

AN ESTIMATE OF THE PROBABILITY
THAT THE QUEUE LENGTH EXCEEDS THE MAXIMUM
FOR A QUEUE THAT IS A GENERALIZED
ORNSTEIN–UHLENBECK STOCHASTIC PROCESS

UDC 519.21

R. E. YAMNENKO

ABSTRACT. We consider the process

$$A(t) = mt + \sigma \int_0^t X(u) du, \quad t \geq 0,$$

describing the queue length, where m and σ are positive constants, $X(u)$ is a φ -sub-Gaussian generalized Ornstein–Uhlenbeck stochastic process, and

$$\varphi(u) = \begin{cases} u^r, & |u| > 1, \\ u^2, & |u| \leq 1, \end{cases}$$

$r \geq 2$. The classes of φ -sub-Gaussian and strictly φ -sub-Gaussian stochastic processes are wider than the class of Gaussian processes and are of interest for modeling stochastic processes appearing in queueing theory and in the mathematics of finance. We obtain an estimate of the probability that the queue length exceeds the maximum allowed for it, namely,

$$\mathbb{P} \left\{ \sup_{t \geq 0} (A(t) - ct) > x \right\} \leq L(\gamma) x^{r/(r-1)} \exp \left\{ -\kappa(\gamma) x^{r/(2(r-1))} \right\},$$

where $c > m$ is the service intensity, $x > 0$ is the maximum queue length, and $L(\gamma)$ and $\kappa(\gamma)$ are some finite constants.

1. INTRODUCTION

Consider the process

$$(1) \quad A(t) = mt + \sigma Y(t), \quad t \geq 0,$$

describing the length of a queue, where m is the mean amount of work needed to serve the customers arriving to the system, $\sigma > 0$ is some constant, $c > m$ is the service intensity of the system, and $Y(t)$ is the amount of work needed to serve the customers arriving to the system in the interval $[0, t)$. If the amount of work needed to serve the arrived customers exceeds the capacity of the system, then some customers are placed in the waiting list (queue). If the queue becomes larger than the maximum, then some customers are excluded from the waiting list; namely, the customers arriving to the system after the maximum of the queue is exceeded are rejected. Consider the probability that the length

2000 *Mathematics Subject Classification.* Primary 60G07; Secondary 60K25.

Key words and phrases. φ -sub-Gaussian stochastic process, generalized Ornstein–Uhlenbeck process, the distribution of the supremum.

$A(t)$ of the queue exceeds $x > 0$:

$$(2) \quad \mathbb{P} \left\{ \sup_{t>0} (A(t) - ct) > x \right\}.$$

Estimation of this probability is of special interest in the theory of information transmission.

Estimates for the probability (2) are known for many classical models. This is the case, for example, if the input process is Gaussian. It is worth mentioning that the assumption that the input is Gaussian is applicable in an asymptotic sense only (if applicable at all). Thus a natural problem is to obtain estimates for the probability (2) in the case of a wider class of input processes. The classes of φ -sub-Gaussian and strictly φ -sub-Gaussian stochastic processes are suitable candidates for this problem. In this paper, we deal with systems whose input is described by a strictly φ -sub-Gaussian generalized Ornstein–Uhlenbeck stochastic process.

The properties of random variables and stochastic processes of the spaces $\text{Sub}_\varphi(\Omega)$ and strictly $\text{Sub}_\varphi(\Omega)$ ($\text{SSub}_\varphi(\Omega)$) can be found in the book [2] and in the papers [3]–[6]. Some related problems are discussed in [7, 8, 11] concerning the estimation of the probability that the trajectories of φ -sub-Gaussian processes cross a continuous curve.

The current paper contains three sections. We define φ -sub-Gaussian generalized Ornstein–Uhlenbeck stochastic processes and discuss the main definitions and properties of random variables and stochastic processes of the spaces $\text{Sub}_\varphi(\Omega)$ and $\text{SSub}_\varphi(\Omega)$ in Section 2. Section 3 contains the main result on the estimate of the probability that the queue length exceeds the maximum for models with φ -sub-Gaussian generalized Ornstein–Uhlenbeck input. Also we compare our estimate with the analogous estimate for the case of the Gaussian Ornstein–Uhlenbeck process.

2. $\text{Sub}_\varphi(\Omega)$ AND $\text{SSub}_\varphi(\Omega)$ RANDOM VARIABLES AND STOCHASTIC PROCESSES

Let $\{\Omega, \mathcal{B}, \mathbb{P}\}$ be a standard probability space and let T be a space of parameters.

Definition 2.1 ([9]). A function $U = \{U(x), x \in \mathbb{R}\}$ is called an Orlicz N -function if U is a continuous, even, and convex function such that $U(0) = 0$, $U(x)$ is increasing for $x > 0$, $U(x)/x \rightarrow 0$ as $x \rightarrow 0$, and $U(x)/x \rightarrow \infty$ as $x \rightarrow \infty$.

Definition 2.2 ([2]). Let φ be an Orlicz N -function such that $\varphi(x) = cx^2$ for $|x| \leq x_0$, where $c > 0$ and $x_0 > 0$ are some constants. A centered random variable ξ belongs to the space $\text{Sub}_\varphi(\Omega)$ if for any $\lambda \in \mathbb{R}$ there exists a constant $r_\xi \geq 0$ such that

$$\mathbb{E} \exp\{\lambda\xi\} \leq \exp\{\varphi(\lambda r_\xi)\}.$$

Theorem 2.1 ([2]). *The space $\text{Sub}_\varphi(\Omega)$ is a Banach space with the norm*

$$(3) \quad \tau_\varphi(\xi) = \sup_{\lambda>0} \frac{\varphi^{(-1)}(\log \mathbb{E} \exp\{\lambda\xi\})}{\lambda},$$

where $\varphi^{(-1)}$ is the inverse function to φ . Moreover

$$\mathbb{E} \exp\{\lambda\xi\} \leq \exp\{\varphi(\lambda\tau_\varphi(\xi))\}$$

for all $\lambda \in \mathbb{R}$, and there exists a constant $c > 0$ such that

$$(4) \quad (\mathbb{E} \xi^2)^{1/2} \leq c\tau_\varphi(\xi).$$

Example ([2]). Any centered Gaussian random variable $\xi = N(0, \sigma^2)$ belongs to the space $\text{Sub}_{x^2/2}(\Omega)$ and moreover $\tau(\xi) = (\mathbb{E} \xi^2)^{1/2}$.

Definition 2.3. A stochastic process $X = (X(t), t \in T)$ is called φ -sub-Gaussian if

$$X(t) \in \text{Sub}_\varphi(\Omega) \quad \text{for all } t \in T$$

and $\sup_{t \in T} \tau_\varphi(X(t)) < \infty$.

