ASYMPTOTIC ANALYSIS OF A MEASURE OF VARIATION

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Abstract. Let $X_i$, $i = 1, \ldots, n$, be a sequence of positive independent identically distributed random variables and define

$$T_n := \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{(X_1 + X_2 + \cdots + X_n)^2}.$$

Utilizing Karamata’s theory of functions of regular variation, we determine the asymptotic behaviour of arbitrary moments $E(T_n^k)$, $k \in \mathbb{N}$, for large $n$, given that $X_i$ satisfies a tail condition, akin to the domain of attraction condition from extreme value theory. As a by-product, the paper offers a new method for estimating the extreme value index of Pareto-type tails.

1. Introduction

Let $X_i$, $i = 1, \ldots, n$, be a sequence of positive independent identically distributed (i.i.d.) random variables with distribution function $F$ and define

$$T_n := \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{(X_1 + X_2 + \cdots + X_n)^2}.$$

The asymptotic behaviour of $E(T_n)$ was investigated in [5], simplifying and generalizing earlier results in [4] and [6].

In this paper we extend several results of [5] and derive the limiting behaviour of arbitrary moments $E(T_n^k)$, $k \in \mathbb{N}$.

This is achieved by using an integral representation of $E(T_n^k)$ in terms of the Laplace transform of $X_1$, which is derived in Section 2.

Most of our results will be derived under the condition that $X_1$ satisfies

$$1 - F(x) \sim x^{-\alpha}f(x), \quad x \uparrow \infty,$$

where $\alpha > 0$ and $f(x)$ is slowly varying, i.e.

$$\lim_{x \to \infty} \frac{\ell(tx)}{\ell(x)} = 1$$

for all $t > 0$; see e.g. [3]. It is well known that condition (2) appears as the essential condition in the domain of attraction problem of extreme value theory. For a recent treatment, see [2]. A distribution satisfying (2) is called of Pareto-type with index $\alpha$.

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When $\alpha < 2$, then the condition coincides with the domain of attraction condition for weak convergence to a nonnormal stable law. It is then obvious that for $\beta > 0$,

\begin{equation}
E(X_1^\beta) := \mu_\beta = \beta \int_0^\infty x^{\beta-1}(1 - F(x)) \, dx \leq \infty
\end{equation}

will be finite if $\beta < \alpha$ but infinite whenever $\beta > \alpha$. For convenience, we define $\mu_0 := 1$ and $\mu := \mu_1$.

The results of this paper are based on the theory of functions of regular variation (see e.g. [3]). Clearly, if $E(\beta) > 0$, both the numerator and the denominator in (1) will exhibit an erratic behaviour, whereas for $E(X_1) < \infty$ and $E(X_1^2) = \infty$, this is the case only for the numerator. The results in Section 3 quantify this effect.

As a by-product, the results of this paper suggest a new method for estimating the extreme value index of Pareto-type distributions from a data set of observations, which is discussed in Section 4.

The quantity $T_n$ is a basic ingredient in the study of the sample coefficient of variation of a given set of independent observations $X_1, \ldots, X_n$ from a random variable $X$, which is a frequently used risk measure in practical applications. In [1], this connection will be used to derive asymptotic properties of the sample coefficient of variation, including a distributional approach.

2. Preliminaries

Let $\varphi(s) := E(e^{-sX_1}) = \int_0^\infty e^{-sx} \, dF(x)$, $s \geq 0$, denote the Laplace transform of $X_1$. Then, following an idea of [3], one can use the identity

\begin{equation}
\frac{1}{x^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-sx}s^{\beta-1} \, ds, \quad \beta > 0,
\end{equation}

and Fubini’s theorem to deduce that

\begin{equation}
E \frac{1}{X_1^\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1}\varphi(s) \, ds.
\end{equation}

More generally, for i.i.d. random variables $X_1, \ldots, X_n$, one obtains the representation

\begin{equation}
E \frac{\prod_{i=1}^n X_i^{k_i}}{(X_1 + X_2 + \cdots + X_n)^\beta} = \frac{(-1)^{k_1+\cdots+k_n}}{\Gamma(\beta)} \int_0^\infty s^{\beta-1} \prod_{i=1}^n \frac{\partial^{k_i} \varphi(s)}{\partial s^{k_i}} \, ds,
\end{equation}

for nonnegative integers $k_i$, $i = 1, \ldots, n$.

