1$ and, for all $w \in \Omega(B)$, the set $\Lambda_B(w) \neq \emptyset$ attracts B and is compact and semiinvariant; that is, $\Lambda_B(\theta_t w) \subset G(t, w)\Lambda_B(w)$ for all $t \in \mathbb{R}_+$.*

Remark 1. The assumption that the set $G(t, w)B_R$ is precompact follows from other assumptions of Lemma 3 if $\dim X < \infty$.

Proof. We have

$$G(t, \theta_{-t}w)B \subset G(1, \theta_{-1}w)G(t-1, \theta_{-(t-1)}\theta_{-1}w)B$$

for all $w \in \Omega$. Let $\Omega(N, R, B) := \{w \mid \sup_{t \geq N} \|G(t-1, \theta_{-(t-1)}\theta_{-1}w)B\| \leq R\}$. Then for all $\varepsilon > 0$, there exist numbers $R = R(\varepsilon)$ and $N = N(B, R, \varepsilon)$ such that

$$\mathbb{P}\{\Omega(N, R, B)\} \geq 1 - \varepsilon$$

and $\Omega(N, R, B) \subset \Omega(N+1, R, B)$. Thus

$$G(t, \theta_{-t}w)B \subset G(1, \theta_{-1}w)B_R$$

for all $w \in \Omega(N, R, B)$ and $t \geq N$. Then $\Lambda_B(w)$ is a nonempty compact set. Hence $\Lambda_B(w)$ is a nonempty compact set for all $w \in \Omega(R, B) := \bigcup_{N \geq N(B, R, \varepsilon)} \Omega(N, R, B)$ and moreover $\mathbb{P}\{\Omega(R, B)\} \geq 1 - \varepsilon$. Since $\Omega(R, B) \subset \Omega(R+1, B)$, one can choose $\varepsilon_i \rightarrow 0$ and $R_i = R(\varepsilon_i) \nearrow \infty$ such that the set $\Lambda_B(w)$ is nonempty and compact for all $w \in \Omega(B) := \bigcup_{i=1}^{\infty} \Omega(R_i, B)$ and moreover $\mathbb{P}\{\Omega(B)\} = 1$.

Now we prove the semiinvariance and attraction for $w \in \Omega(B)$. For all $w \in \Omega(B)$, there exist R and N such that $w \in \Omega(N, R, B)$. Thus

$$G(t_n + t, \theta_{-(t_n+t)}\theta_t w)B \subset G(t, w)G(t_n, \theta_{-t_n} w)B \subset G(t, w)\overline{G(1, \theta_{-1} w)B_R}$$

starting with some n , where $\{t_n\}$ is an arbitrary sequence such that $t_n \nearrow \infty$ and $t \in \mathbb{R}_+$. The latter set is compact, since $G(t, w)$ is upper semicontinuous and compact. Therefore $\Lambda_B(\theta_t w) \neq \emptyset$.

Further, for any $y \in \Lambda(\theta_t w)$ there exists a sequence $\{t_n\}$ such that $y = \lim y_n$ and $y_n \in G(t_n + t, \theta_{-(t_n+t)}\theta_t w)B$. Then there is $\eta_n \in G(t_n, \theta_{-t_n} w)B$ for which $y_n \in G(t, w)\eta_n$. Recall that the mapping $x \mapsto G(t, w)x$ is upper semicontinuous and has compact values. Thus the graph of the mapping $x \mapsto G(t, w)x$ is closed [9]. Since the set $\overline{G(1, \theta_{-1} w)B_R}$ is compact, the limit $\eta = \lim \eta_n \in \Lambda_B(w)$ exists, and $y \in G(t, w)\eta \subset G(t, w)\Lambda_B(w)$. This observation completes the proof of the semiinvariance.

