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ESTIMATES FOR THE DISTRIBUTION OF THE SUPREMUM OF SQUARE-GAUSSIAN STOCHASTIC PROCESSES DEFINED ON NONCOMPACT SETS

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Abstract. Estimates for the distribution of the supremum of square-Gaussian stochastic processes defined on $\mathbb{R}^+$ are found in the paper. Using these results, we find estimates for the deviation in the uniform metric between the correlogram and the correlation function of a real stationary Gaussian stochastic process. A criterion for testing a hypothesis concerning the correlation function is also constructed.

1. Introduction

Throughout the paper we use properties of square-Gaussian stochastic processes. Square-Gaussian processes have been studied by many authors. For example, Kozachenko and Oleshko [1] obtained some results concerning the distribution of the supremum of such processes. Using the metric entropy, Kozachenko and Stadnik [2] found estimates for the distribution of the supremum for a wide class of stochastic processes including square-Gaussian processes. Similar estimates are obtained by Kurchenko [3] and Ponomarenko [4].

In the current paper we consider the problem of estimating the distribution of the supremum of square-Gaussian stochastic processes defined on noncompact sets. Using these results, we find estimates for the distribution of the deviation in the uniform metric in $(0, +\infty)$ between the correlogram and the correlation function for a real-valued Gaussian stationary stochastic process and construct a criterion for testing a hypothesis concerning the correlation function of the process on the interval (a, b).

2. Estimates for the distribution of the supremum of square-Gaussian stochastic processes defined on a separable metric space

Let $\{\Omega, \mathcal{B}, \mathsf{P}\}$ be a standard probability space.

Definition 2.1 ([5])**.** Let T be some set of parameters and $\Xi = \{\xi_t, t \in T\}$ a family of jointly Gaussian random variables with $\mathsf{E}\,\xi_t = 0$ (for example, ξ_t can be a Gaussian stochastic process). The family $SG_\Xi(\Omega)$ of random variables $\zeta \in SG_\Xi(\Omega)$ that are either of the form

$$\zeta = \overline{\xi}^{\,T} A \overline{\xi} - \mathsf{E}\,\overline{\xi}^{\,T} A \overline{\xi} \tag{1}$$

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or are mean square limits of sequences of random variables ζ_n of the form (1):

$$\zeta_n = \overline{\xi}_n^T A_n \overline{\xi}_n - \mathsf{E}\,\overline{\xi}_n^T A_n \overline{\xi}_n, \qquad n \ge 1,$$

is called the space of square-Gaussian random variables, where $\overline{\xi} = (\xi_1, \dots, \xi_N)^T$ is a Gaussian random vector for all $N \ge 1$, $\mathsf{E}\,\overline{\xi} = 0$, the random variables ξ_i, $i = 1, \dots, N$, belong to Ξ, and A is a symmetric matrix.

Definition 2.2 ([5])**.** A stochastic process $\zeta = \{\zeta(t), t \in T\}$ is called square-Gaussian with respect to Ξ if the random variable $\zeta(t)$, $t \in T$, belongs to the space $SG_\Xi(\Omega)$ and $\sup_{t\in T} \mathsf{E}\,\zeta^2(t) < \infty$.

Let $(\mathbf{T}, m)$ be a compact metric space equipped with the metric m and let $X = \{X(t), t \in \mathbf{T}\}$ be a separable square-Gaussian stochastic process.

Assume that there exists a continuous increasing function $\sigma = \{\sigma(h), h > 0\}$ such that $\sigma(h) \to 0$ as $h \to 0$ and

$$\sup_{m(t,s)<h, t,s\in\mathbf{T}} \left(\mathsf{E}(X(t) - X(s))^2\right)^{1/2} \le \sigma(h).$$

Remark 2.1. If a process $X(t)$ is continuous in the norm of the space L_2, then the function

$$\sigma(h) = \sup_{m(t,s)<h, t,s\in\mathbf{T}} \left(\mathsf{E}(X(t) - X(s))^2\right)^{1/2}$$

satisfies the above property if σ is continuous and increasing.

In what follows we use the following notation:

$$\varepsilon_0 = \inf_{t\in\mathbf{T}} \sup_{s\in\mathbf{T}} m(t,s),$$
$$\delta_0 = \sup_{t\in\mathbf{T}} \left(\mathsf{E}\,|X(t)|^2\right)^{1/2},$$

$\sigma^{(-1)}(h)$ is the inverse function to $\sigma(u)$, $t_0 = \sigma(\varepsilon_0)$, $N(\varepsilon)$ is the minimal number of closed balls of radius ε that cover $(\mathbf{T}, m)$, and let $r(u) > 0$, $u \ge 1$, be an increasing function such that $r(u) \to \infty$ as $u \to \infty$ and $r(e^t)$ is convex for $t \ge 0$.

Lemma 4.1 in [6] implies the following result.

Theorem 2.1. *If*

$$\int_0^{t_0} r\left(N\big(\sigma^{(-1)}(v)\big)\right) dv < \infty,$$

then

$$\mathsf{E}\exp\left\{u \sup_{t\in\mathbf{T}} |X(t)|\right\} \le 2\left(R\left(\frac{u\sqrt{2}\delta_0}{1-p}\right)\right)^{1-p} \left(R\left(\frac{u\sqrt{2}t_0}{1-p}\right)\right)^p \times r^{(-1)}\left(\frac{1}{t_0 p}\int_0^{t_0 p} r\left(N\big(\sigma^{(-1)}(v)\big)\right) dv\right) \tag{2}$$

for all p and u such that $0 < p < 1$ *and*

$$0 < u < \frac{1-p}{\sqrt{2}} \min\left\{\frac{1}{\delta_0}, \frac{1}{t_0}\right\},$$

where

$$R(z) = (1-z)^{-1/2} \exp\left\{-\frac{z}{2}\right\}, \qquad 0 \le z < 1. \tag{3}$$

Corollary 2.1. *Let the assumptions of Theorem* 2.1 *hold and* $z_0 = \max(\delta_0, t_0)$. *Then*

$$\mathsf{E}\exp\left\{u\sup_{t\in\mathbf{T}}|X(t)|\right\} \le 2R\left(\frac{u\sqrt{2}z_0}{1-p}\right) r^{(-1)}\left(\frac{1}{t_0 p}\int_0^{t_0 p} r\left(N\big(\sigma^{(-1)}(v)\big)\right) dv\right) \tag{4}$$

for

$$0 < u < \frac{1-p}{z_0\sqrt{2}}.$$

Proof. Inequality (4) follows from Theorem 2.1 since the function $R(z)$ increases for $0 < z < 1$. □

Let $(\mathbf{T}, m)$ be a separable finite-dimensional metric space.