Definition 2.4 ([4]). A family of random variables Δ of the space $\text{Sub}_\varphi(\Omega)$ is called a strictly $\text{Sub}_\varphi(\Omega)$ family if there exists a constant $C_\Delta > 0$ such that

$$\tau_\varphi\left(\sum_{i \in I} \lambda_i \xi_i\right) \leq C_\Delta \left(\mathbb{E}\left(\sum_{i \in I} \lambda_i \xi_i\right)^2\right)^{1/2}$$

for an arbitrary finite set I such that $\xi_i \in \Delta, i \in I$, and for all $\lambda_i \in \mathbb{R}$.

Theorem 2.2 ([4]). *If Δ is a strictly $\text{Sub}_\varphi(\Omega)$ family of random variables, then the linear closure $\overline{\Delta}$ (in the space $L_2(\Omega)$) is also a strictly $\text{Sub}_\varphi(\Omega)$ family of random variables.*

Definition 2.5. The linear closure of families of strictly $\text{Sub}_\varphi(\Omega)$ random variables forms the space of strictly $\text{Sub}_\varphi(\Omega)$ random variables. This space is denoted by $\text{SSub}_\varphi(\Omega)$.

Definition 2.6. A stochastic process $X = (X(t), t \in T)$ is called strictly φ -sub-Gaussian if the corresponding family of random variables $\{X(t), t \in T\}$ belongs to the space $\text{SSub}_\varphi(\Omega)$.

Definition 2.7. A stochastic process $X = (X(t), t \in T)$ is called the φ -sub-Gaussian generalized Ornstein–Uhlenbeck process if X is a φ -sub-Gaussian process with the covariance function

$$(5) \quad B_X(t, s) = e^{-\tau|t-s|}, \quad \tau > 0.$$

3. MAIN RESULTS

Consider the process $A(t)$ defined by (1) and assume that the input is described by the process

$$(6) \quad Y(t) = \int_0^t X(u) du,$$

where $X(u)$ is a strictly φ -sub-Gaussian generalized Ornstein–Uhlenbeck stochastic process whose Orlicz N -function is

$$\varphi(u) = \begin{cases} u^r, & |u| > 1, \\ u^2, & |u| \leq 1, \end{cases} \quad r \geq 2.$$

The covariance function of the process $Y(t)$ is given by

$$(7) \quad B_Y(t, s) = \frac{2 \min(t, s)}{\tau} + \frac{1}{\tau^2} \left(e^{-\tau t} + e^{-\tau s} - e^{-\tau|t-s|} - 1 \right).$$

Put $C = (c - m)/(\sigma C_\Delta)$ and $\varepsilon = x/(\sigma C_\Delta)$.

We study the probability that the length $A(t)$ of the queue exceeds the maximum $x > 0$. The main result of the current paper reads as follows.

Theorem 3.1. *Let $Y(t) = \int_0^t X(u) du, t \geq 0$, where $X(u)$ is a strictly φ -sub-Gaussian Ornstein–Uhlenbeck stochastic process,*

$$\varphi(u) = \begin{cases} u^r, & |u| > 1, \\ u^2, & |u| \leq 1, \end{cases}$$

and $r > 2$. Then

$$(8) \quad \begin{aligned} & \mathbb{P} \left\{ \sup_{t \geq 0} (A(t) - ct) > x \right\} \\ &= \mathbb{P} \left\{ \sup_{t \geq 0} (Y(t) - Ct) > \varepsilon \right\} \leq L(\gamma, \zeta, \varepsilon) \varepsilon^{r/(r-1)} \exp \left\{ -\kappa(\gamma) \varepsilon^{r/(2(r-1))} \right\} \end{aligned}$$

for all $\gamma > 1$ and all

$$(9) \quad \varepsilon \geq \frac{2r\gamma^M(\gamma^{r/2-1} - 1)}{r-2} \max \left\{ \frac{C^{(r-2)/r}}{\tau^{1/r}(\gamma^{r/2} - 1)}, \frac{(\gamma^{r/2} - 1) \max\{2^{r-1}(\gamma - 1)^{1-r}; 1\}}{\tau C(\gamma - 1)^2} \right\},$$

where $\zeta \in (0, \varepsilon\gamma^{-M-1})$, M is a positive integer such that

$$(10) \quad M \geq 1 + \frac{2}{(r-2)\log \gamma} \log \left(\frac{2(r-1)(\gamma-1)}{r(\gamma^{r/2} - 1)} \right),$$

and

$$(11) \quad L(\gamma, \zeta, \varepsilon) = 2e^2 \zeta^{\frac{r}{1-r}} \gamma^{\frac{r(M+1)}{1-r}} (K_0(\gamma) + K_1(\gamma)(1 + S_1(\gamma, \varepsilon) + S_2(\gamma, \varepsilon))) < \infty,$$

$$(12) \quad \kappa(\gamma) = \left(\frac{\tau C}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{2(r-1)(\gamma^{\frac{r}{2}} - \gamma)^{\frac{r-2}{2(r-1)}} (\gamma - 1)^{\frac{1}{r-1}}}{r(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}}},$$

$$(13) \quad \begin{aligned} K_0(\gamma) = \exp \left\{ \left(\frac{\tau C \zeta}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}} \gamma^{\frac{r}{2(r-1)}}}{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}}} \right. \\ \left. \times \left(\gamma^M + \frac{(r-2)(\gamma^{\frac{r}{2}} - 1)}{r(\gamma^{\frac{r}{2}-1} - 1)} + \frac{(r-2)(\gamma^{\frac{r}{2}} - 1)}{4r(\gamma^{\frac{r}{2}-1} - 1)} \right) \right\}, \end{aligned}$$

$$(14) \quad \begin{aligned} K_1(\gamma) = \exp \left\{ \left(\frac{\tau C \zeta}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}} (\gamma - 1)^{\frac{1}{r-1}}}{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}}} \right. \\ \left. \times \left(\gamma^M + \frac{r-2}{r} + \frac{(r-2)(\gamma-1)^{\frac{r}{2}+1}}{2^{\frac{r}{2}} r (\gamma^{\frac{r}{2}-1} - 1)} + \frac{(r-2)(\gamma-1)(\gamma^{\frac{r}{2}} - 1)}{4r(\gamma^{\frac{r}{2}-1} - 1)} \right) \right\}, \end{aligned}$$

$$(15) \quad S_1(\gamma, \varepsilon) = \sum_{k=1}^{M-1} \gamma^{-\frac{r(M-k)}{r-1}} \exp \left\{ -\frac{1}{2} \kappa(\gamma) \varepsilon^{\frac{r}{2(r-1)}} \left(\frac{(r-2)\log \gamma}{2(r-1)} \right)^2 (M-k)^2 \right\},$$

$$(16) \quad S_2(\gamma, \varepsilon) = \sum_{k=M+1}^{\infty} \gamma^{\frac{r(k-M)}{r-1}} \exp \left\{ -\frac{1}{2} \kappa(\gamma) \varepsilon^{\frac{r}{2(r-1)}} \left(\frac{r \log \gamma}{2(r-1)} \right)^2 (k-M)^2 \right\}.$$

Proof. The following result and its proof can be found in [8].