Suppose $\Lambda_B(w)$ does not attract B for some $w \in \Omega(B)$. Then there exist a number $\delta > 0$ and a sequence $\{t_n\}$ such that $t_n \nearrow \infty$ and

$$\text{dist}(G(t_n, \theta_{-t_n} w)B, \Lambda_B(w)) \geq \delta > 0.$$

Hence the limit $\xi_n \in G(t_n, \theta_{-t_n} w)B$ exists and is such that $\text{dist}(\xi_n, \Lambda_B(w)) \geq \delta$. It follows from the above reasoning that the set $\{\xi_n\}$ is precompact, whence it follows that the limit $\xi = \lim \xi_n \in \Lambda_B(w)$ exists. This leads to a contradiction, and thus the lemma is proved. \square

Theorem 1. *Let a mapping $x \mapsto G(t, w)x$ be upper semicontinuous and compact-valued for all $t \in \mathbb{R}_+$ and $w \in \Omega$. Let conditions (G1) and (G2) hold for a multivalued random dynamic system G and let the set $G(t, w)B_R$ be precompact in X for all $w \in \Omega$, $t > 0$, and $R > 0$. Then the set*

$$A(w) = \overline{\bigcup_{n=1}^{\infty} \Lambda_{B_n}(w)}$$

is a random attractor for G . Therefore the attractor is unique, it is a minimal set among closed attracting sets, and it is a maximal set among compact, measurable, semiinvariant sets.

Proof. We prove that $A(w) = \overline{\bigcup_{n=1}^{\infty} \Lambda_{B_n}(w)}$ is a compact set for \mathbb{P} -almost all $w \in \Omega$. Given B_n , consider the set $\Omega(B_n)$ defined in Lemma 3. Since $\Omega(B_{n+1}) \subset \Omega(B_n)$,

$$\Omega_0 := \bigcap_{n=1}^{\infty} \Omega(B_n)$$

is a set of the full measure and $\Lambda_{B_n}(w)$ satisfies the assumptions of Lemma 3 for all $w \in \Omega_0$ and all $n \geq 1$. Consider the set

$$\Omega_0(R, B_n) := \Omega_0 \cap \Omega(R, B_n),$$

where $\Omega(R, B_n)$ is defined in Lemma 3. Then for all $\varepsilon > 0$ there exists $R = R(\varepsilon)$ such that

$$\mathbb{P}\{\Omega_0(R, B_n)\} \geq 1 - \varepsilon,$$

$\Omega_0(R, B_{n+1}) \subset \Omega_0(R, B_n)$ for all $n \geq 1$, and

$$\Lambda_{B_n}(w) \subset \overline{G(1, \theta_{-1} w)B_R}$$

for all $w \in \Omega_0(R, B_n)$. Thus $\mathbb{P}\{\Omega_0(R)\} \geq 1 - \varepsilon$ for $\Omega_0(R) = \bigcup_{n=1}^{\infty} \Omega_0(R, B_n)$ and $A(w) \subset \overline{G(1, \theta_{-1} w)B_R}$ for all $w \in \Omega_0(R)$. Hence $A(w)$ is a compact set. Choose $\varepsilon_i \rightarrow 0$

and $R_i = R(\varepsilon_i) \nearrow \infty$. Since $\Omega_0(R_i) \subset \Omega_0(R_{i+1})$, the set $A(w)$ is compact and $\mathbf{P}\{\Omega^0\} = 1$ for all

$$w \in \Omega^0 := \bigcup_{i=1}^{\infty} \Omega_0(R_i).$$

In what follows we restrict the consideration to subsets of the set Ω^0 . The attraction follows from Lemma 3. Furthermore,

$$A(\theta_t w) \subset \overline{\bigcup_{B \in \beta(X)} G(t, w) \Lambda_B(w)} \subset \overline{G(t, w) \bigcup_{B \in \beta(X)} \Lambda_B(w)} = \overline{G(t, w) A(w)} = G(t, w) A(w),$$