Assume that the space $(\mathbf{T}, m)$ can be represented as a countable union of compact sets B_k, $k = 1, 2, \dots$, that is,

$$\mathbf{T} = \bigcup_{k=1}^{\infty} B_k.$$

Consider a separable square-Gaussian stochastic process $X = \{X(t), t \in \mathbf{T}\}$. Assume that there are increasing functions $\sigma_k = \{\sigma_k(h), h > 0\}$ such that $\sigma_k(h) \to 0$ as $h \to 0$ and

$$\sup_{m(t,s)<h, t,s\in B_k} \big(\mathsf{E}(X(t) - X(s))^2\big)^{1/2} \le \sigma_k(h).$$

Let

$$\varepsilon_{0k} = \inf_{t\in B_k}\sup_{s\in B_k} m(t,s),$$

$$\delta_{0k} = \sup_{t\in B_k}\big(\mathsf{E}\,|X(t)|^2\big)^{1/2},$$

$\sigma_k^{(-1)}$ be the inverse functions to σ_k, $t_{0k} = \sigma_k(\varepsilon_{0k})$, $z_{0k} = \max(\delta_{0k}, t_{0k})$. Let $N_k(u)$ be the minimal number of closed balls of radius u that cover B_k, and let $r(u) > 0$, $u \ge 1$, be an increasing function such that $r(u) \to \infty$ as $u \to \infty$ and $r(e^t)$ is convex for $t \ge 0$.

Theorem 2.1 implies the following result.

Theorem 2.2. *If*

$$\int_0^{t_{0k}} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv < \infty$$

for all k, *then*

$$\begin{aligned}\mathsf{E}\exp\left\{u\sup_{t\in B_k}|X(t)|\right\} &\le 2\left(R\left(\frac{u\sqrt{2}\delta_{0k}}{1-p}\right)\right)^{1-p}\left(R\left(\frac{u\sqrt{2}t_{0k}}{1-p}\right)\right)^{p}\\ &\quad\times r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv\right)\end{aligned} \tag{5}$$

for all p *and* u *such that* $0 < p < 1$ *and*

$$0 < u < \frac{1-p}{\sqrt{2}}\min\left\{\frac{1}{\delta_{0k}}, \frac{1}{t_{0k}}\right\},$$

where $R(z)$ *is defined in* (3) *for* $0 \le z < 1$.

Theorem 2.3. *Let* $c(t)$, $t \in \mathbf{T}$, *be a continuous function such that* $c(t) > 0$ *for all* t. *Put*

$$\gamma_k = \sup_{t\in B_k}|c(t)|.$$

If

(1) $d = \sum_{k=1}^{\infty} \gamma_k z_{0k} < \infty$,
(2) $\int_0^{pt_{0k}} r\big(N_k\big(\sigma_k^{(-1)}(v)\big)\big)\,dv < \infty$,
(3) *for some* $0 < p < 1$,

$$\prod_{k=1}^{\infty} \left(r^{(-1)} \left(\frac{1}{pt_{0k}} \int_0^{pt_{0k}} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv \right)\right)^{\gamma_k z_{0k}/d} < \infty,$$

then

$$\begin{aligned} &\mathsf{E}\exp\left\{ u \sup_{t\in\mathbf{T}} |c(t)X(t)| \right\} \\ &\quad \le 2R\left(\frac{ud\sqrt{2}}{1-p}\right) \prod_{k=1}^{\infty} \left(r^{(-1)} \left(\frac{1}{t_{0k}p} \int_0^{t_{0k}p} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv \right)\right)^{\gamma_k z_{0k}/d} \end{aligned} \tag{6}$$

for all

$$0 < u < \frac{1-p}{d\sqrt{2}}.$$

Proof. It is obvious that

$$\sup_{t\in\mathbf{T}} |c(t)X(t)| \le \sup_k \sup_{t\in B_k} |c(t)|\cdot|X(t)| = \sum_{k=1}^{\infty} \gamma_k \sup_{t\in B_k} |X(t)|.$$

Then

$$\mathsf{E}\exp\left\{ u \sup_{t\in\mathbf{T}} |c(t)X(t)| \right\} \le \mathsf{E}\exp\left\{ u \sum_{k=1}^{\infty} \gamma_k \sup_{t\in B_k} |X(t)| \right\} \tag{7}$$

for $u > 0$. Let $z_{0k} = \max(\delta_{0k}, t_{0k})$. Since the function $R(z)$ increases for $0 < z < 1$, we have

$$\begin{aligned} &\mathsf{E}\exp\left\{ u \sup_{t\in B_k} |X(t)| \right\} \\ &\quad \le 2R\left(\frac{u\sqrt{2}z_{0k}}{1-p}\right) r^{(-1)} \left(\frac{1}{t_{0k}p} \int_0^{t_{0k}p} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv \right) \end{aligned} \tag{8}$$

for all k and

$$0 < u < \frac{1-p}{\sqrt{2}} \frac{1}{z_{0k}}$$

by Theorem 2.2 and Corollary 2.1.

Let $\{q_k\}$ be a sequence of real numbers such that $q_k > 1$ for all $k \ge 1$ and

$$\sum_{k=1}^{\infty} q_k^{-1} = 1.$$

Bounds (7) and (8) together with the Hölder inequality imply that

$$\begin{aligned}
\mathsf{E}\exp\left\{u\sup_{t\in\mathbf{T}}|c(t)X(t)|\right\} &\le \mathsf{E}\prod_{k=1}^{\infty}\exp\left\{u\gamma_k\sup_{t\in B_k}|X(t)|\right\} \\
&\le \prod_{k=1}^{\infty}\left(\mathsf{E}\exp\left\{u\gamma_k q_k\sup_{t\in B_k}|X(t)|\right\}\right)^{1/q_k} \\
&\le \prod_{k=1}^{\infty}\left(2R\left(\frac{u\gamma_k q_k\sqrt{2}z_{0k}}{1-p}\right)r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p} r\left(N_k\left(\sigma_k^{(-1)}(v)\right)\right)\,dv\right)\right)^{1/q_k} \\
&= \prod_{k=1}^{\infty}2^{1/q_k}\prod_{k=1}^{\infty}\left(R\left(\frac{u\gamma_k q_k\sqrt{2}z_{0k}}{1-p}\right)\right)^{1/q_k} \\
&\quad\times\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p} r\left(N_k\left(\sigma_k^{(-1)}(v)\right)\right)\,dv\right)\right)^{1/q_k}
\end{aligned}$$