In particular, by symmetry,

\begin{equation}
E(T_n) = E \frac{X_1^2 + X_2^2 + \cdots + X_n^2}{(X_1 + X_2 + \cdots + X_n)^2} = n \int_0^\infty s\varphi''(s)\varphi^{n-1}(s) \, ds,
\end{equation}

which formed the basis for the analysis in [5]. The representation [5] can be generalized in the following way:

**Lemma 2.1.** For an arbitrary positive integer $k$,

\begin{equation}
E(T_n^k) = \sum_{r=1}^k \sum_{k_1,\ldots,k_r \geq 1 \atop k_1 + \cdots + k_r = k} \frac{k!}{k_1! \cdots k_r!} B(n, k_1, \ldots, k_r)
\end{equation}

with

\begin{equation}
B(n, k_1, \ldots, k_r) = \frac{(n_r)}{\Gamma(2k)} \int_0^\infty s^{2k-1}\varphi(2k_1)(s) \cdots \varphi(2k_r)(s)\varphi^{n-r}(s) \, ds.
\end{equation}
Proof. For an arbitrary positive integer \( k \) we have
\[
E(T_n^k) = E\left(\frac{X_1^k + X_2^k + \cdots + X_n^k}{(X_1 + X_2 + \cdots + X_n)^{2k}}\right) = \sum_{k_1,\ldots,k_n \geq 0 \atop k_1 + \cdots + k_n = k} \frac{k!}{k_1! \cdots k_n!} E\left(\frac{X_1^{2k_1} X_2^{2k_2} \cdots X_n^{2k_n}}{(X_1 + X_2 + \cdots + X_n)^{2k}}\right),
\]
where \( k_i \leq k \) are nonnegative integers. Choose an \( n \)-tuple \( (k_1,\ldots,k_n) \) in the above sum and let \( r \) denote the number of its nonzero elements \( (k_i,\ldots,k_i) \) (clearly \( 1 \leq r \leq k \)). There are exactly \( \binom{n}{r} \) possibilities of extending \( (k_1,\ldots,k_r) \) to an \( n \)-tuple by filling in \( n-r \) zeroes; each of the resulting \( n \)-tuples leads to the same summand in (6). Thus we can write
\[
E(T_n^k) = \sum_{r=1}^{k} \sum_{k_1,\ldots,k_r \geq 1 \atop k_1 + \cdots + k_r = k} \frac{k!}{k_1! \cdots k_r!} E\left(\frac{X_1^{2k_1} X_2^{2k_2} \cdots X_n^{2k_r}}{(X_1 + X_2 + \cdots + X_n)^{2k}}\right) := B(n,k_1,\ldots,k_r)
\]
so that (6) holds in view of (4).

3. Main results

As promised, we will assume in the sequel that \( X_1 \) satisfies condition (2). Recall that when \( \alpha > 1 \), then \( \mu < \infty \) while \( \mu_2 < \infty \) as soon as \( \alpha > 2 \). The finiteness of \( \mu \) and/or \( \mu_2 \) has its influence on the asymptotic behaviour of the summands that make up the statistic \( T_n \). It is therefore not surprising that our results will be heavily dependent on the range of \( \alpha \). We state a first and general result.