Theorem 3.2. *Let*

$$(A(t), t \in [a, b]), \quad 0 \leq a < b < \infty,$$

be the queue length defined by (1) and let the input be of the form (6). Then

$$(17) \quad \mathbb{P} \left\{ \sup_{t \in [a, b]} (A(t) - ct) > x \right\} \leq \frac{2e^2(b-a)}{\tau(\beta p)^2} \inf_{\lambda > \lambda_0} Z(\lambda, p, \beta)$$

for all $p \in (0, 1)$, $\beta \in (0, ((b - a)/\tau)^{1/2}]$, and $x > 0$, where

$$(18) \quad Z(\lambda, p, \beta) = \exp \left\{ \theta(\lambda, p) + \frac{p\lambda^r \beta^r}{(1-p)^r} - \lambda \left(\varepsilon - \frac{C\tau\beta^2 p^2}{2(1-p^2)} \right) \right\},$$

$$(19) \quad \theta(\lambda, p) = \sup_{u \in [a, b]} \left(\frac{\lambda^r \left(\frac{2u}{\tau}\right)^{r/2}}{(1-p)^{r-1}} - \lambda C u \right),$$

$$(20) \quad \lambda_0 = (1-p) \max \left\{ \frac{1}{\beta}; \left(\frac{\tau}{2v}\right)^{1/2} \right\}, \quad v = \begin{cases} a, & \text{if } a > 0, \\ b, & \text{if } a = 0. \end{cases}$$

Remark 3.1 ([8]). If the stochastic process $X(u)$ is strictly φ -sub-Gaussian and $\varphi(u) = u^2$, then $\lambda_0 = 0$.

Since $\varphi(u)$ is strictly convex, it follows for all $\lambda > \lambda_0$ that

$$(21) \quad \theta(\lambda, p) = \begin{cases} \frac{\lambda^r (2a)^{r/2}}{(1-p)^{r-1} \tau^{r/2}} - \lambda C a, & \lambda \leq \lambda^*, \\ \frac{\lambda^r (2b)^{r/2}}{(1-p)^{r-1} \tau^{r/2}} - \lambda C b, & \lambda > \lambda^*, \end{cases}$$

where

$$\lambda^* = (1-p) \left(\frac{C(b-a)}{b^{r/2} - a^{r/2}} \right)^{1/(r-1)} \left(\frac{\tau}{2} \right)^{r/(2(r-1))}.$$

Put $\beta = ((b - a)/\tau)^{1/2}$. Consider the exponential part of estimate (17). Then

$$\begin{aligned} & \inf_{\lambda > \lambda_0} Z(\lambda, p) \leq Z(\lambda^*, p) \\ & = \exp \left\{ (\lambda^*)^r \left(\frac{(2b)^{\frac{r}{2}}}{\tau^{\frac{r}{2}}(1-p)^{r-1}} + \frac{(b-a)^{\frac{r}{2}} p}{\tau^{\frac{r}{2}}(1-p)^r} \right) - \lambda^* \left(Cb + \varepsilon - \frac{C(b-a)p^2}{2(1-p^2)} \right) \right\} \\ & = \exp \left\{ (1-p) \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{C(b-a)}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} \right)^{\frac{1}{r-1}} \right. \\ & \quad \times \left. \left(\left(\frac{\tau}{2} \right)^{\frac{r}{2}} \frac{C(b-a)}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} \left(\frac{(2b)^{\frac{r}{2}}}{\tau^{\frac{r}{2}}} + \frac{(b-a)^{\frac{r}{2}} p}{\tau^{\frac{r}{2}}(1-p)} \right) - Cb - \varepsilon + \frac{C(b-a)p^2}{2(1-p^2)} \right) \right\} \\ & = \exp \left\{ -(1-p) \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{C(b-a)}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} \right)^{\frac{1}{r-1}} \right. \\ (22) \quad & \quad \times \left. \left(\varepsilon + Cab \frac{b^{\frac{r}{2}-1} - a^{\frac{r}{2}-1}}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} + \frac{p}{1-p} \left(\frac{C(b-a)^{\frac{r}{2}+1}}{2^{\frac{r}{2}}(b^{\frac{r}{2}} - a^{\frac{r}{2}})} + \frac{C(b-a)p}{2(1+p)} \right) \right) \right\} \\ & = \exp \left\{ - \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{C(b-a)}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} \right)^{\frac{1}{r-1}} \left(\varepsilon + Cab \frac{b^{\frac{r}{2}-1} - a^{\frac{r}{2}-1}}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} \right) \right\} \\ & \quad \times \exp \left\{ p \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{C(b-a)}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} \right)^{\frac{1}{r-1}} \right. \\ & \quad \times \left. \left(\varepsilon + Cab \frac{b^{\frac{r}{2}-1} - a^{\frac{r}{2}-1}}{b^{\frac{r}{2}} - a^{\frac{r}{2}}} - \frac{C(b-a)^{\frac{r}{2}+1}}{2^{\frac{r}{2}}(b^{\frac{r}{2}} - a^{\frac{r}{2}})} + \frac{C(b-a)p}{2(1+p)} \right) \right\} \\ & = W_{a,b}(\varepsilon) K_{a,b}(p, \varepsilon). \end{aligned}$$

Consider the following partition:

$$[0, \infty) = \bigcup_{k=0}^{\infty} [a_k, b_k],$$

where $a_0 = 0$, $b_0 = \alpha$, $b_k = a_{k+1} = \gamma^k \alpha$ for $k \geq 1$, $\gamma > 1$, and $\alpha > 0$ is a constant to be specified later. Then

$$(23) \quad \begin{aligned} W_k(\gamma, \varepsilon) &= W_{a_k, b_k}(\varepsilon) \\ &= \exp \left\{ - \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{C(\gamma-1)}{\gamma^{r/2}-1} \right)^{\frac{1}{r-1}} \alpha^{\frac{1-r/2}{r-1}} \gamma^{\frac{1-r/2}{r-1}(k-1)} \right. \\ &\quad \left. \times \left(\varepsilon + C\alpha\gamma^k \frac{\gamma^{r/2-1}-1}{\gamma^{r/2}-1} \right) \right\} \end{aligned}$$

for all $k \geq 1$. We find the interval where $W_k(\gamma, \varepsilon)$ attains its maximum. The function

$$(24) \quad J(k) = \gamma^{\frac{1-r/2}{r-1}(k-1)} \left(\varepsilon + C\alpha\gamma^k \frac{\gamma^{r/2-1}-1}{\gamma^{r/2}-1} \right)$$

is continuous with respect to the argument k . Then

$$\begin{aligned} \frac{dJ(k)}{dk} &= \gamma^{\frac{1-r/2}{r-1}(k-1)} \log \gamma \left(\frac{1-\frac{r}{2}}{r-1} \left(\varepsilon + \frac{C\alpha\gamma^k(\gamma^{r/2-1}-1)}{\gamma^{r/2}-1} \right) + \frac{C\alpha\gamma^k(\gamma^{r/2-1}-1)}{\gamma^{r/2}-1} \right) \\ &= \gamma^{\frac{1-r/2}{r-1}(k-1)} \log \gamma \left(\frac{\varepsilon(1-\frac{r}{2})}{r-1} + \frac{C\alpha\gamma^k(\gamma^{r/2-1}-1)\frac{r}{2}}{(\gamma^{r/2}-1)(r-1)} \right) \end{aligned}$$

and

$$(25) \quad \frac{dJ(k)}{dk} = 0 \Leftrightarrow \alpha = \frac{\varepsilon\gamma^{-k}}{C} \frac{r-2}{r} \frac{\gamma^{r/2}-1}{\gamma^{r/2-1}-1}.$$