for $u>0$ such that $0<u\gamma_k q_k<(1-p)/(\sqrt{2}z_{0k})$, $k=1,2,\dots$. The latter bound holds for any u such that

$$0<u<\frac{1-p}{\sqrt{2}}\frac{1}{z_{0k}\gamma_k q_k}$$

for all $k=1,2,\dots$. Put

$$d=\sum_{k=1}^{\infty}\gamma_k z_{0k}$$

and choose

$$q_k=\frac{d}{\gamma_k z_{0k}}\quad\text{such that}\quad q_k>1\quad\text{and}\quad\sum_{k=1}^{\infty}q_k^{-1}=1.$$

Then

$$R\left(u\frac{q_k\gamma_k z_{0k}\sqrt{2}}{1-p}\right)=R\left(u\frac{d\sqrt{2}}{1-p}\right)$$

and

$$\prod_{k=1}^{\infty}\left(R\left(u\frac{q_k\gamma_k z_{0k}\sqrt{2}}{1-p}\right)\right)^{1/q_k}=\left(R\left(u\frac{d\sqrt{2}}{1-p}\right)\right)^{\sum_{k=1}^{\infty}1/q_k}=R\left(u\frac{d\sqrt{2}}{1-p}\right)$$

for any u such that

$$0<u<\frac{1-p}{d\sqrt{2}}.$$

Thus

$$\begin{aligned}
&\mathsf{E}\exp\left\{u\sup_{t\in\mathbf{T}}|c(t)X(t)|\right\} \\
&\qquad\le 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p} r\left(N_k\left(\sigma_k^{(-1)}(v)\right)\right)\,dv\right)\right)^{\gamma_k z_{0k}/d}
\end{aligned}$$

for

$$0<u<\frac{1-p}{d\sqrt{2}}.\qquad\square$$

Theorem 2.4. *If the assumptions of Theorem* 2.3 *hold, then*

$$\mathsf{P}\left\{\sup_{t\in\mathbf{T}}|c(t)X(t)|>x\right\}\le 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1+\frac{\sqrt{2}x(1-p)}{d}\right)^{1/2}\widetilde{\Phi}(p)$$

for arbitrary $x>0$ and for the same $0<p<1$ as in Theorem 2.3, where

$$\widetilde{\Phi}(p)=\prod_{k=1}^{\infty}r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p}r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right)dv\right)^{\gamma_k z_{0k}/d}.$$

Proof. We obtain from Theorem 2.3 and the Chebyshev inequality that

$$\begin{aligned}\mathsf{P}\left\{\sup_{t\in\mathbf{T}}|c(t)X(t)|>x\right\}&\le\frac{\mathsf{E}\exp\{u\sup_{t\in T}|c(t)X(t)|\}}{\exp\{ux\}}\le 2R\left(\frac{ud\sqrt{2}}{(1-p)}\right)\exp\{-ux\}\widetilde{\Phi}(p)\\&=2\left(1-\frac{ud\sqrt{2}}{(1-p)}\right)^{-1/2}\exp\left\{-\frac{ud\sqrt{2}}{2(1-p)}\right\}\exp\{-ux\}\widetilde{\Phi}(p)\end{aligned}$$

for all $x>0$ and

$$0<u<\frac{1-p}{d\sqrt{2}}.$$

Put

$$D=\frac{d\sqrt{2}}{1-p}.$$

Then

$$\mathsf{P}\left\{\sup_{t\in T}|c(t)X(t)|>x\right\}\le Z(u,x)\widetilde{\Phi}(p)$$

for $0<u<D^{-1}$, where

$$Z(u,x)=2(1-uD)^{-1/2}\exp\left\{-\frac{u}{2}(D+2x)\right\}.$$

It is easy to check that this function attains its minimum in the interval $0<u<D^{-1}$ at the point

$$u=\frac{1}{D}-\frac{1}{D+2x}<\frac{1}{D}.$$

Therefore

$$\inf_{0<u<D^{-1}}Z(u,x)=2\exp\left\{-\frac{x}{D}\right\}\left(\frac{D}{D+2x}\right)^{-1/2}.$$

This relation completes the proof of the theorem. □

3. Estimates for the uniform deviation between the correlogram and correlation function of a Gaussian stationary stochastic process

Let $\xi=\{\xi(t),t\ge 0\}$ be a real-valued, mean-square continuous, stationary, Gaussian stochastic process with $\mathsf{E}\,\xi(t)=0$ and the correlation function $\rho(\tau)=\mathsf{E}\,\xi(t+\tau)\xi(t)$. To estimate $\rho(\tau)$ we use the correlogram

$$\widehat{\rho}_T(\tau)=\frac{1}{T}\int_0^T\xi(t+\tau)\xi(t)\,dt.$$

This estimator is unbiased:

$$\mathsf{E}\,\widehat{\rho}_T(\tau)=\rho(\tau).$$

Assume that the spectral density of the stochastic process $\xi(t)$ exists and denote it by

$$f=\{f(\lambda),\lambda\in\mathbb{R}\}.$$

Assume further that $f \in L_2(\mathbb{R})$. In other words, f is square integrable, that is,

$$\int_{-\infty}^{+\infty} f^2(\lambda)\, d\lambda < \infty.$$

This implies that the correlation function $\rho(\tau)$ of the stochastic process ξ is also square-integrable, that is,

$$\|\rho\|_2^2 = \int_{-\infty}^{+\infty} \rho^2(\tau)\, d\tau < \infty.$$

Let

$$X(T,\tau) = \widehat{\rho}_T(\tau) - \rho(\tau).$$

Note that $X(T,\tau)$ is a square-Gaussian stochastic process. We estimate the moments $\mathsf{E}(X(T,\tau))^2$ and

$$\mathsf{E}\big(X(T,\tau) - X(T_1,\tau_1)\big)^2.$$

Define the space $(\mathbf{T}, m)$ as follows:

$$\mathbf{T} = \big\{(T,\tau)\colon A < T < \infty, a \le \tau \le b, 0 \le a < b, A > 0\big\},$$
$$m\big((T',\tau'),(T'',\tau'')\big) = \max\big\{|T'-T''|, |\tau'-\tau''|\big\}.$$