Lemma 3.1. If \( X_1 \) has a regularly varying tail with index \( \alpha > 0 \), i.e.
\[
1 - F(x) \sim x^{-\alpha} \ell(x),
\]
then the asymptotic behaviour of the \( m \)-th derivative of the Laplace transform \( \varphi(s) \) as \( s \downarrow 0 \) is given by
\[
\varphi^{(m)}(s) \sim (-1)^m \alpha \Gamma(m - \alpha) s^{\alpha - m} \ell(1/s), \quad m > \alpha.
\]

Proof. Let \( \chi(s) := \int_0^\infty e^{-sx}(1 - F(x)) \, dx \). Since \( 1 - F(x) \sim x^{-\alpha} \ell(x) \), it follows that for \( k > \alpha - 1 \),
\[
(-1)^k \chi^{(k)}(s) = \int_0^\infty x^k e^{-sx}(1 - F(x)) \, dx \sim \Gamma(k + 1 - \alpha) s^{k-1} \left(1 - F\left(\frac{1}{s}\right)\right) \quad \text{as} \, s \to 0.
\]
Since \( \varphi(s) = 1 - s \chi(s) \), we have for \( m \geq 1 \),
\[
\varphi^{(m)}(s) = -m \chi^{(m-1)}(s) - s \chi^{(m)}(s),
\]
so that for \( m > \alpha \),
\[
\frac{s^m \varphi^{(m)}(s)}{1 - F(1/s)} = -m \frac{s^m \chi^{(m-1)}(s)}{1 - F(1/s)} - \frac{s^{m+1} \chi^{(m)}(s)}{1 - F(1/s)} \sim (-1)^m \left(m \Gamma(m-\alpha) - \Gamma(m+1-\alpha)\right) = (-1)^m \alpha \Gamma(m - \alpha),
\]
from which the assertion follows.

Theorem 3.1. If \( X_1 \) belongs to the domain of attraction of a stable law with index \( \alpha \), \( 0 < \alpha < 1 \), then for all \( k \geq 1 \),
\[
\lim_{n \to \infty} E(T_n^k) = \frac{k!}{\Gamma(2k)} \sum_{r=1}^{k} \frac{\alpha^{r-1}}{r \Gamma(1-\alpha)^r} G(r,k),
\]
where \( G(r, k) \) is the coefficient of \( x^k \) in the polynomial

\[
\sum_{j=1}^{k-r+1} \frac{\Gamma(2j - \alpha)}{j!} x^j. \]

**Proof.** From \( 1 - F(x) \sim x^{-\alpha} \ell(x) \) it follows that \( 1 - \varphi(s) \sim \Gamma(1 - \alpha)s^\alpha \ell(1/s) \) (see e.g. Corollary 8.1.7 in [3]). Moreover, for any sequence \( (a_n)_{n \geq 1} \) with \( a_n \to \infty \) we have

\[
\varphi^n \left( \frac{s}{a_n} \right) = \exp\{n \log(s/a_n)\} \sim \exp\{-n(1 - \varphi(s/a_n))\} \sim \exp\left\{-n \left( \frac{s}{a_n} \right)^\alpha \ell \left( \frac{a_n}{s} \right) \Gamma(1 - \alpha) \right\}. \]

Choose \( (a_n)_{n \geq 1} \) such that

\[
na_n^{-\alpha} \ell(a_n) \Gamma(1 - \alpha) \to 1 \quad \text{as} \quad n \to \infty. \tag{10} \]

Then for all \( s \geq 0 \),

\[
\lim_{n \to \infty} \varphi^n \left( \frac{s}{a_n} \right) = e^{-s^\alpha}. \]