Choose α such that $W_k(\gamma, \varepsilon)$ attains its maximum for $k = M$ where $M \geq 1$. Then

$$(26) \quad a = \frac{\varepsilon\gamma^{-M}}{C} \frac{r-2}{r} \frac{\gamma^{r/2}-1}{\gamma^{r/2-1}-1}.$$

Substituting α defined in (26) into (23) we get

$$(27) \quad \begin{aligned} W_k^{(M)}(\gamma, \varepsilon) &= \exp \left\{ - \left(\frac{\tau C \varepsilon}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \gamma^{\frac{r-2}{2(r-1)}(M-k+1)} \right. \\ &\quad \left. \times \frac{(\gamma^{r/2-1}-1)^{\frac{r-2}{2(r-1)}} (\gamma-1)^{\frac{1}{r-1}}}{(\gamma^{r/2}-1)^{\frac{r}{2(r-1)}}} \left(1 + \frac{r-2}{r} \gamma^{k-M} \right) \right\} \end{aligned}$$

for all $k \geq 1$, and

$$(28) \quad W_0^{(M)}(\gamma, \varepsilon) = \exp \left\{ - \left(\frac{\tau C \varepsilon}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \gamma^{\frac{(r-2)M}{2(r-1)}} \left(\frac{\gamma^{\frac{r}{2}-1}-1}{\gamma^{\frac{r}{2}-1}} \right)^{\frac{r-2}{2(r-1)}} \right\}.$$

Put

$$(29) \quad W(\gamma, \varepsilon) = W_M^{(M)}(\gamma, \varepsilon) = \exp \left\{ -\kappa(\gamma) \varepsilon^{\frac{r}{2(r-1)}} \right\},$$

where

$$(30) \quad \kappa(\gamma) = \left(\frac{\tau C}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{2(r-1)(\gamma^{r/2}-\gamma)^{\frac{r-2}{2(r-1)}} (\gamma-1)^{\frac{1}{r-1}}}{r(\gamma^{r/2-1}-1)^{\frac{r-2}{2(r-1)}}}.$$

It is obvious that $W(\gamma, \varepsilon) \geq W_0^{(M)}(\gamma, \varepsilon)$ for

$$(31) \quad M \geq 1 + \frac{2}{(r-2) \log \gamma} \log \left(\frac{2(r-1)(\gamma-1)}{r(\gamma^{r/2}-1)} \right).$$

Consider the ratio

$$(32) \quad \begin{aligned} \frac{W_k^{(M)}(\gamma, \varepsilon)}{W(\gamma, \varepsilon)} &= \exp \left\{ - \left(\frac{\tau C \varepsilon}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \right. \\ &\quad \times \frac{(\gamma^{r/2} - \gamma)^{\frac{r-2}{2(r-1)}} (\gamma - 1)^{\frac{1}{r-1}} 2(r-1)}{(\gamma^{r/2} - 1)^{\frac{r}{2(r-1)}} r} \\ &\quad \left. \times \left(\frac{r}{2(r-1)} \gamma^{(M-k)\frac{r-2}{2(r-1)}} + \frac{r-2}{2(r-1)} \gamma^{(k-M)\frac{r}{2(r-1)}} - 1 \right) \right\} \\ &= W(\gamma, \varepsilon) S_k^{(M)}(\gamma), \end{aligned}$$

where

$$(33) \quad S_k^{(M)}(\gamma) = \frac{r}{2(r-1)} \gamma^{\frac{r-2}{2(r-1)}(M-k)} + \frac{r-2}{2(r-1)} \gamma^{\frac{r}{2(r-1)}(k-M)} - 1.$$

It is easy to see that $S_k^{(M)}(\gamma) > 0$ for all $\gamma > 1$ and $k \geq 1$.

Using the known inequalities $e^x \geq 1 + x + \frac{x^2}{2}$ and $e^{-x} \geq 1 - x$ for $x \geq 0$, we obtain

$$(34) \quad S_k^{(M)}(\gamma) \geq \underline{S}_k(\gamma) = \frac{1}{2} \left(\frac{(r-2) \log \gamma}{2(r-1)} \right)^2 (M-k)^2, \quad k < M,$$

$$(35) \quad S_k^{(M)}(\gamma) \geq \overline{S}_k(\gamma) = \frac{1}{2} \left(\frac{r \log \gamma}{2(r-1)} \right)^2 (k-M)^2, \quad k > M.$$

Consider the following sums:

$$(36) \quad S_1(\gamma, \varepsilon) = \sum_{k=1}^{M-1} \gamma^{-\frac{rk}{r-1}} W(\gamma, \varepsilon) \underline{S}_k(\gamma),$$

$$(37) \quad S_2(\gamma, \varepsilon) = \sum_{k=M+1}^{\infty} \gamma^{\frac{rk}{r-1}} W(\gamma, \varepsilon) \overline{S}_k(\gamma).$$

Since $W_k^{(M)}(\gamma, \varepsilon) \leq W(\gamma, \varepsilon)$ for all $k \geq 0$, relations (32)–(35) imply that $S_1(\gamma, \varepsilon) < \infty$ and $S_2(\gamma, \varepsilon) < \infty$ for all $\gamma > 1$.