Lemma 3.1. *Assume that*

$$\int_{-\infty}^{+\infty} f^2(\lambda)\, d\lambda < \infty. \tag{9}$$

Then

$$\sup_{(T,\tau)\in \boldsymbol{T}} \mathsf{E}(X(T,\tau))^2 = \frac{C_1}{T},$$

where

$$C_1 = \left(1+\sqrt{2}\right)\|\rho\|_2^2.$$

Proof. Since

$$\int_0^T\int_0^T f(t-s)\, dt\, ds = 2\int_0^T (T-u) f(u)\, du$$

for every even function f, we obtain from the Isserlis formula [7] that

$$\begin{aligned}\mathsf{E}(X(T,\tau))^2 &= \mathsf{E}\,\widehat{\rho}^{\,2}(\tau) - (\mathsf{E}\,\widehat{\rho}(\tau))^2 \\ &= \frac{2}{T^2}\int_0^T (T-u)\big(\rho^2(u) + \rho(u-\tau)\rho(u+\tau)\big)\, du.\end{aligned}$$

It follows from $\rho \in L_2(\mathbb{R})$ that

$$\int_0^{+\infty} \rho^2(u)\, du = \frac{1}{2}\int_{-\infty}^{+\infty} \rho^2(u)\, du < \infty$$

as $\rho(\tau)$ is even. Then

$$\int_0^{+\infty} \rho(u-\tau)\rho(u+\tau)\, du \le \sqrt{2}\int_0^{+\infty} \rho^2(v)\, dv < \infty$$

for $\tau > 0$, whence

$$\mathsf{E}(X(T,\tau))^2 \le \frac{2+2\sqrt{2}}{T}\int_0^{+\infty} \rho^2(u)\, du = \frac{(1+\sqrt{2})\|\rho\|_2^2}{T}. \qquad \square$$

Consider a partition

$$\mathbf{T} = \bigcup_{k=1}^{\infty} B_k,$$

where

$$B_k = \{(T,\tau)\colon T_k \le T \le T_{k+1}, a \le \tau \le b\}$$

is such that $T_k < T_{k+1}$, $T_{k+1} - T_k > \text{const} > 1$, and $T_k \to \infty$ as $k \to \infty$.

Lemma 3.2. *Let*

$$\int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln(1+|\lambda|)\right)^{2\alpha}\, d\lambda < \infty \tag{10}$$

for $\alpha > 0$. Then

$$\sup_{(T,\tau),(T',\tau')\in B_k, m((T,\tau),(T',\tau'))<h} \left(\mathsf{E}\left(X(T,\tau) - X(T',\tau')\right)^2\right)^{1/2} \le \sigma_k(h),$$

where

$$\sigma_k(h) = \frac{C_2^{1/2}}{T_k^{1/2}\left(\ln\left(e^\alpha + C/h\right)\right)^{\alpha/2}},$$

$C > 0$ is an arbitrary constant,

$$\widetilde{f} = \int_{-\infty}^{+\infty} f^2(\lambda)\big(\ln(e^\alpha + C|\lambda|/2)\big)^{2\alpha}\, d\lambda,$$

$$\begin{aligned} C_2 = 8\pi &\left[\widetilde{f}\left(\ln\left(e^\alpha + \frac{C}{b-a}\right)\right)^{-\alpha} + \|f\|_2 \widetilde{f}^{1/2}\right] \\ &+ 2\|\rho\|_2^2 \left(1 + \frac{6T_{k+1}}{T_k} + \frac{T_{k+1}^2}{T_k^2}\right)\frac{T_{k+1}-T_k}{T_k}\left(\ln\left(e^\alpha + \frac{1}{T_{k+1}-T_k}\right)\right)^{\alpha}. \end{aligned} \tag{11}$$

Proof. Let $(T,\tau) \in B_k$ and $(T',\tau') \in B_k$ be such that $T \le T'$. Then

$$\begin{aligned} &\mathsf{E}\left(X(T,\tau) - X(T',\tau')\right)^2 \\ &\quad\le \Bigg| \frac{1}{T^2}\int_0^T\!\!\int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau)\rho(t-s-\tau)\right] dt\, ds \\ &\qquad - \frac{2}{T^2}\int_0^T\!\!\int_0^T \left[\rho(t-s)\rho(t-s+\tau-\tau') + \rho(t-s+\tau)\rho(t-s-\tau')\right] dt\, ds \\ &\qquad\qquad + \frac{1}{T^2}\int_0^T\!\!\int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dt\, ds \Bigg| \\ &\qquad + \Bigg| \frac{2}{T^2}\int_0^T\!\!\int_0^T \left[\rho(t-s)\rho(t-s+\tau-\tau') + \rho(t-s+\tau)\rho(t-s-\tau')\right] dt\, ds \\ &\qquad\qquad - \frac{2}{TT'}\int_0^T\!\!\int_0^{T'} \left[\rho(t-s)\rho(t-s+\tau-\tau') + \rho(t-s+\tau)\rho(t-s-\tau')\right] dt\, ds \Bigg| \\ &\qquad + \Bigg| \frac{1}{T'^2}\int_0^{T'}\!\!\int_0^{T'} \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dt\, ds \\ &\qquad\qquad - \frac{1}{T^2}\int_0^T\!\!\int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dt\, ds \Bigg| \\ &\quad= I + A_1 + A_2. \end{aligned}$$