We will now make use of the representation [9] for \( E(T_n^k) \). We have to investigate the asymptotic behaviour of \( B(n, k_1, \ldots, k_r) \). The change of variables \( s = t/a_n \) together with an application of Potter’s theorem [3, Th. 1.5.6], Lebesgue’s dominated convergence theorem and Lemma 3.1 leads to

\[
B(n, k_1, \ldots, k_r) = \frac{\binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left( \frac{t}{a_n} \right)^{2k-1} \varphi(2k_1) \left( \frac{t}{a_n} \right) \cdots \varphi(2k_r) \left( \frac{t}{a_n} \right) \varphi^n \left( \frac{t}{a_n} \right) dt
\]

\[
\sim \frac{\alpha^r \binom{n}{r}}{\Gamma(2k)} \int_0^\infty \left( \frac{t}{a_n} \right)^{2k-1} \left( \frac{t}{a_n} \right)^{r \alpha - 2k} \ell^r \left( \frac{a_n}{t} \right) \left( \prod_{j=1}^r \Gamma(2k_j - \alpha) \right) e^{-t^\alpha} dt
\]

\[
\sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j - \alpha)}{\Gamma(2k)} \frac{n^{r \alpha}}{a_n^{r \alpha}} \int_0^\infty t^{r \alpha - 1} e^{-t^\alpha} dt
\]

\[
\sim \frac{\alpha^{-r-1} \prod_{j=1}^r \Gamma(2k_j - \alpha)}{\Gamma(1-\alpha)r! \Gamma(2k)}. \]

Summing over all \( r = 1, \ldots, k \) in (9), we arrive at

\[
\lim_{n \to \infty} E(T_n^k) = \frac{k!}{(2k - 1)!} \sum_{r=1}^{k} \frac{\alpha^{-r-1}}{r \Gamma(1-\alpha)^r} \sum_{k_1, \ldots, k_r \geq 1 \atop k_1 + \cdots + k_r = k} \prod_{j=1}^r \frac{\Gamma(2k_j - \alpha)}{k_j!}. \tag{11} \]

Now observe that

\[
G(r, k) := \sum_{k_1, \ldots, k_r \geq 1 \atop k_1 + \cdots + k_r = k} \prod_{j=1}^r \frac{\Gamma(2k_j - \alpha)}{k_j!}
\]

can be determined by generating functions. Concretely, if we look at the \( r \)-fold product

\[
\left( \Gamma(2-\alpha)x + \frac{\Gamma(4-\alpha)}{2!}x^2 + \cdots + \frac{\Gamma(2m-\alpha)}{m!}x^m \right)^r
\]
for $m$ sufficiently large, then $G(r, k)$ can be read off as its coefficient of $x^k$, since the $k$th power exactly comprises all contributions of combinations $k_1, \ldots, k_r \geq 1$ with

$$k_1 + \cdots + k_r = k$$

in the above sum. It suffices to choose $m = k - r + 1$, since larger powers do not contribute to the coefficient of $x^k$ any more. Hence Theorem 3.1 follows from (11). □

**Remark 3.1.** For $k = 1$, we obtain $\lim_{n \to \infty} E(T_n) = 1 - \alpha$, which is Theorem 5.3 of [5]. The limit of moments of higher order can now be calculated from (9):

$$\lim_{n \to \infty} E(T_n^2) = \frac{1}{3}(1 - \alpha)(3 - 2\alpha),$$

$$\lim_{n \to \infty} E(T_n^3) = \frac{1}{15}(1 - \alpha)(15 - 17\alpha + 5\alpha^2),$$

$$\lim_{n \to \infty} E(T_n^4) = \frac{1}{105}(1 - \alpha)(105 - 155\alpha + 79\alpha^2 - 14\alpha^3),$$

$$\lim_{n \to \infty} E(T_n^5) = \frac{1}{945}(1 - \alpha)(945 - 1644\alpha + 1106\alpha^2 - 344\alpha^3 + 42\alpha^4).$$

![Figure 1. $\lim_{n \to \infty} E(T_n^k)$ as a function of $\alpha$ ($k = 1, \ldots, 5$ from top to bottom)](image_url)

The following result generalizes Theorem 5.5 of [5], where the case $k = 1$ was covered:

**Theorem 3.2.** If $X_1$ belongs to the domain of attraction of a stable law with index $\alpha = 1$ and $E(X_1) = \infty$, then for all $k \geq 1$,

$$E(T_n^k) \sim \frac{1}{2k - 1} \frac{\ell(a_n)}{\ell(a_n)},$$

where $\ell(x) = \int^x (\ell(t)/t) \, dt$ and $(a_n)_{n \geq 1}$ is a sequence satisfying $a_n \sim n\ell(a_n)$.