Consider the other factor on the right hand side of (22). Substituting α defined by (26) we get

$$\begin{aligned}
 K_k(\gamma, p, \varepsilon) &= K_{a_k, b_k}(\gamma, p, \varepsilon) \\
 &= \exp \left\{ p \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{C(b_k - a_k)}{b_k^{\frac{r}{2}} - a_k^{\frac{r}{2}}} \right)^{\frac{1}{r-1}} \right. \\
 &\quad \times \left(\varepsilon + C a_k b_k \frac{b_k^{\frac{r}{2}-1} - a_k^{\frac{r}{2}-1}}{b_k^{\frac{r}{2}} - a_k^{\frac{r}{2}}} + \frac{C(b_k - a_k)^{\frac{r}{2}+1}}{2^{\frac{r}{2}}(b_k^{\frac{r}{2}} - a_k^{\frac{r}{2}})} + \frac{C(b_k - a_k)p}{2(1+p)} \right) \left. \right\} \\
 &= \exp \left\{ p \varepsilon^{\frac{r}{2(r-1)}} \left(\frac{\tau C}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \right. \\
 &\quad \times \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}} (\gamma - 1)^{\frac{1}{r-1}}}{(\gamma^{\frac{r}{2}} - 1)^{\frac{r}{2(r-1)}} \gamma^{\frac{(r-2)(k-M-1)}{2(r-1)}}} \\
 &\quad \times \left(1 + \frac{r-2}{r} \gamma^{k-M} + \frac{(r-2)(\gamma-1)^{\frac{r}{2}+1} \gamma^{k-M-1}}{2^{\frac{r}{2}} r (\gamma^{\frac{r}{2}-1} - 1)} \right. \\
 &\quad \left. \left. + \frac{(r-2)(\gamma-1)(\gamma^{\frac{r}{2}} - 1) \gamma^{k-M-1} p}{2r(\gamma^{\frac{r}{2}-1} - 1)(1+p)} \right) \right\} \\
 &= \exp \left\{ p \varepsilon^{\frac{r}{2(r-1)}} \left(\frac{\tau C}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \right. \\
 &\quad \times \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}} (\gamma - 1)^{\frac{1}{r-1}}}{(\gamma^{\frac{r}{2}} - 1)^{\frac{r}{2(r-1)}}} \\
 &\quad \times \left(\gamma^{\frac{(r-2)(M-k+1)}{2(r-1)}} + \frac{r-2}{r} \gamma^{\frac{r(k-M-1)}{2(r-1)}} \right. \\
 &\quad \left. + \frac{(r-2)(\gamma-1)^{\frac{r}{2}+1}}{2^{\frac{r}{2}} r (\gamma^{\frac{r}{2}-1} - 1)} \gamma^{\frac{r(k-M-1)}{2(r-1)}} \right. \\
 &\quad \left. \left. + \frac{(r-2)(\gamma-1)(\gamma^{\frac{r}{2}} - 1)p}{2r(\gamma^{\frac{r}{2}-1} - 1)(1+p)} \gamma^{\frac{r(k-M-1)}{2(r-1)}} \right) \right\}
 \end{aligned} \tag{38}$$

for all $k \geq 1$, and

$$\begin{aligned}
 K_0(\gamma, p, \varepsilon) &= K_{a_0, b_0}(\gamma, p, \varepsilon) \\
 &= \exp \left\{ p \varepsilon^{\frac{r}{2(r-1)}} \left(\frac{\tau C}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}}}{(\gamma^{\frac{r}{2}} - 1)^{\frac{r-2}{2(r-1)}}} \gamma^{\frac{(r-2)M}{2(r-1)}} \right. \\
 &\quad \left. \times \left(1 + \frac{(r-2)(\gamma^{\frac{r}{2}} - 1)}{r(\gamma^{\frac{r}{2}-1} - 1)\gamma^M} + \frac{(r-2)(\gamma^{\frac{r}{2}} - 1)p}{2r(\gamma^{\frac{r}{2}-1} - 1)(1+p)\gamma^M} \right) \right\}
 \end{aligned} \tag{39}$$

for $k = 0$. For any interval $[a_k, b_k]$, $k \geq 0$, put

$$p = p_k = \frac{\zeta^{\frac{r}{2(r-1)}}}{\varepsilon^{\frac{r}{2(r-1)}} \gamma^{\frac{r(k-M-1)}{2(r-1)}}}, \tag{40}$$

where ζ is some constant such that $p_k < 1$ for all intervals, that is,

$$\zeta \in (0, \varepsilon/\gamma^{M+1}).$$

Since $p/(1+p) < \frac{1}{2}$ for all $p \in (0, 1)$, we obtain

$$\begin{aligned}
 &K_k(\gamma, p_k, \varepsilon) \\
 (41) \quad &= \exp \left\{ \left(\frac{\tau C \zeta}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}} (\gamma - 1)^{\frac{1}{r-1}}}{(\gamma^{\frac{r}{2}} - 1)^{\frac{r}{2(r-1)}}} \right. \\
 &\quad \left. \times \left(\gamma^{M-k+1} + \frac{r-2}{r} + \frac{(r-2)(\gamma-1)^{\frac{r}{2}+1}}{2^{\frac{r}{2}} r (\gamma^{\frac{r}{2}-1} - 1)} + \frac{(r-2)(\gamma-1)(\gamma^{\frac{r}{2}} - 1)}{4r(\gamma^{\frac{r}{2}-1} - 1)} \right) \right\}.
 \end{aligned}$$

It is obvious that $K_k(\gamma, p_k, \varepsilon) \leq K_1(\gamma, p_1, \varepsilon) = K_1(\gamma)$ for all $k \geq 1$. Similarly

$$\begin{aligned}
 &K_0(\gamma, p_0, \varepsilon) = K_0(\gamma) \\
 (42) \quad &= \exp \left\{ \left(\frac{\tau C \zeta}{2} \right)^{\frac{r}{2(r-1)}} \left(\frac{r}{r-2} \right)^{\frac{r-2}{2(r-1)}} \frac{(\gamma^{\frac{r}{2}-1} - 1)^{\frac{r-2}{2(r-1)}}}{(\gamma^{\frac{r}{2}} - 1)^{\frac{r-2}{2(r-1)}}} \gamma^{\frac{r}{2(r-1)}} \right. \\
 &\quad \left. \times \left(\gamma^M + \frac{(r-2)(\gamma^{\frac{r}{2}} - 1)}{r(\gamma^{\frac{r}{2}-1} - 1)} + \frac{(r-2)(\gamma^{\frac{r}{2}} - 1)}{4r(\gamma^{\frac{r}{2}-1} - 1)} \right) \right\}.
 \end{aligned}$$

Condition (20) holds if $\lambda^* > \lambda_0$ for all intervals; that is,

$$(43) \quad (1-p) \left(\frac{C(b_k - a_k)}{b_k^{r/2} - a_k^{r/2}} \right)^{\frac{1}{r-1}} \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \geq (1-p) \max \left\{ \frac{\tau^{1/2}}{(b_k - a_k)^{1/2}}; \left(\frac{\tau}{2v} \right)^{1/2} \right\},$$

where

$$v = \begin{cases} \alpha, & \text{for } k = 0, \\ a_k, & \text{for } k \geq 1. \end{cases}$$