Now we estimate A_1, A_2, and I. We have

$$
\begin{aligned}
A_1 &= \bigg| \frac{2}{T^2} \int_0^T \int_0^T \big[\rho(t-s)\rho(t-s+\tau-\tau') + \rho(t-s+\tau)\rho(t-s-\tau')\big]\, dt\, ds \\
&\qquad - \frac{2}{TT'} \int_0^T \int_0^{T'} \big[\rho(t-s)\rho(t-s+\tau-\tau') + \rho(t-s+\tau)\rho(t-s-\tau')\big]\, dt\, ds \bigg| \\
&\le \bigg| \frac{2}{T^2} \int_T^{T'} \int_0^T \big[\rho(t-s)\rho(t-s+\tau-\tau') + \rho(t-s+\tau)\rho(t-s-\tau')\big]\, dt\, ds \bigg| \\
&\quad + \bigg| \left(\frac{2}{T^2} - \frac{2}{TT'} \right) \int_0^T \int_0^{T'} \big[\rho(t-s)\rho(t-s+\tau-\tau') \\
&\qquad\qquad + \rho(t-s+\tau)\rho(t-s-\tau')\big]\, dt\, ds \bigg| \\
&\le \frac{2}{T^2} \bigg| \int_T^{T'} \left(\int_0^T \rho^2(t-s) dt \int_0^T \rho^2(t-s+\tau-\tau')\, dt \right)^{1/2} ds \\
&\qquad + \int_T^{T'} \left(\int_0^T \rho^2(t-s+\tau)\, dt \int_0^T \rho^2(t-s-\tau')\, dt \right)^{1/2} ds \bigg| \\
&\quad + \bigg| \left(\frac{2}{T^2} - \frac{2}{TT'} \right) \bigg[\int_0^T \left(\int_0^{T'} \rho^2(t-s)\, dt \int_0^{T'} \rho^2(t-s+\tau-\tau')\, dt \right)^{1/2} ds \\
&\qquad + \int_0^T \left(\int_0^{T'} \rho^2(t-s+\tau)\, dt \int_0^{T'} \rho^2(t-s-\tau')\, dt \right)^{1/2} ds \bigg] \bigg| \\
&\le \frac{4}{T^2} |T'-T| \int_{-\infty}^{+\infty} \rho^2(u)\, du + \left| \frac{2}{T^2} - \frac{2}{TT'} \right| 2T \int_{-\infty}^{+\infty} \rho^2(u)\, du \\
&= \frac{4(T'+T)|T'-T|}{T^2 T'} \|\rho\|_2^2 \\
&\le 4\|\rho\|_2^2 \frac{2T_{k+1}}{T_k} \frac{|T'-T|}{T^2}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
A_2 &= \bigg| \frac{1}{T^2} \int_0^T \int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right]\, dt\, ds \\
&\qquad - \frac{1}{T'^2} \int_0^{T'} \int_0^{T'} \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right]\, dt\, ds \bigg| \\
&\le 2\|\rho\|_2^2 \frac{|T'-T|}{T^2} \left(1 + \frac{T'+T}{T'} + \frac{T^2}{T'^2} \right) \\
&\le 2\|\rho\|_2^2 \frac{|T'-T|}{T^2} \left(1 + \frac{2T_{k+1}}{T_k} + \frac{T_{k+1}^2}{T_k^2} \right).
\end{aligned}
$$

Using a lemma in [7] we get

$$\mathsf{E}(X(T,\tau)-X(T,\tau'))^2 \le \frac{8\pi}{T}\left(\left[\int_{-\infty}^{+\infty} f^2(\lambda)\sin^2\frac{\lambda(\tau-\tau')}{2}\,d\lambda\right] + \|f\|_2\left[\int_{-\infty}^{+\infty} f^2(\lambda)\sin^2\frac{\lambda(\tau-\tau')}{2}\,d\lambda\right]^{1/2}\right), \tag{12}$$

where

$$\|f\|_2^2 = \int_{-\infty}^{+\infty} f^2(\lambda)\,d\lambda < \infty.$$

Since

$$\left|\sin\frac{u}{v}\right| \le \left(\frac{\ln(e^\alpha+u)}{\ln(e^\alpha+v)}\right)^\alpha$$

for all $u \ge 0$, $v > 0$, and $\alpha > 0$ (see [8]), we have

$$\left|\sin\frac{\lambda(\tau-\tau')}{2}\right| \le \frac{(\ln(e^\alpha + C|\lambda|/2))^\alpha}{(\ln(e^\alpha + C/|\tau-\tau'|))^\alpha},$$

where $C > 0$ is an arbitrary constant. Then we rewrite (12) in the following form:

$$I \le \frac{8\pi}{T}\left[\int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln\left(e^\alpha + \frac{C|\lambda|}{2}\right)\right)^{2\alpha} d\lambda \frac{1}{(\ln(e^\alpha + C/|\tau-\tau'|))^{2\alpha}} + \|f\|_2\left(\int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln\left(e^\alpha + \frac{C|\lambda|}{2}\right)\right)^{2\alpha} d\lambda\right)^{1/2} \frac{1}{(\ln(e^\alpha + C/|\tau-\tau'|))^{\alpha}}\right].$$

Inequality (10) implies that

$$\widetilde{f} = \int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln\left(e^\alpha + \frac{C|\lambda|}{2}\right)\right)^{2\alpha} d\lambda < \infty.$$

Thus

$$I \le \frac{\widetilde{C_2}}{T(\ln(e^\alpha + C/|\tau-\tau'|))^\alpha},$$

where

$$\widetilde{C_2} = 8\pi\left[\widetilde{f}\left(\ln\left(e^\alpha + \frac{C}{b-a}\right)\right)^{-\alpha} + \|f\|_2\widetilde{f}^{1/2}\right].$$

Hence

$$\mathsf{E}\big(X(T,\tau)-X(T',\tau')\big)^2 \le \frac{\widetilde{C_2}}{T(\ln(e^\alpha + C/|\tau-\tau'|))^\alpha} + \frac{C^*|T'-T|}{T^2}$$

for

$$C^* = 2\|\rho\|_2^2\left(1 + \frac{6T_{k+1}}{T_k} + \frac{T_{k+1}^2}{T_k^2}\right).$$

Since

$$\sup_{h<T_{k+1}-T_k} h\left(\ln\left(e^\alpha + \frac{1}{h}\right)\right)^\alpha = (T_{k+1}-T_k)\left(\ln\left(e^\alpha + \frac{1}{T_{k+1}-T_k}\right)\right)^\alpha,$$

we obtain

$$\sup_{(T,\tau),(T',\tau')\in B_k,\, m((T,\tau),(T',\tau'))<h} \left(\mathsf{E}\big(X(T,\tau)-X(T',\tau')\big)^2\right)^{1/2} \le \sigma_k(h),$$

where

$$\sigma_k(h) = \frac{C_2^{1/2}}{T_k^{1/2}\left(\ln\left(e^\alpha + C/h\right)\right)^{\alpha/2}},$$
$$C_2 = \widetilde{C_2} + C^* \frac{T_{k+1}-T_k}{T_k}\left(\ln\left(e^\alpha + \frac{1}{T_{k+1}-T_k}\right)\right)^\alpha. \qquad \square$$