**Proof.** Since $X_1$ belongs to the domain of attraction of a stable law with index $\alpha = 1$, we have $1 - F(x) \sim x^{-1}\ell(x)$ for some slowly varying function $\ell(x)$. Moreover

$$1 - \varphi(s) \sim s\ell\left(\frac{1}{s}\right)$$
with \(\ell(x) = \int^x (\ell(t)/t) \, dt\) (see e.g. [3]). Note that \(\tilde{\ell}(x)\) is again a slowly varying function.

For any sequence \((a_n)_{n \geq 1}\) with \(a_n \to \infty\) we have

\[
\varphi^n \left( \frac{s}{a_n} \right) = \exp \{ n \log \varphi(s/a_n) \} \sim \exp \{ -n(1 - \varphi(s/a_n)) \} \sim \exp \left\{ -n \left( \frac{s}{a_n} \right) \tilde{\ell} \left( \frac{a_n}{s} \right) \right\}.
\]

If we choose \(a_n\) such that

\[
(13) \quad na_n^{-1} \tilde{\ell}(a_n) \to 1 \quad \text{as} \quad n \to \infty,
\]

then

\[
(14) \quad \lim_{n \to \infty} \varphi^n \left( \frac{s}{a_n} \right) = e^{-s}.
\]

Take \(a_n\) as in (13) and replace \(s\) by \(t/a_n\) in the representation (6). An application of Potter’s theorem, Lebesgue’s dominated convergence theorem and Lemma 3.1 yields

\[
B(n, k_1, \ldots, k_r) = \frac{\binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left( \frac{t}{a_n} \right)^{2k-1} \varphi^{(2k_1)} \left( \frac{t}{a_n} \right) \cdots \varphi^{(2k_r)} \left( \frac{t}{a_n} \right) \varphi^{n-r} \left( \frac{t}{a_n} \right) \, dt
\]

\[
\sim \frac{\binom{n}{r}}{a_n \Gamma(2k)} \int_0^\infty \left( \frac{t}{a_n} \right)^{2k-1} \left( \frac{t}{a_n} \right)^{r-2k} \tilde{\ell}^{r} \left( \frac{a_n}{t} \right) \left( \prod_{j=1}^r \Gamma(2k_j - 1) \right) e^{-t} \, dt
\]

\[
\sim \frac{\prod_{j=1}^r \Gamma(2k_j - 1)}{r! \Gamma(2k)} \frac{a_n^r \tilde{\ell}^{r}(a_n)}{r \Gamma(2k)} \int_0^\infty t^{r-1} e^{-t} \, dt
\]

\[
\sim \frac{\prod_{j=1}^r \Gamma(2k_j - 1)}{r \Gamma(2k)} \left( \frac{\tilde{\ell}(a_n)}{\tilde{\ell}(a_n)} \right)^r.
\]

Note that \(\ell(a_n)/\tilde{\ell}(a_n) \to 0\) for \(n \to \infty\) and thus, opposed to the case \(\alpha < 1\), only the summand with \(r = 1\) contributes to the dominating asymptotic term of (6). Therefore we obtain

\[
\mathbb{E}(T_n^k) \sim \frac{1}{2k-1} \frac{\ell(a_n)}{\tilde{\ell}(a_n)}.
\]

**Theorem 3.3.** Let \(X_1\) belong to the domain of attraction of a stable law with index \(\alpha\), \(1 \leq \alpha < 2\) and \(\mu := \mathbb{E}(X_1) < \infty\). Then for all \(k \geq 1\),

\[
(15) \quad \mathbb{E} \left( T_n^k \right) \sim \frac{\Gamma(2k - \alpha) \Gamma(1 + \alpha)}{\Gamma(2k) \mu^\alpha} n^{1-\alpha} \ell(n).
\]

