Теорія Ймовір. та Матем. Статист.
Вип. 76, 2007

Theor. Probability and Math. Statist.
No. 76, 2008, Pages 15–22
S 0094-9000(08)00727-8
Article electronically published on July 10, 2008

THE INVARIANCE PRINCIPLE FOR THE ORNSTEIN–UHLENBECK PROCESS WITH FAST POISSON TIME: AN ESTIMATE FOR THE RATE OF CONVERGENCE

UDC 519.21

B. V. BONDAREV AND A. V. BAEV

Abstract. We consider the invariance principle for

$$\varsigma_n(t) = n^{-1/2} \int_0^{Z(nt)} \xi(s)\, ds,$$

where $\xi(s)$ is the Ornstein–Uhlenbeck process and $Z(t)$, $t \geq 0$, is the Poisson process such that $\mathsf{E}Z(t) = \lambda(t)$. We prove that

$$\mathsf{P}\left\{ \sup_{0 \leq t \leq T} \left| \varsigma_n(t) - \frac{\sigma}{\gamma} n^{-1/2} W(\lambda(nt)) \right| > r_n \right\} \leq \alpha_n,$$

where $r_n \to 0$ and $\alpha_n \to 0$ as $n \to +\infty$.

1. Introduction

Let a stochastic process $\xi(s)$, $s \geq 0$, be defined on a complete probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ equipped with a filtration $\mathfrak{F}_0^t$, $t \geq 0$,

$$\mathfrak{F}_0^t = \bigcap_{\varepsilon > 0} \sigma\left\{\xi_s, 0 \leq s \leq t + \varepsilon\right\}.$$

The invariance principle for this setting means that the sequence of stochastic processes

$$\frac{X(nt)}{\sqrt{n}}, \qquad t \in [0, T]\,,\ n \geq 1,$$

weakly converges in the Skorokhod topology to the Wiener process $W(t)$, $t \in [0, T]$, where

$$X(t) = \int_0^t \xi(s)\, ds, \qquad t \in [0, T]. \tag{1}$$

The proofs of the invariance principle in the papers [1, 2, 3] are based on the assumption that the process $X(t)$, $t \in [0, +\infty)$, is "close" to a martingale. To be precise, the proofs in [1, 2, 3] use the decomposition

$$X(t) = \mu(t) + \rho(t), \tag{2}$$

where $\mu(t)$, $t \in [0, +\infty)$, is a martingale and $\rho(nt)/\sqrt{n}$, $t \in [0, T]$, is an asymptotically negligible process as $n \to +\infty$. Decomposition (2) holds under some conditions posed on

2000 *Mathematics Subject Classification.* Primary 60E15, 60H10; Secondary 60F17.
Key words and phrases. Ornstein–Uhlenbeck process, distribution of the supremum, Poisson process.

the process $\xi(s)$. These conditions are similar to the weak dependence of the process $\xi(s)$. Below is an example of such a condition:

$$\int_t^{+\infty} \left(\mathsf{E}\left[\mathsf{E}\left\{ \xi(s) \,/\, \mathfrak{F}_0^t \right\} \right]^2 \right)^{1/2} ds < +\infty \tag{3}$$

(see [2]).

Consider the process

$$v(t) = \int_t^{+\infty} \mathsf{E}\left\{ \xi(s) \,/\, \mathfrak{F}_0^t \right\} ds.$$

Note that v is well defined if, for example, condition (3) holds. Let

$$\mu(t) = \mathsf{E}\left\{ \int_0^{+\infty} \xi(s)\, ds \,\Big/\, \mathfrak{F}_0^t \right\} - v(0) \tag{4}$$

be a martingale. The stochastic process $\mu(t)$ is well defined if condition (3) holds and $\mathsf{E}\,|\xi(s)| < +\infty$. Using the properties of conditional expectation we obtain the following representation:

$$\mu(t) = \mathsf{E}\left\{ \int_0^{+\infty} \xi(s)\, ds \,\Big/\, \mathfrak{F}_0^t \right\} - v(0) = \int_0^t \xi(s)\, ds + v(t) - v(0),$$

whence

$$\int_0^t \xi(s)\, ds = \mu(t) + \rho(t), \qquad \rho(t) = v(0) - v(t). \tag{5}$$