since $A(w)$ is a compact set and the mapping $x \mapsto G(t, w)x$ is upper semicontinuous [9]. The measurability of $A(w)$ with respect to the σ -algebra Φ follows from the $\overline{\Phi}$ -measurability of $A(w) = \bigcup_{n=1}^{\infty} \bigcap_{\tau_m \geq 0} \overline{\bigcup_{t \geq \tau_m} G(t, \theta_{-t} w) B_n}$. In the proof of the measurability we use the following well-known result [6]: for an arbitrary compact $\overline{\Phi}$ -measurable set $A(w)$, there exists a compact Φ -measurable set $\tilde{A}(w)$ that coincides with $A(w)$ almost everywhere with respect to \mathbf{P} .

To prove the uniqueness we follow the method of [8]. Let $E(w)$ be a semiinvariant, compact, and measurable set. Let $F = \{w \mid E(w) \subset B\}$ for a nonrandom set $B \in \beta(X)$. Then the Poincaré recurrence theorem implies that $\mathbf{P}\{F_{\infty}\} \geq \mathbf{P}\{F\}$ for

$$F_{\infty} = \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} \theta_n F = \{w \mid \theta_{-n} w \in F \text{ infinitely often}\}.$$

If

$$w \in F_{\infty},$$

then $E(\theta_{-n} w) \subset B$ for infinitely many $n \in \mathbb{N}$. Since $E(w)$ is semiinvariant, we have $E(w) \subset G(n, \theta_{-n} w) E(\theta_{-n} w) \subset G(n, \theta_{-n} w) B$; that is, $E(w) \subset \Lambda_B(w)$. Thus

$$\mathbf{P}\{E(w) \subset \Lambda_B(w)\} \geq \mathbf{P}\{E(w) \subset B\}.$$

On the other hand, it follows from [6] that, for an arbitrary compact measurable set $E(w)$ and for an arbitrary $\varepsilon > 0$, there exists a nonrandom compact set $K_{\varepsilon} \subset X$ such that

$$\mathbf{P}\{E(w) \subset K_{\varepsilon}\} > 1 - \varepsilon.$$

Hence $\mathbf{P}\{E(w) \subset A(w)\} \geq \mathbf{P}\{E(w) \subset \Lambda_{K_{\varepsilon}}(w)\} > 1 - \varepsilon$, whence

$$\mathbf{P}\{E(w) \subset A(w)\} = 1.$$

Therefore a random attractor is unique and is the maximal set among semiinvariant, compact, and measurable sets.

Now let $K(w)$ be a closed attracting set. We prove that

$$\Lambda_B(w) \subset K(w)$$

for all $B \in \beta(X)$ and $w \in \Omega^0$. Were this wrong we would find $w \in \Omega^0$, $y \in \Lambda_B(w)$, and $\delta > 0$ such that $\text{dist}(y, K(w)) > \delta$. At the same time, there exist $t_n \nearrow \infty$ and $y_n \in G(t_n, \theta_{-t_n} w) B$ such that $y = \lim y_n$. Thus there exists a number $n(w)$ such that $\text{dist}(y_n, K(w)) < \delta/2$ for all $n \geq n(w)$ and this leads to a contradiction. The theorem is proved. \square

Remark 2. The assumptions (G1) and (G2) are necessary to check only for all $B = B_r$, $r > 0$.

The following result allows one to check assumption (G1).

Lemma 4. *Let Ω be a metric space and Φ be a Borel σ -algebra. Assume that a multivalued mapping $G: R_+ \times \Omega \times X \mapsto C(X)$ satisfies the following condition: if $x_n \rightarrow x_0$ weakly in X as $t_n \rightarrow t_0 > 0$, $w_n \rightarrow w_0$ in Ω , and $y_n \in G(t_n, w_n)x_n$, then $y_n \rightarrow y_0 \in G(t_0, w_0)x_0$ in X for some subsequence.*

Then assumption (G1) holds for the mapping G .