Keeping the above notation, we have

(1)
$$\begin{aligned}\varepsilon_{0k} &= \inf_{(T',\tau')\in B_k}\ \sup_{(T'',\tau'')\in B_k} m\big((T',\tau'),(T'',\tau'')\big)\\ &= \max\big\{(T_{k+1}-T_k)/2, (b-a)/2\big\},\end{aligned}$$

(2) $\delta_{0k} = \sup_{(T,\tau)\in B_k}\left(\mathsf{E}(X(T,\tau))^2\right)^{1/2} = C_1^{1/2}/T_k^{1/2}$,

(3) $\sigma_k^{(-1)}(v)$ is the inverse function for $\sigma_k(h)$,

$$\sigma_k^{(-1)}(v) = \frac{C}{\exp\left\{\left(C_2/(v^2T_k)\right)^{1/\alpha}\right\} - \exp\{\alpha\}},$$

(4) $t_{0k} = \sigma_k(\varepsilon_{0k})$,

(5) $z_{0k} = \max\{t_{0k}, \delta_{0k}\}$,

(6) $N_k(\varepsilon)$ is the minimal number of closed balls of radius ε that cover B_k.

In what follows we choose $C = \sqrt{b-a}$ and assume that

$$T_{k+1} - T_k > b - a > 1.$$

Lemma 3.3. *Let*

$$X(T,\tau) = \widehat{\rho}_T(\tau) - \rho(\tau)$$

and let $c = \{c(T), T \in [A;+\infty)\}$ *be a continuous function such that* $c(T) > 0$. *Put*

$$\gamma_k = \max_{T\in[T_k;T_{k+1}]} c(T).$$

If

(1) $\sum_{k=1}^\infty \gamma_k z_{0k}\ln(T_{k+1}-T_k) < \infty$, *and*

(2) $\int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln(1+|\lambda|)\right)^{2\alpha} d\lambda < \infty$ *for some* $\alpha > 2$,

then

$$\begin{aligned}&\mathsf{E}\exp\left\{u\sup_{(T,\tau)\in\mathbf{T}}|c(T)X(T,\tau)|\right\}\\ &\le 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\exp\left\{\frac{1}{d}\sum_{k=1}^\infty \gamma_k z_{0k}\ln(T_{k+1}-T_k) + \frac{2\widetilde{P}}{p^{2/\alpha}(1-2/\alpha)}\right\}\end{aligned} \tag{13}$$

for all real numbers p *and* u *such that* $0 < p < 1$ *and*

$$0 < u < \frac{1-p}{d\sqrt{2}},$$

where

$$d = \sum_{k=1}^\infty \gamma_k z_{0k}, \qquad \widetilde{P} = \sup_k\left(\ln\left(e^\alpha + 2\sqrt{b-a}/(T_{k+1}-T_k)\right)\right).$$

Proof. Let

$$r(v) = (\ln v)^f, \qquad v > e,\ 1 < f < \alpha/2.$$

Since

$$r(xy) = (\ln x + \ln y)^f \le 2^{f-1}\left((\ln x)^f + (\ln y)^f\right)$$

and

$$N_k\big(\sigma_k^{(-1)}(v)\big) \le \left(\frac{T_{k+1}-T_k}{2\sigma_k^{(-1)}(v)} + 1\right)\left(\frac{b-a}{2\sigma_k^{(-1)}(v)} + 1\right) \le \frac{(T_{k+1}-T_k)(b-a)}{\big(\sigma_k^{(-1)}(v)\big)^2}$$

for $v < t_{0k}$, we have

$$\begin{aligned}
\frac{1}{pt_{0k}}\int_0^{pt_{0k}} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv &\le \frac{1}{pt_{0k}}\int_0^{pt_{0k}} r\left(\frac{(b-a)(T_{k+1}-T_k)}{\big(\sigma_k^{(-1)}(v)\big)^2}\right) dv \\
&\le \frac{1}{pt_{0k}}\int_0^{pt_{0k}} r\left(\frac{(b-a)(T_{k+1}-T_k)}{C^2}\exp\left\{2\left(\frac{C_2}{v^2T_k}\right)^{1/\alpha}\right\}\right) dv \\
&= \frac{1}{pt_{0k}}\int_0^{pt_{0k}} \left(\ln\frac{(b-a)(T_{k+1}-T_k)}{C^2} + 2\left(\frac{C_2}{v^2T_k}\right)^{1/\alpha}\right)^f dv \\
&\le 2^{f-1}\left[\left(\ln\frac{(b-a)(T_{k+1}-T_k)}{C^2}\right)^f + \frac{2^f}{pt_{0k}}\left(\frac{C_2}{T_k}\right)^{f/\alpha}\int_0^{pt_0k} v^{-2f/\alpha}\,dv\right] \\
&= 2^{f-1}\left[\left(\ln\frac{(b-a)(T_{k+1}-T_k)}{C^2}\right)^f + \frac{2^fC_2^{f/\alpha}}{T_k^{f/\alpha}pt_{0k}}\frac{(pt_{0k})^{1-2f/\alpha}}{1-2f/\alpha}\right] \\
&= 2^{f-1}\left[\left(\ln\frac{(b-a)(T_{k+1}-T_k)}{C^2}\right)^f + \frac{2^fC_2^{f/\alpha}}{T_k^{f/\alpha}}\frac{(pt_{0k})^{-2f/\alpha}}{1-2f/\alpha}\right].
\end{aligned}$$

Since

$$t_{0k} = \sigma_k(\varepsilon_{0k}) = \frac{C_2^{1/2}}{T_k^{1/2}\left(\ln\left(e^\alpha + \frac{2C}{T_{k+1}-T_k}\right)\right)^{\alpha/2}},$$

we get

$$\begin{aligned}
&\frac{1}{pt_{0k}}\int_0^{pt_{0k}} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right) dv \\
&\quad\le 2^{f-1}\left[(\ln(T_{k+1}-T_k))^f + \frac{2^fC_2^{f/\alpha}}{T_k^{f/\alpha}p^{2f/\alpha}\left(1-\frac{2f}{\alpha}\right)}\frac{T_k^{f/\alpha}\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)^f}{C_2^{f/\alpha}}\right] \\
&\quad= 2^{f-1}\left[(\ln(T_{k+1}-T_k))^f + \frac{2^f}{p^{2f/\alpha}\left(1-\frac{2f}{\alpha}\right)}\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)^f\right]
\end{aligned}$$

for $1 < f < \alpha/2$.