If $\xi(s)$, $s \geq 0$, is a stationary process, then $v(t)$ is a strongly stationary process. Equality (5) implies, for example, that

$$\varsigma_n(t) = \frac{X\,(nt)}{\sqrt{n}} = \frac{\mu(nt)}{\sqrt{n}} + \frac{1}{\sqrt{n}}\left(v(0) - v(nt)\right).$$

Moreover the strongly stationary process $v(t)$ is such that

$$\sup_{0 \leq t \leq T} \frac{|v(0) - v(nt)|}{\sqrt{n}} \to 0, \qquad n \to +\infty,$$

in probability (see [2]).

To this end we note that representation (5) may also hold in the case of a nonstationary process $\xi(t)$, $t \geq 0$, if μ and ρ have the same properties as above, namely if μ is a martingale and ρ is negligible. This is the case we consider below.

In this paper, we study the invariance principle for normalized integrals with fast Poisson time. Namely, we consider the problem of estimating the rate of approximation in probability of the process

$$\varsigma_n(t) = \frac{1}{\sqrt{n}} \int_0^{Z(nt)} \xi(s)\, ds, \qquad t \in [0, T],$$

by the family of processes $\sigma\gamma^{-1}n^{-1/2}W(\lambda(nt))$, $t \in [0, T]$, as $n \to +\infty$. Here $Z(t)$ is a Poisson process such that $\mathsf{E}\, Z(t) = \lambda(t)$, $\lambda(0) = 0$.

2. An estimate of the rate of convergence of a compound Poisson process to a normalized integral of the Ornstein–Uhlenbeck process

Consider a process $\xi(t)$ to be a solution of the equation

$$\xi(t) = \xi(0) - \gamma \int_0^t \xi(s)\, ds + \sigma W(t) \tag{6}$$

where $\gamma > 0$ and $\sigma > 0$ are constants and $W(t)$, $t \geq 0$, is a standard Wiener process such that $W(t)$, $t \geq 0$, does not depend on $\xi(0)$. We also assume that $\operatorname{Var}\xi(0) < +\infty$. Then $\xi(t)$ is called the Ornstein–Uhlenbeck process.

Let $\xi_{t,x}(s)$ be a stochastic process starting from a point x at the moment $t \geq 0$ and let $\xi_{t,x}(s)$ have the following stochastic differential:

$$d\xi_{t,x}(\tau) = -\gamma\xi_{t,x}(\tau)\,d\tau + \sigma W_t, \qquad \xi_{t,x}(\tau)\big|_{\tau=t} = x. \tag{7}$$

We understand equality (7) in the sense that

$$\xi_{t,x}(\tau) = x - \gamma\int_t^\tau \xi_{t,x}(s)\,ds + \sigma\,[W(\tau) - W(t)]. \tag{8}$$

It is well known that

$$\mathsf{E}\left\{\xi(s)\ /\ \mathfrak{F}_0^t\right\} = \xi(t)e^{-\gamma(s-t)}, \qquad s \geq t. \tag{9}$$

Since

$$\xi(t) = e^{-\gamma t}\left[\xi(0) + \sigma\int_0^t e^{\gamma s}\,dW(s)\right],$$

we have

$$\mathsf{E}\,\xi(\tau) = \mathsf{E}\,\xi(0)e^{-\gamma(\tau-t)},$$

$$\begin{aligned}\operatorname{Var}\xi(t) &= \mathsf{E}\left[\xi(t) - \mathsf{E}\,\xi(0)e^{-\gamma t}\right]^2 = e^{-2\gamma t}\,\mathsf{E}\left[\xi(0) - \mathsf{E}\,\xi(0) + \sigma e^{-\gamma t}\int_0^t e^{\gamma s}\,dW(s)\right]^2\\ &= e^{-2\gamma t}\,\mathsf{E}\left[\xi(0) - \mathsf{E}\,\xi(0)\right]^2 + \mathsf{E}\,\sigma^2 e^{-2\gamma\tau}\left[\int_0^t e^{\gamma s}\,dW(s)\right]^2\\ &= e^{-2\gamma t}\operatorname{Var}\xi(0) + \frac{\sigma^2}{2\gamma}\left[1 - e^{-2\gamma\tau}\right] \leq \operatorname{Var}\xi(0) + \frac{\sigma^2}{2\gamma} < +\infty.\end{aligned}$$

