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A LOCATION INVARIANT MOMENT-TYPE ESTIMATOR. I

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Abstract. The moment's estimator (Dekkers et al., 1989) has been used in extreme value theory to estimate the tail index, but it is not location invariant. The location invariant Hill-type estimator (Fraga Alves, 2001) is only suitable to estimate positive indices. In this paper, a new moment-type estimator is studied, which is location invariant. This new estimator is based on the original moment-type estimator, but is made location invariant by a random shift. Its weak consistency and strong consistency are derived, in a semiparametric setup.

1. Introduction

Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with common distribution function (d.f.) $F(x)$ and let $X_{1,n} \le X_{2,n} \le \cdots \le X_{n,n}$ be the associated order statistics. If there exist some numbers $a_n > 0$, $b_n \in \mathbb{R}$ and some non-degenerate distribution $G(x)$ such that

$$\mathsf{P}(X_{n,n} \le a_n x + b_n) = F^n(a_n x + b_n) \xrightarrow{d} G(x) \quad \text{as } n \to \infty, \tag{1.1}$$

then $G(x)$ must be equivalent to

$$G_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\}, & 1+\gamma x > 0,\ \gamma \ne 0, \\ \exp\{-\exp(-x)\}, & x \in \mathbb{R},\ \gamma = 0. \end{cases}$$

If F satisfies (1.1) we say $F(x)$ belongs to the domain of attraction of an extreme value d.f. G_γ, denoted by $F \in D(G_\gamma)$ and γ is referred to as the extreme value index (EVI). In the last two decades many estimators of the extreme value index $\gamma \in \mathbb{R}$ have been proposed that use upper order statistics; see, for example, Hill [16], Pickands [20], Dekkers et al. [6], Drees [7, 8], and Drees and Kaufmann [9]. For maximum likelihood estimators of γ, see Hall [15], Smith [24, 25], and Smith and Weissman [26]. For $\gamma > 0$, Hill [16] proposed the estimator given by

$$\hat{\gamma}_n^H(k) = \frac{1}{k}\sum_{i=0}^{k-1} \log X_{n-i,n} - \log X_{n-k,n} \tag{1.2}$$

for $k = 1, \ldots, n-1$. For $\gamma \in \mathbb{R}$, Dekkers et al. [6] proposed the moment-type estimators

$$\hat{\gamma}_n^M = M_n^{(1)} + 1 - \frac{1}{2}\left\{1 - \frac{\left(M_n^{(1)}\right)^2}{M_n^{(2)}}\right\}^{-1}, \tag{1.3}$$

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where

$$M_n^{(j)} = \frac{1}{k}\sum_{i=0}^{k-1}\left(\log\frac{X_{n-i,n}}{X_{n-k,n}}\right)^j$$

for $j = 1, 2$. Pan [17] discussed the asymptotic expansion of the distribution of the Hill and moment estimators. Cheng and de Haan [1] and Cuntz et al. [3] obtained penultimate forms of the Hill estimator.

The above estimators are scale invariant but not location invariant. Indeed there are mathematical as well as practical reasons to require location invariance properties. Since an affine transformation of the r.v.'s X_i merely leads to a change of the normalizing constants a_n and b_n, it influences neither the extreme value index nor the accuracy of the approximation (1.1). Moreover, in practice, the observations (e.g., sea levels, temperatures) depend on an arbitrarily chosen zero-point, which equally should not affect the estimator. In fact, the prominent Pickands [20] estimator given by

$$\hat{\gamma}_n^P = \frac{1}{\log 2}\log\frac{X_{n-k+1,n} - X_{n-2k+1,n}}{X_{n-2k+1,n} - X_{n-4k+1,n}} \tag{1.4}$$

is both scale and location invariant, where $k = k(n)$ is an intermediate integer sequence, i.e., $k = k(n) \to \infty$ and $k/n \to 0$. Qi and Cheng [21] and Peng [19] discussed the asymptotic behavior of various Pickands-type estimators. Segers [23] proposed a general Pickands estimator given by

$$\hat{\gamma}_{n,k}(c, v) = \frac{1}{\log v}\log\left(\frac{X_{n-[ck],n} - X_{n-k,n}}{X_{n-[cvk],n} - X_{n-[vk],n}}\right)$$

and proved its consistency, asymptotic normality and discussed an optimal choice of c and v in the sense of minimum square error (MSE). Drees [8] proposed a general class of estimators which have the scale invariant property. For $\gamma > 0$, Fraga Alves [11] established a location invariant Hill estimator given by

$$\hat{\gamma}_n^H(k_0, k) = \frac{1}{k_0}\sum_{i=0}^{k_0-1}\log\left(\frac{X_{n-i,n} - X_{n-k,n}}{X_{n-k_0,n} - X_{n-k,n}}\right), \tag{1.5}$$

where $k \to \infty$, $k_0 \to \infty$, $k/n \to 0$, $k_0/k \to 0$, and discussed its weak consistency, asymptotic expansion and the optimal choice of the sample fraction k_0.