It is easy to check that condition (43) is equivalent to

$$(44) \quad \varepsilon \geq \frac{2rC^{(r-2)/r} \gamma^M (\gamma^{r/2-1} - 1)}{\tau^{1/r} (r-2) (\gamma^{r/2} - 1)}$$

for the interval $[a_0, b_0] = [0, \alpha]$. It follows from (43) that

$$\begin{aligned}
 &\left(\frac{C(\gamma-1)(\alpha\gamma^{k-1})^{1-\frac{r}{2}}}{\gamma^{\frac{r}{2}} - 1} \right)^{\frac{1}{r-1}} \left(\frac{\tau}{2} \right)^{\frac{r}{2(r-1)}} \geq \tau^{1/2} \max \left\{ ((\gamma-1)\alpha\gamma^{k-1})^{-\frac{1}{2}}; (2\alpha\gamma^{k-1})^{-\frac{1}{2}} \right\}, \\
 (45) \quad &\left(\frac{C(\gamma-1)}{\gamma^{\frac{r}{2}} - 1} \right)^{\frac{1}{r-1}} \frac{\tau^{\frac{1}{2(r-1)}}}{2^{\frac{1}{2(r-1)}}} (\alpha\gamma^{k-1})^{\frac{1}{2(r-1)}} \geq \max \left\{ (\gamma-1)^{-\frac{1}{2}}; 2^{-\frac{1}{2}} \right\}
 \end{aligned}$$

for the intervals $[a_k, b_k] = [\alpha\gamma^{k-1}, \alpha\gamma^k]$, $k \geq 1$. If inequality (45) holds for $k = 1$, it holds for $k > 1$, too. Thus (26) implies that

$$(46) \quad \varepsilon \geq \frac{2r r \gamma^M (\gamma^{r/2-1} - 1) (\gamma^{r/2} - 1) \max \{ (\gamma-1)^{1-r}; 2^{1-r} \}}{\tau C (r-2) (\gamma-1)^2}.$$

Inequality (9) follows from (44) and (46). Therefore

$$\begin{aligned}
 (47) \quad & \mathbb{P} \left\{ \sup_{t \geq 0} (A(t) - ct) > x \right\} = \mathbb{P} \left\{ \sup_{t \geq 0} (Y(t) - Ct) > \varepsilon \right\} \\
 & \leq \sum_{k \geq 0} \mathbb{P} \left\{ \sup_{t \in [a_k, b_k]} (Y(t) - Ct) > \varepsilon \right\} = \sum_{k \geq 0} \frac{2e^2}{p_k^2} W_k^{(M)}(\gamma, \varepsilon) K_k(\gamma, p_k, \varepsilon) \\
 & \leq \varepsilon^{\frac{r}{r-1}} 2e^2 \zeta^{\frac{-r}{r-1}} \left(\gamma^{\frac{-r(M+1)}{r-1}} W_0^{(M)}(\gamma, \varepsilon) K_0(\gamma) \right. \\
 & \quad \left. + \sum_{k=1}^{M-1} \gamma^{\frac{r(k-M-1)}{r-1}} W_k^{(M)}(\gamma, \varepsilon) K_1(\gamma) \right. \\
 & \quad \left. + W(\gamma, \varepsilon) K_1(\gamma) + \sum_{k=M+1}^{\infty} \gamma^{\frac{r(k-M-1)}{r-1}} W_k^{(M)}(\gamma, \varepsilon) K_1(\gamma) \right) \\
 & \leq \varepsilon^{\frac{r}{r-1}} 2e^2 \zeta^{\frac{-r}{1-r}} \gamma^{\frac{r(M+1)}{1-r}} W(\gamma, \varepsilon) \\
 & \quad \times \left(K_0(\gamma) + K_1(\gamma) \left(\sum_{k=1}^{M-1} \gamma^{\frac{rk}{r-1}} W(\gamma, \varepsilon) S_k(\gamma) + 1 + \sum_{k=M+1}^{\infty} \gamma^{\frac{rk}{r-1}} W(\gamma, \varepsilon) \bar{S}_k(\gamma) \right) \right) \\
 & = \varepsilon^{\frac{r}{r-1}} W(\gamma, \varepsilon) 2e^2 \zeta^{\frac{-r}{1-r}} \gamma^{\frac{r(M+1)}{1-r}} (K_0(\gamma) + K_1(\gamma)(1 + S_1(\gamma, \varepsilon) + S_2(\gamma, \varepsilon))). \quad \square
 \end{aligned}$$

Theorem 3.3. Let $Y(t) = \int_0^t X(u) du$, $t \geq 0$, where $X(u)$ is a strictly φ -sub-Gaussian generalized Ornstein–Uhlenbeck stochastic process with parameter $\tau > 0$ and $\varphi(u) = u^2$. Then

$$(48) \quad \mathbb{P} \left\{ \sup_{t \geq 0} (Y(t) - Ct) > \varepsilon \right\} \leq \frac{2e^2 \varepsilon^2}{\zeta^2} \exp \left\{ -\frac{\tau C \varepsilon}{2} \right\} \left(\tilde{K}_0(\zeta, \gamma) + \tilde{K}_1(\zeta, \gamma) S(\gamma, \varepsilon) \right)$$

for all $\gamma > 1$, $\varepsilon > 0$, and $\zeta \in (0, \varepsilon \gamma^{-1})$, where

$$(49) \quad \tilde{K}_0(\zeta, \gamma) = \exp \left\{ \frac{3\zeta \tau C \gamma}{8} \right\},$$

$$(50) \quad \tilde{K}_1(\zeta, \gamma) = \exp \left\{ \frac{\zeta \tau}{8} (\chi_{\{\gamma < 3\}}(6C + \gamma - 1) + \chi_{\{\gamma \geq 3\}}((C + 1)(\gamma - 1) + 4C)) \right\},$$

$$(51) \quad S(\gamma, \varepsilon) = \sum_{k \geq 1} \gamma^{2(k-1)} \exp \left\{ -\frac{\tau C \varepsilon}{8} (\gamma^{k-1} - 2 + \gamma^{1-k}) \right\} < \infty.$$

Proof. The bound

$$(52) \quad \mathbb{P} \left\{ \sup_{t \in [a, b]} (A(t) - ct) > x \right\} \leq \frac{2e^2}{p^2} \inf_{\lambda > 0} Z(\lambda, p),$$

follows from Theorem 3.2 with $\beta = ((b - a)/\tau)^{1/2}$, where

$$(53) \quad Z(\lambda, p) = \exp \left\{ \theta(\lambda, p) + \frac{\lambda^2 p(b - a)}{\tau(1 - p)^2} - \lambda \left(\varepsilon - \frac{C\tau(b - a)p^2}{2(1 - p^2)} \right) \right\},$$

$$(54) \quad \theta(\lambda, p) = \sup_{u \in [a, b]} \left(\frac{\lambda^2 2u}{\tau(1 - p)} - \lambda C u \right).$$

It is obvious that $\theta(\lambda, p) = \lambda^2 2a/(\tau(1 - p)) - \lambda Ca$ for $\lambda \leq \lambda^* = (1 - p)\tau C/2$.