This implies the following bound:

$$\mathsf{E}\,|\xi(t)|^2 \leq \operatorname{Var}\xi(0) + \frac{\sigma^2}{2\gamma} + [\mathsf{E}\,\xi(0)]^2.$$

Combining the latter result with (7) we finally get

$$\begin{aligned}\int_t^\infty \left(\mathsf{E}\left[\mathsf{E}\{\xi(s)\ /\ \mathfrak{F}_0^t\}\right]^2\right)^{1/2} ds &= \int_t^\infty \left(\mathsf{E}\left[\xi(t)e^{-\gamma(s-t)}\right]^2\right)^{1/2} ds\\ &= \left(\mathsf{E}\,|\xi(t)|^2\right)^{1/2}\int_t^\infty e^{-\gamma(s-t)}\,ds = \left(\mathsf{E}\,|\xi(t)|^2\right)^{1/2}\frac{1}{\gamma}\\ &\leq \frac{1}{\gamma}\left(\operatorname{Var}\xi(0) + \frac{\sigma^2}{2\gamma} + [\mathsf{E}\,\xi(0)]^2\right)^{1/2}.\end{aligned}$$

This means that condition (3) holds. Thus the stochastic process

$$v(t) = \int_t^{+\infty} \mathsf{E}\left\{\xi(s)\ /\ \mathfrak{F}_0^t\right\}\,ds$$

is well defined. It is clear that the process

$$\mu(t) = \mathsf{E}\left\{\int_0^{+\infty} \xi(s)\,ds\ \Big/\ \mathfrak{F}_0^t\right\} - v(0)$$

also is well defined. Therefore both processes $v(t)$ and $\mu(t)$ in the definition of the Ornstein–Uhlenbeck process are well defined and moreover

$$v(t) = \int_t^{+\infty} \mathsf{E}\left\{\xi(s) \,/\, \mathfrak{F}_0^t\right\} ds = \int_t^{+\infty} \xi(t)\ell^{-\gamma(s-t)}\, ds = \frac{\xi(t)}{\gamma},$$

$$\mu(t) = \mathsf{E}\left\{\int_0^{+\infty} \xi(s)\, ds \,\Big/\, \mathfrak{F}_0^t\right\} - v(0) = \int_0^t \xi(s)\, ds + \frac{\xi(t)}{\gamma} - \frac{\xi(0)}{\gamma}$$

$$= \frac{1}{\gamma}\xi(0) + \frac{1}{\gamma}\sigma W(t) - \frac{\xi(0)}{\gamma} = \frac{1}{\gamma}\sigma W(t).$$

This yields

$$\int_0^t \xi(s)\, ds = \frac{1}{\gamma}\sigma W(t) - \frac{\xi(t)}{\gamma} + \frac{\xi(0)}{\gamma}. \tag{10}$$

Note that representation (10) is obvious in the case of the Ornstein–Uhlenbeck process.

Theorem 2.1. *Let $\xi(t)$, $t \geq 0$, be the Ornstein–Uhlenbeck process such that*

$$\operatorname{Var}\xi(0) < +\infty,$$

and suppose that $W(t)$, $t \geq 0$, does not depend on $\xi(0)$ and $Z(t)$. Then

$$\frac{1}{\sqrt{n}}\int_0^{Z(nt)} \xi(s)\, ds = \frac{\sigma}{\gamma}\frac{1}{\sqrt{n}}W(Z(nt)) + \rho_n(t), \qquad t \in [0, T]\,, \tag{11}$$

where

$$\sup_{0\leq t\leq T} |\rho_n(t)| \to 0, \qquad n \to +\infty,$$

in probability. Moreover

$$\mathsf{P}\left\{\sup_{0\leq t\leq T} |\rho_n(t)| > \rho\right\} \leq \frac{32\operatorname{Var}\xi(0)}{\rho^2\gamma^2 n} + 2n\delta\exp\left\{-\frac{\rho^2\gamma^3 n}{16\sigma^2}\left[e^{2\gamma T} - 1\right]^{-1}\right\}. \tag{12}$$