Proof. According to Remark 2, one only needs to prove that the mapping

$$(t, w) \mapsto \overline{G(t, w)B_r}$$

is measurable. For an arbitrary closed set $C \subset X$ and for $n \geq 1$, consider the set

$$L_n := \left\{ (t, w) \mid t \geq \frac{1}{n}, \overline{G(t, w)B_r} \cap C \neq \emptyset \right\}.$$

Then L_n is closed and

$$L = \begin{cases} \bigcup_{n \geq 1} L_n & \text{if } B_r \cap C = \emptyset, \\ \bigcup_{n \geq 1} L_n \cup \{0 \times \Omega\} & \text{if } B_r \cap C \neq \emptyset, \end{cases}$$

where $L := \{(t, w) \mid \overline{G(t, w)B_r} \cap C \neq \emptyset\}$. Thus L is a Borel set, and therefore the lemma is proved. \square

APPLICATION TO PROBLEM (1)

The results obtained above can be applied to the family of mappings (5). Indeed, we prove that (5) generates a multivalued random dynamic system for which a random attractor exists. The main result of the paper reads as follows.

Theorem 2. *Let $\xi(t, w) = w(t)$ be a process and $\delta > 0$ be some number. If for all $\varepsilon > 0$ there exists $T > 0$ such that*

$$(6) \quad \mathbb{P} \left\{ \sup_{t \geq T} \frac{1}{t} \int_{-t}^0 |w(p)| dp \leq \frac{a\lambda_1}{2C_1 + \gamma} - \delta \right\} > 1 - \varepsilon,$$

and for all $\varepsilon > 0$ there exists D such that

$$(7) \quad \sup_{t \geq 0} \mathbb{P} \left\{ \int_{-t}^0 |w(s)| e^{\delta(2C_1 + \gamma)s} ds \leq D \right\} > 1 - \varepsilon,$$

where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ and C_1 is a constant such that

$$|g(u)| \leq C_1|u| + C_2, \quad C_1 > 0,$$

then the family of mappings (5) forms a multivalued random dynamic system for which a random attractor exists.

Proof. It easily follows from Lemmas 1 and 4 that the family of mappings (5) forms a multivalued random dynamic system (Definition 1 is checked analogously to the paper [8, Proposition 4]). Condition (G1) holds for this multivalued random dynamic system and the set $G(t, w)B_r$ is precompact. Moreover, Lemmas 1 and 4 also imply that the mapping $x \mapsto G(t, w)x$ is upper semicontinuous and has compact values. Thus it remains to check condition (G2), and then we are ready to apply Theorem 1.

Lemma 1 implies that, for all $T_1 > T_2 > 0$ and all $w \in \Omega$, there exist $t(w) \in [T_1, T_2]$ and $x_0(w) \in B_r$ such that

$$\sup_{t \in [T_1, T_2]} \|G(t, \theta_{-t}w)B_r\| = \|u(t(w), \theta_{-t(w)}w)x_0(w)\|.$$

Since $w \mapsto \sup_{t \in [T_1, T_2]} \|G(t, \theta_{-t}w)B_r\|$ is $\overline{\Phi}$ -measurable,

$$\|u(t(w), \theta_{-t(w)}w)x_0(w)\|$$

is $\overline{\Phi}$ -measurable too.