**Proof.** Since \(\mu\) is finite, it follows that

\[
(16) \quad \lim_{n \to \infty} \varphi^n(t/n) = e^{-\mu t} \quad \text{for all} \quad t \geq 0.
\]

However, in view of (16), we will use the change of variables \(s = t/n\) in the representation (6). By virtue of Potter’s theorem, Lebesgue’s dominated convergence theorem
and Lemma 3.1 we then obtain
\[ B(n, k_1, \ldots, k_r) = \frac{n!}{n\Gamma(2k)} \int_0^{\infty} \left( \frac{t}{n} \right)^{2k-1} \varphi^{(2k_1)} \left( \frac{t}{n} \right) \cdots \varphi^{(2k_r)} \left( \frac{t}{n} \right) \varphi^{-r} \left( \frac{t}{n} \right) \, dt \]
\[ \sim \frac{\alpha^r \Gamma(n)}{\Gamma(2k)} \int_0^{\infty} \left( \frac{t}{n} \right)^{2k-1} \left( \frac{t}{n} \right)^{r\alpha - 2k} \ell^r \left( \frac{n}{r} \right) \left( \prod_{j=1}^r \Gamma(2k_j - \alpha) \right) e^{-\mu t} \, dt \]
\[ \sim \frac{\alpha^r \prod_{j=1}^r \Gamma(2k_j - \alpha)}{\Gamma(2k)} \frac{\Gamma(n)}{n^{r\alpha}} \int_0^{\infty} t^{r\alpha - 1} e^{-\mu t} \, dt \]
\[ \sim \frac{\alpha^r \Gamma(r\alpha) \prod_{j=1}^r \Gamma(2k_j - \alpha)}{r! \mu^r \Gamma(2k)} n^{(1-\alpha)\ell(n)}. \]

Hence the first-order asymptotic behaviour of (10) is solely determined by the term with \( r = 1 \) and we obtain
\[ \mathbb{E} \left( T_n^k \right) \sim \frac{\Gamma(2 - \alpha)\Gamma(1 + \alpha)}{\Gamma(2k)\mu^\alpha} n^{1-\alpha} \ell(n). \]

\[ \square \]

**Remark 3.2.** For the special case \( k = 1 \), (15) yields
\[ \mathbb{E} \left( T_n \right) \sim \frac{\Gamma(2 - \alpha)\Gamma(1 + \alpha)}{\mu^\alpha} n^{1-\alpha} \ell(n), \]

which is Theorem 5.1 of [3].

We pass to the case \( \alpha > 2 \).

**Theorem 3.4.** Let \( 1 - F(x) \sim x^{-\alpha} \ell(x) \) for some slowly varying function \( \ell(x) \) and \( \alpha > 2 \). Then for all integers \( k < \alpha - 1 \),
\[ \mathbb{E} \left( T_n^k \right) \sim \left( \frac{\mu^2}{\mu^2} \right)^k n^{-k} \]
and for \( k > \alpha - 1 \),
\[ \mathbb{E} \left( T_n^k \right) \sim \frac{\Gamma(2 - \alpha)\Gamma(1 + \alpha)}{\Gamma(2k)\mu^\alpha} n^{1-\alpha} \ell(n). \]

If \( k = \alpha - 1 \), then
\[ (i) \] holds if \( \ell(x) = o(1) \) (and in particular if \( \mathbb{E}(X_1^{k+1}) < \infty \)),
\[ (ii) \]
\[ \mathbb{E}(T_n^{k-1}) \sim \left( \frac{\mu^2}{\mu^2} \right)^k + C \frac{\Gamma(k-1)\Gamma(k+2)}{\Gamma(2k)\mu^{k+1}} \right) n^{-k} \]
holds if \( \ell(x) \sim C \) for a constant \( C \),
\[ (iii) \] else (18) holds.