Proof. Using representation (10) we obtain

$$\frac{1}{\sqrt{n}}\int_0^{Z(nt)} \xi(s)\, ds = \frac{\sigma}{\gamma}\frac{1}{\sqrt{n}}W(Z(nt)) + \frac{1}{\gamma\sqrt{n}}\big(\xi(0) - \xi(Z(nt))\big). \tag{13}$$

Let

$$\begin{gathered} \xi_0(t) = e^{-\gamma t}\sigma\int_0^t e^{\gamma s}\, dW(s), \\ \xi_{kT}(t) = e^{-\gamma kT}\sigma\int_{kT}^t e^{\gamma s}\, dW(s), \\ kT \leq t < (k+1)T. \end{gathered} \tag{14}$$

Then

$$\mathsf{P}\left\{\sup_{kT\leq t\leq (k+1)T} |\xi_{kT}(t)| > r\right\} \leq 2\exp\left\{-r^2\gamma\left[e^{2\gamma T} - 1\right]^{-1}\right\}. \tag{15}$$

Indeed, the process

$$\xi_{kT}(t) = e^{-\gamma kT}\sigma\int_{kT}^t e^{\gamma s}\, dW(s), \qquad kT \leq t < (k+1)T,$$

is a martingale, since

$$\begin{aligned}\mathsf{E}\left\{\xi_{kT}(t) \,/\, \mathfrak{F}_0^\tau\right\} &= e^{-\gamma kT}\,\mathsf{E}\left\{\sigma\int_{kT}^t e^{\gamma s}\,dW(s) \,\Big/\, \mathfrak{F}_0^\tau\right\}\\ &= e^{-\gamma kT}\sigma\int_{kT}^\tau e^{\gamma s}\,dW(s) = \xi_{kT}(\tau),\\ kT \le \tau \le t &< (k+1)T.\end{aligned}$$

The characteristics of ξ_{kT} do not exceed

$$e^{-2\gamma kT}\int_{kT}^{(k+1)T} e^{2\gamma s}\,ds = \frac{1}{2\gamma}\left[e^{2\gamma T}-1\right].$$

Applying an estimate of [4, p. 172] we prove representation (13). Further,

$$\begin{aligned}&\mathsf{P}\left\{\sup_{0\le t\le T}\left|\frac{1}{\gamma\sqrt{n}}\big(\xi(0)-\xi(Z(nt))\big)\right| > \rho\right\}\\ &\quad\le \mathsf{P}\left\{|\xi(0)| > \frac{\rho}{2}\gamma\sqrt{n}\right\} + \mathsf{P}\left\{\sup_{0\le t\le T}|\xi(Z(nt))| > \frac{\rho}{2}\gamma\sqrt{n}\right\}\\ &\quad\le \mathsf{P}\left\{|\xi(0)| > \frac{\rho}{2}\gamma\sqrt{n}\right\} + \mathsf{P}\left\{|\xi(0)| > \frac{\rho}{4}\gamma\sqrt{n}\right\} + \mathsf{P}\left\{\sup_{0\le t\le Z(nT)}|\xi_0(t)| > \frac{\rho}{4}\gamma\sqrt{n}\right\}\\ &\quad\le 2\,\mathsf{P}\left\{|\xi(0)| > \frac{\rho\sqrt{n}\gamma}{4}\right\}\\ &\qquad + \sum_{m=0}^{+\infty}\mathsf{P}\left\{\sup_{0\le k\le m}\sup_{kT\le t\le(k+1)T}|\xi_{kT}(t)| > \frac{\rho}{4}\gamma\sqrt{n}\right\}\mathsf{P}\left\{Z(nT)=m\right\}\\ &\quad\le \frac{32\operatorname{Var}\xi(0)}{\rho^2\gamma^2 n} + \sum_{m=0}^{+\infty}\sum_{k=0}^{m}\mathsf{P}\left\{\sup_{kT\le t\le(k+1)T}|\xi_{kT}(t)| > \frac{\rho}{4}\gamma\sqrt{n}\right\}\mathsf{P}\{Z(nT)=m\}\\ &\quad\le \frac{32\operatorname{Var}\xi(0)}{\rho^2\gamma^2 n} + 2\exp\left\{-\frac{\rho^2\gamma^3 n}{16\sigma^2}\left[e^{2\gamma T}-1\right]^{-1}\right\}\sum_{m=0}^{+\infty} m\,\mathsf{P}\left\{Z(nT)=m\right\}\\ &\quad= \frac{32\operatorname{Var}\xi(0)}{\rho^2\gamma^2 n} + 2\exp\left\{-\frac{\rho^2\gamma^3 n}{16\sigma^2}\left[e^{2\gamma T}-1\right]^{-1}\right\}\lambda(nT).\end{aligned}$$