Note that $\ln r^{(-1)}(z) = z^{1/f}$. Letting $f \uparrow 1$ we obtain

$$\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right)dv\right)\right)^{\gamma_k z_{0k}/d}$$

$$= \exp\left\{\sum_{k=1}^{\infty}\frac{\gamma_k z_{0k}}{d}\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}} r\left(N_k\big(\sigma_k^{(-1)}(v)\big)\right)dv\right)^{1/f}\right\}$$

$$\le \exp\Bigg\{2^{1-1/f}\sum_{k=1}^{\infty}\frac{\gamma_k z_{0k}}{d}\Bigg[\ln(T_{k+1}-T_k) + \frac{2}{p^{2/\alpha}\left(1-\frac{2f}{\alpha}\right)^{1/f}}\left(\ln\left(e^{\alpha}+\frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)\Bigg]\Bigg\} \tag{14}$$

$$\le \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln(T_{k+1}-T_k) + \frac{2\widetilde{P}}{p^{2/\alpha}(1-2/\alpha)}\right\},$$

where

$$\widetilde{P} = \sup_k\left(\ln\left(e^{\alpha}+\frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right).$$

The latter inequality together with (6) completes the proof of the theorem. □

Theorem 3.1. *Let $X(T,\tau) = \widehat{\rho}_T(\tau) - \rho(\tau)$ and let $c = \{c(T), T \in [A;+\infty)\}$ be a continuous function such that $c(T) > 0$. Put*

$$\gamma_k = \max_{T\in[T_k;T_{k+1}]} c(T).$$

If

(1) $\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln(T_{k+1}-T_k) < \infty$, *and*
(2) $\int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln(1+|\lambda|)\right)^{2\alpha}d\lambda < \infty$ *for some* $\alpha > 2$,

then

$$\mathsf{P}\left\{\sup_{(T,\tau)\in\mathbf{T}}|c(T)X(T,\tau)| > x\right\} \le 2e\exp\left\{-\frac{x}{d\sqrt{2}} + \frac{2\widetilde{P}}{(1-2/\alpha)}\left(\frac{x}{d\sqrt{2}}\right)^{2/\alpha}\right\}\left(\frac{x\sqrt{2}}{d}\right)^{1/2}\widetilde{\Phi}_5,$$

where

$$d = \sum_{k=1}^{\infty}\gamma_k z_{0k}, \qquad \widetilde{P} = \sup_k\left(\ln\left(e^{\alpha}+\frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right),$$

$$\widetilde{\Phi}_5 = \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln(T_{k+1}-T_k)\right\}$$

for any $x > d\sqrt{2}$.

Proof. The above inequality can easily be proved by substituting $p = d\sqrt{2}/x$, $x > d\sqrt{2}$, in (14) and applying the Chebyshev inequality and Theorem 2.4. Indeed

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{(T,\tau)\in\mathbf{T}} |c(T)X(T,\tau)| > x\right\} \\
&\quad\le 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1+\frac{x\sqrt{2}(1-p)}{d}\right)^{1/2}\exp\left\{\frac{2\widetilde{P}}{p^{2/\alpha}\,(1-2/\alpha)}\right\}\widetilde{\Phi}_5 \\
&\quad= 2\exp\left\{-\frac{x}{d\sqrt{2}}\left(1-\frac{d\sqrt{2}}{x}\right)\right\}\left(1+\frac{x\sqrt{2}}{d}\left(1-\frac{d\sqrt{2}}{x}\right)\right)^{1/2} \\
&\qquad\times\exp\left\{\frac{2\widetilde{P}}{(1-\frac{2}{\alpha})}\left(\frac{x}{d\sqrt{2}}\right)^{2/\alpha}\right\}\widetilde{\Phi}_5 \\
&\quad\le 2e\exp\left\{-\frac{x}{d\sqrt{2}}+\frac{2\widetilde{P}}{(1-2/\alpha)}\left(\frac{x}{d\sqrt{2}}\right)^{2/\alpha}\right\}\left(\frac{x\sqrt{2}}{d}\right)^{1/2}\widetilde{\Phi}_5. \qquad \square
\end{aligned}$$

Theorem 3.2. *Let a function $c(T) = T^{1/2}/(\ln T)^{\beta}$ be defined for all $T > e^m$ (here $m > 4$ is a fixed number) and let $2 < \beta < m/2$.*

If

$$\int_{-\infty}^{+\infty} f^2(\lambda)\,(\ln(1+|\lambda|))^{2\alpha}\,d\lambda < \infty$$

for some $\alpha > 2$, then

$$\mathsf{P}\left\{\sup_{(T,\tau)\in\mathbf{T}} |c(T)X(T,\tau)| > x\right\} \le 2e\exp\left\{-\frac{x}{d\sqrt{2}}+D_\alpha\left(\frac{x}{d\sqrt{2}}\right)^{2/\alpha}\right\}\left(\frac{x\sqrt{2}}{d}\right)^{1/2} D$$

for any $x > d\sqrt{2}$, where

$$D_\alpha = \frac{2\left(\ln\left(e^\alpha+\frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)}{(1-2/\alpha)}, \qquad D = \exp\left\{\frac{\sum_{k=1}^{\infty}\frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^{\infty}\frac{1}{(m+k+1)^{\beta}}}\right\},$$

$$d = C_0 e^{1/2}\sum_{k=1}^{\infty}\frac{1}{(m+k+1)^{\beta-1}},$$

and C_0 is a known constant.