**Proof.** Let us look at the quantity \( B(n, k_1, \ldots, k_r) \). By Lemma 3.1 and the Bingham–Doney lemma (see e.g. [3] Th. 8.1.6]) the asymptotic behaviour of \( \varphi^{(m)}(s) \) at the origin is given by
\[ (-1)^m \varphi^{(m)}(s) \sim \begin{cases} \alpha \Gamma(m-\alpha)s^{\alpha-m}f(1/s) & \text{if } m > \alpha, \\ \alpha \ell(1/s) & \text{if } m = \alpha \text{ and } \mathbb{E}(X_1^{m}) = \infty, \\ \mu_m & \text{if } m \leq \alpha \text{ and } \mathbb{E}(X_1^{m}) < \infty, \end{cases} \]
where \( \bar{\ell}(x) = \int_0^x (\ell(u)/u) \, du \) is itself a slowly varying function. For simplicity, let us first assume that \( \alpha \notin \mathbb{N} \). Then one can conclude in an analogous way as in the proof of Theorem 3.3 that the asymptotic behaviour of \( B(n, k_1, \ldots, k_r) \) is given by

\[
B(n, k_1, \ldots, k_r) \sim C_1 n^{-\alpha r - 2(k - u_1)} r_1(n),
\]

where \( r_1 \) is the number of integers among \( k_1, \ldots, k_r \) that are greater than \( \alpha/2 \), \( u_1 \) is the sum of these and \( C_1 \) is some constant. It remains to determine the dominating asymptotic term among all possible \( B(n, k_1, \ldots, k_r) \): If \( r_1 > 0 \), then \( r_1 = 1, u_1 = k \) and thus \( r = 1 \) yields the largest exponent, so that the asymptotic order is \( n^{1-\alpha} \ell(n) \). Note that \( r_1 > 0 \) is possible for \( 2k > \alpha \) only. For \( r_1 = 0 \), on the other hand, \( r = k \) and thus \( k_1 = \cdots = k_r = 1 \) dominates, leading to asymptotic order \( n^{-k} \). Hence the asymptotically dominating power among \( B(n, k_1, \ldots, k_r) \) is given by \( \max(1-\alpha, -k) \). From this we see that for \( k < \alpha - 1 \), \( r = k \) dominates and we obtain

\[
E(T_n^k) \sim k! n^k \mu_2^2 \Gamma(2k) k! \Gamma(2k)n^{2k} \mu^{2k} \sim \left( \frac{\mu_2^2}{\mu^2} \right)^k n^{-k}.
\]

Alternatively, if \( k > \alpha - 1 \), the term with \( r = 1 \) dominates and we obtain (18) in just the same way as in Theorem 3.3.

Finally, the above conclusions also hold for \( \alpha \in \mathbb{N} \) except when \( k = \alpha - 1 \). In the latter case the slowly varying function \( \ell(x) \) determines which of the two terms \( n^{1-\alpha} \ell(n) \) (corresponding to \( r = 1 \)) and \( n^{-k} \) (corresponding to \( r = k \)) dominates the asymptotic behaviour: if \( \ell(x) = \alpha(1) \) (which due to \( E(X_1^{k+1}) \sim (k + 1) \int_0^x x^{-1} \ell(x) \, dx \) is in particular fulfilled for \( E(X_1^{k+1}) < \infty \)), the second one dominates. If \( \ell(x) \sim \text{const} \), then both terms matter and the assertion of the theorem follows.

\[ \square \]

**Corollary 3.1.** If \( 1 - F(x) \sim x^{-2} \ell(x) \), then for \( k \geq 2 \),

\[
E(T_n^k) \sim \frac{1}{(k-1)(2k-1)\mu^2} \frac{\ell(n)}{n}
\]

and

\[
E(T_n) \sim \begin{cases} \frac{\ell_2(n)}{\mu} & \text{if } E(X_1^2) < \infty, \\ \frac{\ell(n)}{\mu^2} & \text{if } E(X_1^2) = \infty. \end{cases}
\]

\[ \square \]