This implies the upper bound (12). □

Remark 2.1. If the initial distribution possesses an exponential moment, that is, if

$$\mathsf{E}\exp\{\alpha\xi(0)\} \le C < +\infty$$

for some $0 < \alpha < +\infty$, then a sharper bound holds. Namely,

$$\begin{aligned}&\mathsf{P}\left\{\sup_{0\le t\le T}\left|\frac{1}{\gamma\sqrt{n}}\big(\xi(0)-\xi(Z(nt))\big)\right| > \rho\right\}\\ &\qquad\le C\exp\left\{-\frac{\alpha\rho\sqrt{n}\gamma}{4}\right\} + 2n\delta\exp\left\{-\frac{\rho^2\gamma^3 n}{16\sigma^2}\left[e^{2\gamma T}-1\right]^{-1}\right\}.\end{aligned}$$

Now we prove that the following representation holds:

$$\frac{1}{\sqrt{n}}\int_0^{Z(nt)}\xi(s)\,ds = \frac{\sigma}{\gamma}\frac{1}{\sqrt{n}}W(\lambda(nt)) + \rho_n'(t), \qquad t\in[0,T]. \tag{16}$$

3. An estimate of the rate of approximation of sums of increments of a Wiener process with fast Poisson time by sums of increments of the Wiener process with fast nonrandom time

The main result of this section reads as follows.

Theorem 3.1. *Let $Z(t)$, $0 \le t \le T$, be a Poisson process such that $\mathsf{E}\, Z(t) = \lambda(t)$. Assume that*

$$0 < \rho T n \le \lambda(nT) \le \chi T n < +\infty.$$

Let $\rho(n) \to 0$ and $\delta(n) \to 0$ such that $n\delta(n) \to +\infty$, $\rho^2(n)/h(n) \to +\infty$, and

$$\frac{\rho(n)}{\sqrt{T\chi}} h^{-3/2}(n) \exp\left\{-\frac{\rho^2(n)}{2h(n)T\chi}\right\} \to 0$$

as $n \to +\infty$. Then

$$\begin{aligned} &\mathsf{P}\left\{\sup_{0\le t\le T} |W(Z(nt)) - W(\lambda(nt))| > \rho(n)\sqrt{n}\right\} \\ &\qquad \le 20\frac{\rho(n)}{\sqrt{T\chi}} h^{3/2}(n) \exp\left\{-\frac{\rho^2(n)}{2h(n)T\chi}\right\} + 2\exp\left\{-T\rho h(n) n + \frac{1}{6}\right\}. \end{aligned} \tag{17}$$

Proof. Put

$$\widetilde{W}(t) = \frac{W(t\lambda(nT))}{\sqrt{\lambda(nT)}}.$$

Then $\widetilde{W}(t)$ is a Wiener process and

$$\begin{aligned} &\mathsf{P}\left\{\sup_{0\le t\le T} |W(Z(nt)) - W(\lambda(nt))| > \rho(n)\sqrt{n}\right\} \\ &\quad = \mathsf{P}\left\{\sup_{0\le t\le T} \left|\widetilde{W}\left(\frac{\lambda(nt)}{\lambda(nT)} + \frac{Z(nt)-\lambda(nt)}{\lambda(nT)}\right) - \widetilde{W}\left(\frac{\lambda(nt)}{\lambda(nT)}\right)\right| > \frac{\rho(n)}{\sqrt{\chi T}}\right\} \\ &\quad \le \mathsf{P}\left\{\sup_{0\le s\le 1}\sup_{|\tau|\le h(n)} \left|\widetilde{W}(s+\tau) - \widetilde{W}(s)\right| > \frac{\rho(n)}{\sqrt{T\chi}}\right\} \\ &\qquad + \mathsf{P}\left\{\sup_{0\le t\le T} \left|\frac{Z(nt)-\lambda(nt)}{\lambda(nT)}\right| > h(n)\right\}. \end{aligned} \tag{18}$$