Fix $\varepsilon > 0$ and consider (4) for $t \in [T, T + N]$. Let

$$L^N := \left\{ w : \sup_{t \in [T, T+N]} \|G(t, \theta_{-t}w)B_r\|^2 > R^2 \right\} = \left\{ w : \|u(t(w), \theta_{-t(w)}w)x_0(w)\|^2 > R^2 \right\}.$$

Then estimate (4) yields

$$\begin{aligned} L^N \subset \left\{ w : r^2 \exp \left(\frac{1}{t(w)} \int_{-t(w)}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1 + \gamma} \right) (2C_1 + \gamma) t(w) \right. \\ \left. + \int_{-t(w)}^0 \exp \left(\left(\frac{1}{s} \int_s^0 |w(p)| dp + \frac{a\lambda_1}{2C_1 + \gamma} \right) (2C_1 + \gamma) s \right) \right. \\ \left. \times \left(2C + \frac{1}{a\lambda_1} \|h\|^2 + C_\gamma |w(s)| \right) ds > R^2 \right\}. \end{aligned}$$

Put

$$\begin{aligned} A_1 &:= \left\{ w : r^2 \exp \left(\frac{1}{t(w)} \int_{-t(w)}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1 + \gamma} \right) (2C_1 + \gamma) t(w) \geq 1 \right\} \\ &\subset \left\{ w : \left(\frac{1}{t(w)} \int_{-t(w)}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1 + \gamma} \right) \geq \frac{1}{(2C_1 + \gamma) t(w)} \ln \left(\frac{1}{r^2} \right) \right\}. \end{aligned}$$

Now we pick up a number $T = T(r)$ such that

$$\frac{1}{(2C_1 + \gamma)T} \ln \left(\frac{1}{r^2} \right) > -\delta.$$

Then

$$\begin{aligned} A_1 &\subset \left\{ w : \frac{1}{t(w)} \int_{-t(w)}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1 + \gamma} > -\delta \right\} \\ &\subset \left\{ w : \sup_{t \geq T} \frac{1}{t} \int_{-t}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1 + \gamma} > -\delta \right\}. \end{aligned}$$

Condition (6) implies that there exist a number $T_1 = T_1(\varepsilon)$ and a random event $A_2 \subset \Omega$ such that $\mathbf{P}\{A_2\} < \frac{\varepsilon}{4}$ and

$$\sup_{t \geq T_1} \frac{1}{t} \int_{-t}^0 |w(p)| dp - \frac{a\lambda_1}{2C_1 + \gamma} \leq -\delta$$

for all $w \in \Omega \setminus A_2$. Thus there exists $T_2 = T_2(\varepsilon, r) \geq T_1 + T(r)$ such that

$$\mathbf{P}\{A_1\} < \frac{\varepsilon}{4}$$

for all $t(w) \in [T_2, T_2 + N]$.

Then

$$\begin{aligned}
L^N &\subset A_1 \cup \left\{ w: \int_{-t(w)}^0 \exp \left(\left(\frac{1}{s} \int_s^0 |w(p)| dp + \frac{a\lambda_1}{2C_1 + \gamma} \right) (2C_1 + \gamma)s \right) \right. \\
&\quad \left. \times \left(2C + \frac{1}{a\lambda_1} \|h\|^2 + C_\gamma |w(s)| \right) ds > R^2 - 1 \right\} \\
&\subset A_1 \cup A_2 \cup \left\{ w: \int_{-T_1}^0 \left(2C + \frac{1}{a\lambda_1} \|h\|^2 + C_\gamma |w(s)| \right) \right. \\
&\quad \times \exp \left\{ \left(\frac{1}{s} \int_s^0 |w(p)| dp + \frac{a\lambda_1}{2C_1 + \gamma} \right) (2C_1 + \gamma)s \right\} ds \\
&\quad \left. + \int_{-T_2-N}^0 \left(2C + \frac{1}{a\lambda_1} \|h\|^2 + C_\gamma |w(s)| \right) e^{\delta(2C_1 + \gamma)s} ds > R^2 - 1 \right\} \\
&= A_1 \cup A_2 \cup \left\{ w: f_\varepsilon(w) + \int_{-T_2-N}^0 |w(s)| e^{\delta(2C_1 + \gamma)s} ds > \frac{R^2 - A}{B} \right\},
\end{aligned}$$