The function $Z(\lambda, p)$ attains its minimum at the point

$$\tilde{\lambda} = \frac{Ca + \varepsilon - \frac{Cp^2(b-a)}{2(1-p^2)}}{2\left(\frac{2a}{\tau(1-p)} + \frac{p(b-a)}{\tau(1-p)^2}\right)},$$

whence we obtain

$$\begin{aligned} \inf_{\lambda > 0} Z(\lambda, p) &= \exp \left\{ -\frac{\left(Ca + \varepsilon - \frac{Cp^2(b-a)}{2(1-p^2)}\right)^2}{4\left(\frac{2a}{\tau(1-p)} + \frac{p(b-a)}{\tau(1-p)^2}\right)} \right\} \\ &= \exp \left\{ -\frac{\tau(Ca + \varepsilon)^2}{4\left(2a + \frac{p(b-a)}{1-p}\right)} \right\} \\ (55) \quad &\times \exp \left\{ p\tau \frac{(Ca + \varepsilon)^2 + \frac{p(Ca + \varepsilon)(b-a)}{1+p} - \frac{C^2p^3(b-a)^2}{4(1-p^2)(1+p)}}{4\left(2a + \frac{p(b-a)}{1-p}\right)} \right\} \\ &\leq \exp \left\{ -\frac{\tau(Ca + \varepsilon)^2}{8a} \right\} \\ &\times \exp \left\{ p\tau \left(\frac{(Ca + \varepsilon)^2(b-a)}{8a(1-p)\left(2a + \frac{p(b-a)}{1-p}\right)} + \frac{(Ca + \varepsilon)^2 + \frac{(Ca + \varepsilon)(b-a)}{2}}{4\left(2a + \frac{p(b-a)}{1-p}\right)} \right) \right\} \\ &= \Gamma_{a,b}(\varepsilon)K_{a,b}(p, \varepsilon) \end{aligned}$$

if $\tilde{\lambda} \leq \lambda^*$ or, equivalently, if

$$(56) \quad \varepsilon \leq Ca + \frac{C(b-a)p}{1-p} + \frac{Cp^2(b-a)}{2(1-p^2)}.$$

Since $\Gamma_{a,b}(\varepsilon)$ attains the maximum at the point $a = \varepsilon/C$, we consider the partition $[0, \infty) = \bigcup_{k=0}^{\infty} [a_k, b_k]$, where $[a_0, b_0] = [0, \varepsilon/C]$, $[a_k, b_k] = [\varepsilon\gamma^{k-1}/C, \varepsilon\gamma^k/C]$, $k \geq 1$, and $\gamma > 1$. Condition (56) holds for any interval of this partition. Estimate (22) holds for $k = 0$:

$$\begin{aligned} \inf_{\lambda > 0} Z(\lambda, p) &\leq Z(\lambda^*, p) \\ (57) \quad &= \exp \left\{ -\frac{\tau C \varepsilon}{2} \right\} \exp \left\{ \frac{p\tau C}{2} \left(\varepsilon - \frac{C(b_0 - a_0)}{2} + \frac{C(b_0 - a_0)p}{2(1+p)} \right) \right\} \\ &= \exp \left\{ -\frac{\tau C \varepsilon}{2} \right\} \exp \left\{ \frac{p\tau C \varepsilon}{4} \left(1 + \frac{p}{1+p} \right) \right\} \leq \exp \left\{ -\frac{\tau C \varepsilon}{2} \right\} \exp \left\{ \frac{3p\tau C \varepsilon}{8} \right\} \\ &= \Gamma(\varepsilon)K_0(p, \varepsilon). \end{aligned}$$

Put

$$\Gamma_k(\gamma, \varepsilon) = \Gamma_{a_k, b_k}(\varepsilon).$$

Then

$$\begin{aligned} (58) \quad \frac{\Gamma_k(\gamma, \varepsilon)}{\Gamma(\varepsilon)} &= \exp \left\{ -\frac{\tau C \varepsilon}{2} \left(\frac{(\gamma^{k-1} + 1)^2}{4\gamma^{k-1}} - 1 \right) \right\} = \exp \left\{ -\frac{\tau C \varepsilon}{2} \frac{(\gamma^{k-1} - 1)^2}{4\gamma^{k-1}} \right\} \\ &= \exp \left\{ -\frac{\tau C \varepsilon}{8} (\gamma^{k-1} - 2 + \gamma^{1-k}) \right\}, \end{aligned}$$

$$(59) \quad S(\gamma, \varepsilon) = \sum_{k \geq 1} \gamma^{2(k-1)} \exp \left\{ -\frac{\tau C \varepsilon}{8} (\gamma^{k-1} - 2 + \gamma^{1-k}) \right\} < \infty$$

for $k \geq 1$ and all $\gamma > 1$ and $\varepsilon > 0$. Further,

$$(60) \quad \begin{aligned} K_k(\gamma, p, \varepsilon) &= K_{a_k, b_k}(p, \varepsilon) \\ &= \exp \left\{ \frac{p\varepsilon\tau}{8\gamma^{k-1}} \left(\frac{C(\gamma^{k-1} + 1)^2(\gamma - 1)}{2(1-p) + p(\gamma - 1)} \right. \right. \\ &\quad \left. \left. + \frac{2C(\gamma^{k-1} + 1)^2 + (\gamma^{k-1} + 1)\gamma^{k-1}(\gamma - 1)}{2 + \frac{p(\gamma-1)}{1-p}} \right) \right\}. \end{aligned}$$

Let

$$p = p_k = \zeta / (\varepsilon\gamma^{k-1}) < 1$$

for all $k \geq 0$, that is, $\zeta \in (0, \varepsilon\gamma^{-1})$. Then

$$(61) \quad \begin{aligned} &K_k(\gamma, p_k, \varepsilon) \\ &\leq \exp \left\{ \frac{\zeta\tau}{8} \left(\frac{C(\gamma^{1-k} + 1)(\gamma - 1)}{2 + \frac{\zeta(\gamma-3)}{\varepsilon\gamma^{k-1}}} + \frac{2C(\gamma^{1-k} + 1)^2 + (\gamma^{1-k} + 1)(\gamma - 1)}{2} \right) \right\} \\ &\leq \exp \left\{ \frac{\zeta\tau}{8} \left(\frac{2C(\gamma - 1)}{2 + \frac{\zeta(\gamma-3)}{\varepsilon\gamma^{k-1}}} + \frac{8C + 2(\gamma - 1)}{2} \right) \right\}. \end{aligned}$$

Thus

$$(62) \quad K_k(\gamma, p_k, \varepsilon) \leq \exp \left\{ \frac{\zeta\tau}{8} ((C + 1)(\gamma - 1) + 4C) \right\}$$

for $\gamma \geq 3$. If $1 < \gamma < 3$, then $\zeta(3 - \gamma)(\varepsilon\gamma^{k-1}) < 3 - \gamma$ and

$$(63) \quad K_k(\gamma, p_k, \varepsilon) \leq \exp \left\{ \frac{\zeta\tau}{8} (6C + \gamma - 1) \right\}.$$

Put

$$(64) \quad \tilde{K}_1(\zeta, \gamma) = \exp \left\{ \frac{\zeta\tau}{8} (\chi_{\{\gamma < 3\}}(6C + \gamma - 1) + \chi_{\{\gamma \geq 3\}}((C + 1)(\gamma - 1) + 4C)) \right\}.$$