Let $\omega(h)$ be the modulus of continuity of the process $\widetilde{W}(t)$, $0 \le t \le 1$:

$$\omega(h) = \sup\left\{\left|\widetilde{W}(t+s) - \widetilde{W}(t)\right| : 0 \le t, t+s \le 1, |s| \le h\right\}.$$

The following estimate is well known (see [5]):

$$\mathsf{P}\{\omega(h) > x\} \le 20 x h^{-3/2} \exp\left\{-x^2/2h\right\}. \tag{19}$$

Since the stochastic processes

$$\eta^{\pm}(t) = \exp(z[Z(t) - \lambda(t)] - \lambda(t)[e^{\mp z} - 1 \mp z])$$

are martingales, we have the following bound:

$$
\begin{aligned}
&\mathsf{P}\left\{\sup_{0\le t\le T}\left|\frac{Z(nt)-\lambda(nt)}{\lambda(nT)}\right|>h(n)\right\}\\
&\quad\le \mathsf{P}\left\{\sup_{0\le t\le T}[Z(nt)-\lambda(nt)]>h(n)\lambda(nT)\right\}\\
&\qquad+\mathsf{P}\left\{\sup_{0\le t\le T}[-Z(nt)+\lambda(nt)]>\delta_n\lambda(nT)\right\}\\
&\quad\le \mathsf{P}\Bigg\{\sup_{0\le t\le nT}\exp\bigl(z[Z(t)-\lambda(t)]-\lambda(t)[e^z-1-z]\bigr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad>\exp\bigl(zh(n)\lambda(nT)-[e^z-1-z]\bigr)\Bigg\}\\
&\qquad+\mathsf{P}\Bigg\{\sup_{0\le t\le nT}\exp\bigl(z[-Z(t)+\lambda(t)]-\lambda(t)[e^{-z}-1+z]\bigr)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad>\exp\bigl(zh(n)\lambda(nT)-[e^{-z}-1+z]\bigr)\Bigg\}\\
&\quad\le 2\exp\left\{-T\rho h(n)n+\frac{1}{6}\right\}.
\end{aligned}
\tag{20}
$$

Inequality (17) follows from the bounds (18), (19), and (20).

Theorem 3.1 is proved. □

Below we provide an estimate of the rate of approximation of a normalized integral of the Ornstein–Uhlenbeck process with fast Poisson time by a family of Wiener processes.

Theorem 3.2. *Let $Z(t)$, $0\le t\le T$, be a Poisson process such that $\mathsf{E}\,Z(t)=\lambda(t)$. Assume that*

$$0<\rho Tn\le\lambda(nT)\le\chi Tn<+\infty.$$

Let $\rho(n)\to 0$ and $\delta(n)\to 0$ such that $n\delta(n)\to+\infty$, $\rho^2(n)/h(n)\to+\infty$, and

$$\frac{\rho(n)}{\sqrt{T\chi}}h^{-3/2}(n)\exp\left\{-\frac{\rho^2(n)}{2h(n)T\chi}\right\}\to 0$$

as $n\to+\infty$. Further let $\xi(t)$, $t\ge 0$, be an Ornstein–Uhlenbeck process with a random initial value $\xi(0)$ and suppose that $W(t)$, $t\ge 0$, does not depend on $\xi(0)$. Finally, let $\operatorname{Var}\xi(0)<+\infty$. *Then*

$$\frac{1}{\sqrt{n}}\int_0^{Z(nt)}\xi(s)\,ds=\frac{\sigma}{\gamma}\frac{1}{\sqrt{n}}W(\lambda(nt))+\rho_n'(t),\qquad t\in[0,T].$$