where $A > 0$ and $B > 0$ are some constants, and $f_\varepsilon: \Omega \mapsto \mathbb{R}$ is a measurable and \mathbb{P} -almost everywhere bounded function. Hence there exist a number $R_1 = R_1(\varepsilon)$ and a random event $A_3 \subset \Omega$ such that $f_\varepsilon(w) > R_1$ and $\mathbb{P}\{A_3\} < \frac{\varepsilon}{4}$ for all $w \in A_3$. Thus

$$L^N \subset A_1 \cup A_2 \cup A_3 \cup \left\{ w: \int_{-T-N}^0 |w(s)| e^{\delta(2C_1 + \gamma)s} ds > \frac{R^2 - A}{B} - R_1(\varepsilon) \right\}.$$

Condition (7) implies that there exists $D = D(\varepsilon)$ such that

$$\mathbb{P} \left\{ w: \int_{-t}^0 |w(s)| e^{\delta(2C_1 + \gamma)s} ds > D \right\} < \frac{\varepsilon}{4}$$

for all $t > 0$.

Choose $R = R(\varepsilon)$ such that

$$\frac{R^2 - A}{B} - R_1(\varepsilon) > D.$$

Then, for all $\varepsilon > 0$ one can find $R = R(\varepsilon)$ for which, whatever B_r is, there exists $T = T(\varepsilon, R, r)$ such that

$$\mathbb{P}\{L^N\} = \mathbb{P} \left\{ w: \sup_{t \in [T, T+N]} \|G(t, \theta_{-t}w)B_r\|^2 > R^2 \right\} < \varepsilon$$

for all $N \geq 1$. Since $L^N \subset L^{N+1}$, we have $\mathbb{P}\{L\} < \varepsilon$ and

$$\left\{ w: \sup_{t \geq T} \|G(t, \theta_{-t}w)B_r\|^2 > R^2 \right\} = L,$$

where $L := \bigcup_{N=1}^{\infty} L^N$. Therefore condition (G2) holds and the theorem is proved. \square

Theorem 2 holds under weaker restrictions imposed on the function g if one assumes stronger conditions concerning the process $\xi(t, w)$. Namely, let the function g be such that

$$(8) \quad |g(u)| \leq C_1 |u|^r + C_2$$

for $r < p - 1$. In this case condition (7) must be replaced with the following: for all $\varepsilon > 0$ there exists $D > 0$ such that

$$(9) \quad \sup_{t \geq 0} \mathbb{P} \left\{ \int_{-t}^0 |w(p)|^{p/(p-r-1)} e^{a\lambda_1 s} ds \leq D \right\} > 1 - \varepsilon.$$

Theorem 3. *Suppose all the assumptions of Theorem 2 hold and additionally the function g satisfies condition (8), while the process $\xi(t, w) = w(t)$ satisfies conditions (6) and (9). Then the family of mappings (5) forms a multivalued random dynamic system for which a random attractor exists.*

Proof. Multiplying both sides of (1) by u we get

$$\begin{aligned} \frac{d \|u(t)\|^2}{dt} &\leq -\lambda_1 a \|u(t)\|^2 - \alpha \|u(t)\|_{L_p}^p + 2C_1 |w(t)| \|u(t)\|_{L_{r+1}}^{r+1} + 2C_2 |w(t)| \cdot \|u(t)\|_{L_1} \\ &\quad + 2C + \frac{\|h\|^2}{a\lambda_1} \\ &\leq -\lambda_1 a \|u(t)\|^2 + C_3 |w(t)|^{p/(p-r-1)} + C_4. \end{aligned}$$

Then inequalities (3) and (4) can be rewritten as follows:

$$(10) \quad \|u(t)\|^2 \leq \|u(s)\|^2 + C_3 \int_s^t |w(s)|^{p/(p-r-1)} ds + C_4 (t - s),$$

$$(11) \quad \|u(t)\|^2 \leq e^{-\lambda_1 a t} \|u_0\|^2 + \int_0^t e^{-\lambda_1 a (t-s)} (C_3 |w(s)|^{p/(p-r-1)} + C_4) ds. \quad \square$$