Now

$$(65) \quad K_0(p_0, \varepsilon) \leq \exp \left\{ \frac{3\zeta\tau C\gamma}{8} \right\} = \tilde{K}_0(\zeta, \gamma),$$

and therefore

$$(66) \quad \begin{aligned} &\mathbb{P} \left\{ \sup_{t>0} (Y(t) - Ct) > \varepsilon \right\} \leq \sum_{k \geq 0} \mathbb{P} \left\{ \sup_{t \in [a_k, b_k]} (Y(t) - Ct) > \varepsilon \right\} \\ &= \sum_{k \geq 0} \frac{2e^2\varepsilon^2\gamma^{2(k-1)}}{\zeta^2} \Gamma_k(\gamma, \varepsilon) K_k(\gamma, p_k, \varepsilon) \\ &= \frac{2e^2\varepsilon^2}{\zeta^2} \Gamma(\varepsilon) \left(K_0(p_0, \varepsilon) + \sum_{k \geq 1} \gamma^{2(k-1)} \frac{\Gamma_k(\gamma, \varepsilon)}{\Gamma(\varepsilon)} K_k(\gamma, p_k, \varepsilon) \right) \\ &\leq \frac{2e^2\varepsilon^2}{\zeta^2} \Gamma(\varepsilon) \left(\tilde{K}_0(\zeta, \gamma) + \tilde{K}_1(\zeta, \gamma) S(\gamma, \varepsilon) \right). \quad \square \end{aligned}$$

Remark 3.2. A similar problem is considered for the Gaussian Ornstein–Uhlenbeck stochastic process in a number of papers (see, for example, [1, 10]). In particular,

$$(67) \quad \mathbb{P} \left\{ \sup_{t>0} (Y(t) - Ct) > \varepsilon \right\} \sim \exp \{-I(\varepsilon)\}$$

for the model studied in the current paper with $\tau = 1$ and sufficiently large ε , where

$$(68) \quad I(\varepsilon) = \inf_{t>0} \frac{(\varepsilon + Ct)^2}{2\sigma_t^2},$$

$$\sigma_t^2 = 2(t - 1 + e^{-t}).$$

It is obvious that the solution of problem (68) is $I(\varepsilon) = \kappa\varepsilon$, where $\kappa > 0$ is a constant. Moreover the estimate obtained in Theorem 3.3 implies that $\kappa > C/2$ for a φ -sub-Gaussian Ornstein–Uhlenbeck stochastic process with $\varphi(u) = u^2$, since

$$I(\varepsilon) \geq \inf_{t>0} \frac{(\varepsilon + Ct)^2}{4t} = C\varepsilon.$$

This means that our result obtained for the class of φ -sub-Gaussian stochastic processes (including the class of Gaussian stochastic processes) coincides, up to a multiplicative factor, with the corresponding result for the Gaussian case.

4. CONCLUDING REMARKS

A model for the queueing system is considered in the paper. The queue length is continuous and the input is a strictly φ -sub-Gaussian generalized Ornstein–Uhlenbeck stochastic process defined on the positive semiaxis. The classes of φ -sub-Gaussian and strictly φ -sub-Gaussian stochastic processes include the class of Gaussian processes and can be used when modeling stochastic processes. We obtained an estimate for the probability that the queue length exceeds the maximum for this model. Similar results can be obtained for other types of stochastic processes.

BIBLIOGRAPHY

1. D. D. Botvich and N. G. Duffield, *Large deviations, the shape of the loss curve, and economies of scale in large multiplexers*, Queueing Systems Theory Appl. **20** (1995), 293–320. MR1356879
2. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, Translation of Math. Monographs, vol. 188, American Mathematical Society, Providence, RI, 2000. MR1743716 (2001g:60089)
3. Yu. V. Kozachenko and E. I. Ostrovskii, *Banach spaces of random variables of sub-Gaussian type*, Teor. Veroyatn. i Matem. Stat. **32** (1985), 42–53; English transl. in Theory Probab. and Math. Statist. **32** (1986), 45–56. MR882158 (88e:60009)
4. Yu. V. Kozachenko and Yu. A. Kovalchuk, *Boundary value problems with random initial conditions, and functional series from $\text{Sub}_\varphi(\Omega)$* . I, Ukrain. Mat. Zh. **50** (1998), no. 4, 504–515; English transl. in Ukrainian Math. J. **50** (1999), no. 4, 572–585. MR1698149 (2000f:60029)
5. Yu. Kozachenko, T. Sottinen, and O. Vasylyk, *Self-similar processes with stationary increments in the spaces $\text{SSub}_\varphi(\Omega)$* , Teor. ĭmovir. Mat. Stat. **65** (2001), 67–78; English transl. in Theory Probab. Math. Statist. **65** (2002), 77–88. MR1936131 (2004a:60078)
6. Yu. V. Kozachenko and O. I. Vasylyk, *On the distribution of suprema of $\text{Sub}_\varphi(\Omega)$ random processes*, Theory Stoch. Process. **4(20)** (1998), no. 1–2, 147–160. MR2026624 (2004k:60094)
7. Yu. Kozachenko, O. Vasylyk, and R. Yamnenko, *On the probability of exceeding some curve by φ -sub-Gaussian random process*, Theory Stoch. Process. **9(25)** (2003), no. 3–4, 70–80.
8. Yu. Kozachenko, O. Vasylyk, and R. Yamnenko, *Upper estimate of overrunning by $\text{Sub}_\varphi(\Omega)$ random process the level specified by continuous function*, Random Oper. Stochastic Equations **13** (2005), no. 2, 111–128. MR2152102 (2006b:60207)
9. M. A. Krasnosel'skiĭ and Ya. B. Rutitskiĭ, *Convex Functions in the Orlicz spaces*, Fizmatgiz, Moscow, 1958; English transl., P. Noordhoff Ltd., Groningen, 1961. MR0126722 (23:A4016)

10. Q. Ren and H. Kobayashi, *Diffusion approximation modeling for Markov modulated bursty traffic and its applications to bandwidth allocation in ATM networks*, IEEE Journal on Selected Areas in Communications **16** (1998), no. 5, 679–691.
11. R. E. Yamnenko, *On the probability of exceeding some curve by trajectories of φ -sub-Gaussian stochastic process*, Visnyk Kyiv. Univ. Ser. Fiz.-Mat. Nauk **3** (2004), 82–86.

DEPARTMENT OF PROBABILITY THEORY AND MATHEMATICAL STATISTICS, FACULTY FOR MECHANICS AND MATHEMATICS, NATIONAL TARAS SHEVCHENKO UNIVERSITY, ACADEMICIAN GLUSHKOV AVENUE 6, KIEV 03127, UKRAINE

E-mail address: `rostyslav_yamnenko@yahoo.com`

Received 26/DEC/2004

Translated by S. KVASKO