Moreover

$$
\begin{aligned}
\mathsf{P}\left\{\sup_{0\le t\le T}|\rho_n'(t)|>\rho\right\}\le{}&\frac{32\operatorname{Var}\xi(0)}{\rho^2\gamma^2 n}+2n\delta\exp\left\{-\frac{\rho^2\gamma^3 n}{16\sigma^2}\left[\ell^{2\gamma T}-1\right]^{-1}\right\}\\
&+20\frac{\gamma\rho(n)}{\sigma\sqrt{T\chi}}h^{-3/2}(n)\exp\left\{-\rho^2(n)\gamma^2/\bigl(2\sigma^2h(n)T\chi\bigr)\right\}\\
&+2\exp\left\{-T\rho h(n)n+\frac{1}{6}\right\}.
\end{aligned}
$$

4. Concluding remarks

We considered the problem on estimating the rate of approximation in probability as $n \to +\infty$ of the process

$$\varsigma_n(t) = \frac{1}{\sqrt{n}} \int_0^{Z(nt)} \xi(s)\, ds, \qquad t \in [0, T],$$

by the family of processes $\sigma\gamma^{-1}n^{-1/2}W(\lambda(nt)$, $t \in [0, T]$, where $Z(t)$ is a Poisson process such that

$$\mathsf{E}\, Z(t) = \lambda(t), \qquad \lambda(0) = 0,$$

and $W(t)$, $t \geq 0$, is a standard Wiener process. We obtained sufficient conditions for the inequality

$$\mathsf{P}\left\{ \sup_{0 \leq t \leq T} \left| \varsigma_n(t) - \frac{\sigma}{\gamma} n^{-1/2} W(\lambda(nt)) \right| > r_n \right\} \leq \alpha_n,$$

where $r_n \to 0$ and $\alpha_n \to 0$ as $n \to +\infty$.

The statistical analysis of logarithms of the prices of some stocks leads to the following empirical conclusions: the moments of jumps in prices are Poisson, while the heights of the jumps are Gaussian. Moreover, there exists a positive correlation between jumps. The process $\varsigma_n(t)$ considered above possesses these properties. Therefore $\varsigma_n(t)$ can be used as the main process when modeling the evolution of prices of stocks. Note that this model is an ε-martingale for sufficiently large n (see [6] concerning the notion of ε-martingales).

The idea of the above proof of Theorem 3.1 belongs to Yu. S. Mishura. This idea allowed us to shorten the original proof. The authors are grateful to Professor Yu. S. Mishura and to a reviewer whose remarks allowed us to improve the presentation of our results.

Bibliography

1. D. O. Chikin, *A functional limit theorem for stationary processes: A martingale approach*, Teor. Veroyatnost. i Primenen. **34** (1989), no. 4, 731–741; English transl. in Theory Probab. Appl. **34** (1989), no. 4, 668–678. MR1036712 (91e:60106)
2. S. V. Anulova, A. Yu. Veretennikov, N. V. Krylov, R. Sh. Liptser, and A. N. Shiryaev, *Stochastic Calculus*, Itogi nauki i tekhniki [Progress in Science and Technology], Current problems in mathematics, Fundamental directions, vol. 45, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, pp. 5–260. (Russian) MR1039617 (91i:60002b)
3. J. Jacod and A. N. Shiryaev, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, 1987. MR959133 (89k:60044)
4. I. I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations and their Applications*, "Naukova dumka", Kiev, 1982. (Russian) MR678374 (84j:60003)
5. Yu. V. Prokhorov (ed.), *Probability and Mathematical Statistics*, "Bol'shaya Rossiĭskaya Entsiklopediya", Moscow, 2003. (Russian) MR1785437 (2001j:62001)
6. B. Gnedenko, *On an extension of the concept of a martingale*, Teor. Veroyatnost. i Primenen. **50** (2005), no. 4, 763–767; English transl. in Theory Probab. Appl. **50** (2006), no. 4, 659–662. MR2331987 (2008g:60127)

Department of Probability Theory and Mathematical Statistics, Faculty for Mathematics, Donetsk National University, Universiets'ka Street, 24, 83055 Donetsk, Ukraine
E-mail address: bvbondarev@cable.netlux.org

Department of Probability Theory and Mathematical Statistics, Faculty for Mathematics, Donetsk National University, Universiets'ka Street, 24, 83055 Donetsk, Ukraine
E-mail address: tv@matfak.dongu.donetsk.ua

Received 6/JAN/2006

Translated by S. KVASKO