Proof. Theorem 3.2 follows from Theorem 3.1. Note that the function $c(T) > 0$ is increasing for $\beta < \ln T/2$. Since $\beta > 2$, one can choose $A = e^m$ for some $m > 4$. Now we check the assumptions of Theorem 3.1 and obtain bounds for the probability

$$\mathsf{P}\left\{\sup_{(T,\tau)\in\mathbf{T}} |c(T)X(T,\tau)| > x\right\}.$$

First we choose the points T_k of the partition such that $T_k = e^{m+k}$, $k = 1, 2, \dots$. It is clear that

$$T_{k+1} - T_k = e^{m+k}(e-1) > 1$$

and

$$\gamma_k = c(T_{k+1}) = \frac{T_{k+1}^{1/2}}{(\ln T_{k+1})^{\beta}} = \frac{e^{(m+k+1)/2}}{(m+k+1)^{\beta}}, \qquad k = 1, 2, \dots,$$

$$z_{0k} = \max\{\delta_{0k}, t_{0k}\} = \max\left\{\frac{C_1^{1/2}}{T_k^{1/2}}, \frac{C_2^{1/2}}{T_k^{1/2}\left(\ln\left(e^{\alpha} + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)^{\alpha/2}}\right\} = \frac{C_0}{e^{(m+k)/2}},$$

where

$$C_0 = \max\left\{C_1^{1/2}, \frac{{C'}_2^{1/2}}{\left(\ln\left(e^{\alpha} + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)^{\alpha/2}}\right\},$$

$$C_2' = 8\pi\left[\widetilde{f}\left(\ln\left(e^{\alpha} + \frac{C}{b-a}\right)\right)^{-\alpha} + \|f\|_2 \widetilde{f}^{1/2}\right] + 2\|\rho\|_2^2\big(1 + 6e + e^2\big)(e-1)\left(\ln\left(e^{\alpha} + \frac{1}{e^m(e-1)}\right)\right)^{\alpha}.$$

Thus

$$d = \sum_{k=1}^{\infty} \gamma_k z_{0k} = C_0 e^{1/2} \sum_{k=1}^{\infty} \frac{1}{(m+k+1)^{\beta}} < \infty \quad \text{for } \beta > 1,$$

$$\sum_{k=1}^{\infty} \gamma_k z_{0k} \ln(T_{k+1} - T_k) \le C_0 e^{1/2} \sum_{k=1}^{\infty} \frac{1}{(m+k+1)^{\beta-1}} < \infty \quad \text{for } \beta > 2;$$

that is, assumption 1) of Theorem 3.1 holds.

Now we estimate $\widetilde{\Phi}_5$ and $\widetilde{P}$. We have

$$\widetilde{\Phi}_5 = \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty} \gamma_k z_{0k} \ln(T_{k+1} - T_k)\right\} \le \exp\left\{\frac{\sum_{k=1}^{\infty} \frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^{\infty} \frac{1}{(m+k+1)^{\beta}}}\right\},$$

$$\widetilde{P} = \sup_{k \ge m}\left(\ln\left(e^{\alpha} + \frac{2\sqrt{b-a}}{e^k(e-1)}\right)\right) = \left(\ln\left(e^{\alpha} + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right),$$

whence

$$\mathsf{P}\left\{\sup_{(T,\tau)\in\mathbf{T}} |c(T)X(T,\tau)| > x\right\} \le 2e\exp\left\{-\frac{x}{d\sqrt{2}} + D_{\alpha}\left(\frac{x}{d\sqrt{2}}\right)^{2/\alpha}\right\}\left(\frac{x\sqrt{2}}{d}\right)^{1/2} D$$

for

$$D_{\alpha} = \frac{2\left(\ln\left(e^{\alpha} + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)}{(1 - 2/\alpha)}, \qquad D = \exp\left\{\frac{\sum_{k=1}^{\infty} \frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^{\infty} \frac{1}{(m+k+1)^{\beta}}}\right\}. \qquad \square$$

Theorem 3.2 allows one to construct a criterion for testing a hypothesis concerning the correlation function of a stochastic process.

Let $\xi = \{\xi(t), t \ge 0\}$ be a real-valued, mean square continuous, stationary, Gaussian stochastic process with $\mathsf{E}\,\xi(t) = 0$. Denote its spectral density by $f(\lambda)$ and its correlation function by

$$\rho(\tau) = \mathsf{E}\,\xi(t+\tau)\xi(t), \qquad a \le \tau \le b.$$

Assume that the stochastic process ξ is observed on the interval $[0, \widetilde{T} + b]$, where $\widetilde{T} > e^m$ for some $m > 4$. We treat the correlogram $\widehat{\rho}_T(\tau)$ as an estimator for the correlation function and assume that $T > e^m$ $(m > 4)$.

Let H be the hypothesis that the correlation function of the stochastic process $\xi(t)$ coincides with $\rho(\tau)$ for all $a \le \tau \le b$. To test the hypothesis H we propose the following criterion.

Criterion 3.1. *Given γ, $0 < \gamma < 1$, find x_γ such that*

$$A(x_\gamma) = 2e \exp\left\{ -\frac{x_\gamma}{d\sqrt{2}} + D_\alpha \left(\frac{x_\gamma}{d\sqrt{2}}\right)^{2/\alpha} \right\} \left(\frac{x_\gamma \sqrt{2}}{d}\right)^{1/2} D = \gamma,$$

where $\alpha > 2$ satisfies

$$\int_{-\infty}^{+\infty} f^2(\lambda)(\ln(1+|\lambda|))^{2\alpha}\, d\lambda < \infty,$$

$$D_\alpha = \frac{2\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)}{(1-2/\alpha)}, \qquad D = \exp\left\{ \frac{\sum_{k=1}^\infty \frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^\infty \frac{1}{(m+k+1)^{\beta}}} \right\}, \qquad 2 < \beta < \frac{m}{2},$$

and the number d is defined in Theorem 3.2. The hypothesis H is accepted if

$$\sup_{e^m < T < \widetilde{T}, a \le \tau \le b} \frac{T^{1/2}}{(\ln T)^\beta} |\widehat{\rho}_T(\tau) - \rho(\tau)| < x_\gamma$$

for $e^m < T < \widetilde{T}$ and $a \le \tau \le b$. Otherwise the hypothesis is rejected.

Note that the error of the first kind does not exceed γ for this criterion.

4. Concluding remarks

The bounds for the distribution of the supremum of a square Gaussian process are found and a criterion for testing a hypothesis pertaining to the correlation function of a stationary Gaussian stochastic process is constructed in this paper. The criterion is useful for sufficiently large T (say, for $T > e^m$, where $m > 4$). Its error of the first kind does not exceed γ. If one simultaneously uses Criterion 3.1 and other criteria constructed in other papers (say, in [9]), then one can decrease essentially the error of the second kind when testing hypotheses concerning the correlation function of a Gaussian stochastic process by using observations of its trajectory on a finite interval of an arbitrary length.

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