**Proof.** One can easily verify that Theorem 3.3 remains true for \( \alpha = 2 \) except for \( k = 1 \) in the case \( E(X_1^2) = \infty \). In the latter case obviously \( r = 1 \) and one obtains (using \( \varphi''(s) \sim 2\ell(1/s) \))

\[
E(T_n) \sim B(n, 1) \sim \frac{2n\ell(n)}{n^2} \int_0^\infty t e^{-nt} \, dt \sim \frac{2}{\mu^2} \frac{\ell(n)}{n},
\]

which is already contained in [3, Theorem 5.2].

**Remark 3.3.** One might wonder whether a general limit result for \( E(T_n^k) \) for \( X_1 \) in the domain of attraction of a normal law (in the spirit of Theorem 5.2 of [3] for \( k = 1 \)) can be obtained with the integral representation approach used in this paper. This is however not the case: From \( \int_0^x y^2 dF(y) \sim \ell_2(x) \) (where \( \ell_2(x) \) is a slowly varying function) it follows by partial integration that \( \varphi^{(2k)}(s)/\ell_2(1/s) = o(s^{2-2k}) \) for \( k > 1 \) as \( s \to 0 \), but the latter is not strong enough to identify the dominating term among the \( B(n, k_1, \ldots, k_r) \) without any further assumptions on the distribution of \( X_1 \).

As an illustration of the results of this paper, Table I gives the first order asymptotic terms of \( E(T_n) \), \( \text{Var}(T_n) \) and the dispersion \( \text{Var}(T_n)/E(T_n) \) as a function of \( \alpha \). Note that the entries for \( \alpha > 2 \) have been obtained by calculating second-order asymptotic terms. The result for \( \alpha > 4 \) in the table actually holds whenever \( \mu_4 < \infty \), since in this case
the derivation of second-order terms does not rely on the assumption of regular variation and one obtains

\[ E(T_n^2) = \frac{\mu^2}{2} n^2 1 + \left( \frac{10\mu^3 - 3\mu_2^2\mu^2 - 8\mu_2\mu_3 + \mu^2\mu_4}{\mu^6} \right) \frac{1}{n^3} + O \left( \frac{1}{n^4} \right) \]
and

\[ E^2(T_n) = \frac{\mu^2}{\mu^2} n^2 1 + \left( \frac{6\mu^2 - 4\mu_2\mu_3 - 2\mu_2^2}{\mu^6} \right) \frac{1}{n^3} + O \left( \frac{1}{n^4} \right). \]

From Table 1 we see that the dispersion of \( T_n \) is a continuous function in \( \alpha \) with its maximum in \( \alpha = 1 \) (see Figure 2).

\[ \lim \frac{\text{Var}(T_n^k)}{E(T_n^k)} \]

**Figure 2.** Limit of the dispersion of \( T_n \) as a function of \( \alpha \)

4. **Estimation of the extreme value index for Pareto-type tails**

The results of Section 3 also give rise to an alternative and seemingly new method for estimating the extreme value index \( 1/\alpha \) for Pareto-type tails \( 1 - F(x) \sim x^{-\alpha} \ell(x) \) with \( 0 < \alpha < 2 \) from a given data set of independent observations (see e.g. [2] for other
estimators of the extreme value index). In fact, plotting $nT_n$ against $n$ will tend to a line with slope $1 - \alpha$, if $0 < \alpha < 1$ and plotting $\log(nT_n)$ against $\log n$ will tend to a line with slope $2 - \alpha$, if $1 < \alpha < 2$. The asymptotic behaviour of higher order moments of $nT_n$ available from Section 3 can then be used to increase the efficiency of the estimation procedure.

At the same time, this provides a technique to test the finiteness of the mean of a distribution in the domain of attraction of a stable law.

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