The proof of Lemma 1 is the same in this case, since

$$J(t, w) = \|u(t)\|^2 - C_4 t - C_3 \int_0^t |w(s)|^{p/(p-r-1)} ds.$$

Condition (G2) is checked similarly to the proof of the preceding theorem by observing that

$$\|y\|^2 \leq e^{-\lambda_1 a t} r^2 + \int_{-t}^0 e^{\lambda_1 a p} (C_3 |w(p)|^{p/(p-r-1)} + C_4) dp$$

for all B_r and $y \in G(t, \theta_{-t} w) B_r$. □

CONCLUDING REMARKS

In this paper, we proved that a random attractor exists and is unique for a multivalued random dynamic system that is dissipative in probability. Using this result, we proved that solutions of the reaction-diffusion equation perturbed by a stochastic càdlàg process form a multivalued random dynamic system for which a random attractor exists in the phase space.

BIBLIOGRAPHY

1. L. Arnold, *Random Dynamical Systems*, Springer, Berlin, 1998. MR1723992 (2000m:37087)
2. I. I. Gikhman and A. V. Skorokhod, *Introduction to the Theory of Random Processes*, “Nauka”, Moscow, 1977; English transl. of the first edition, Scripta Technica, Inc. W. B. Saunders Co., Philadelphia, Pa.–London–Toronto, Ont., 1969. MR0488196 (58:7758)
3. A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, “Nauka”, Moscow, 1989; English transl., North-Holland Publishing Co., Amsterdam, 1992. MR1007829 (92f:58101)
4. A. V. Kapustyan, *Global attractors of a nonautonomous reaction-diffusion equation*, Differ. Uravn. **38** (2002), no. 10, 1378–1382; English transl. in Differ. Equ. **38** (2002), no. 10, 1467–1471. MR1984456 (2004g:35028)
5. H. Crauel and F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields **100** (1994), 365–393. MR1305587 (95k:58092)

6. H. Crauel, *Global random attractors are uniquely determined by attracting deterministic compact sets*, Ann. Mat. Pura Appl. **126** (1999), no. 4, 57–72. MR1746535 (2000m:37090)
7. K. R. Schenk-Hoppe, *Random attractors—general properties, existence and applications to stochastic bifurcation theory*, Discrete Contin. Dyn. Syst. **4** (1998), no. 1, 99–130. MR1485366 (98h:34103)
8. T. Caraballo, J. A. Langa, and J. Valero, *Global attractors for multivalued random dynamical systems*, Nonlinear Anal. **48** (2002), 805–829. MR1878338 (2003a:37068)
9. J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990. MR1048347 (91d:49001)

DEPARTMENT OF INTEGRAL AND DIFFERENTIAL EQUATIONS, FACULTY FOR MATHEMATICS AND MECHANICS, NATIONAL TARAS SHEVCHENKO UNIVERSITY, ACADEMICIAN GLUSHKOV AVENUE 6, KYIV 03127, UKRAINE

E-mail address: alexkap@univ.kiev.ua

UNIVERSIDAD MIGUEL HERNANDEZ, CENTRO DE INVESTIGATION OPERATIVA, AVDA. DEL FERROCARRIL S/N 03202 ELCHE (ALICANTE), SPAIN

E-mail address: jvalero@umh.es

DEPARTMENT OF INTEGRAL AND DIFFERENTIAL EQUATIONS, FACULTY FOR MATHEMATICS AND MECHANICS, NATIONAL TARAS SHEVCHENKO UNIVERSITY, ACADEMICIAN GLUSHKOV AVENUE 6, KYIV 03127, UKRAINE

E-mail address: perol@ua.fm

Received 24/AUG/2004

Translated by S. KVASKO