Although both (1.4) and (1.5) are location invariant, (1.4) has poor efficiency and it is difficult to decide on the optimal sample fraction k. Also (1.5) is only valid for $\gamma > 0$. In this paper, we propose a general estimator for $\gamma \in \mathbb{R}$ based on the invariant Hill estimator and the moment-type estimator. It is given by

$$\hat{\gamma}_n^M(k_0, k) = M_n^{(1)}(k_0, k) + 1 - \frac{1}{2}\left\{1 - \frac{\left(M_n^{(1)}(k_0, k)\right)^2}{M_n^{(2)}(k_0, k)}\right\}^{-1}, \tag{1.6}$$

where

$$M_n^{(j)}(k_0, k) = \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left(\log\frac{X_{n-i,n} - X_{n-k,n}}{X_{n-k_0,n} - X_{n-k,n}}\right)^j$$

for $j = 1, 2$ and $k = k(n)$, $k_0 = k_0(n)$ are integer sequences that satisfy $0 < k \le n$, $0 < k_0 \le k$. We derive the weak and strong consistencies of this new estimator (Section 2).

2. Weak and strong consistency

In what follows we assume that $F \in D(G_\gamma)$, $\gamma \in \mathbb{R}$, which is equivalent to supposing that $U := (1/(1-F))^{\leftarrow}$ is a general regularly varying function with index γ (denoted by $U \in GRV_\gamma$); i.e., there exists a measurable function $a(t) > 0$ such that as $t \to \infty$,

$$\frac{U(tx) - U(t)}{a(t)} \to \frac{x^\gamma - 1}{\gamma} \quad (= \log x \text{ for } \gamma = 0) \tag{2.1}$$

holds for all $x > 0$ (de Haan [4]). In fact, the auxiliary function $a(t)$ is a regularly varying function at infinity with index γ, denoted by $a(t) \in RV_\gamma$.

Let $Y_1, \dots, Y_n$ be independent random variables (r.v.'s) with common d.f.

$$F_Y(y) = 1 - 1/y, \qquad y \geq 1,$$

and let $Y_{1,n} \leq Y_{2,n} \leq \cdots \leq Y_{n,n}$ be the associated order statistics. Then $X_{(i,n)} \stackrel{d}{=} U(Y_{i,n})$ for $i = 1, \dots, n$. Notice also that the following relations are true:

$$\left\{ \frac{Y_{n-i,n}}{Y_{n-k_0,n}} \right\}_{i=0}^{k_0-1} \stackrel{d}{=} \{Y_{k_0-i,k_0}\}_{i=0}^{k_0-1} \tag{2.2}$$

and

$$\log Y_{i,n} \stackrel{d}{=} E_{i,n}$$

for $i = 1, \dots, n$, where $E_{i,n}$ is the ith order statistic associated with a random sample $E_1, \dots, E_n$ from the standard exponential distribution. Notice further that for any intermediate sequence $k = k(n)$, as $n \to \infty$, $(k/n)Y_{n-k,n} \to 1$ in probability, and that for any integer sequence $k(n)$ satisfying $k(n)/n \to 0$ and $k(n)/(\log n)^\delta \to \infty$ for some $\delta > 0$, as $n \to \infty$, $(k/n)Y_{n-k,n} \to 1$ almost surely.

Now we present the main results concerning the proposed estimator's consistency.

Theorem 2.1. *If $U \in GRV_\gamma$, $x^*(F) > 0$ (the right endpoint of F), $k(n)/n \to 0$, $k_0/k \to 0$, $k(n) \to \infty$ and $k_0(n) \to \infty$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \hat{\gamma}_n^M(k_0, k) = \gamma \tag{2.3}$$

in probability.

Theorem 2.2. *If $U \in GRV_\gamma$, $x^*(F) > 0$, $k(n)/n \to 0$, $k_0/k \to 0$, $k/(\log n)^{\delta_1} \to \infty$ and $k_0/(\log k)^{\delta_2} \to \infty$ for some δ_1, $\delta_2 > 0$, then*

$$\lim_{n\to\infty} \hat{\gamma}_n^M(k_0, k) = \gamma \tag{2.4}$$

almost surely.

For the proof we need some lemmas.

Lemma 2.1. *Let $U(t) \in GRV_\gamma$ and $x_* = \inf\{x \colon F(x) > 0\}$. There exists a function $f(t_2, t_1)$ such that as $t_1 \to \infty$ and $t_2/t_1 \to \infty$*

$$\frac{1}{f(t_2,t_1)} \log \frac{U(t_2 s) - U(t_1)}{U(t_2) - U(t_1)} \longrightarrow \begin{cases} \log s, & \gamma \geq 0, \\ (s^\gamma - 1)/\gamma, & \gamma < 0 \end{cases} \tag{2.5}$$

holds locally uniformly for all $s > 0$. Moreover for every $\varepsilon > 0$, there exist N_1, N_2 such that $t_1 > N_1$, $t_1 s \geq N_1$, $t_2/t_1 > N_2$,

$$(1-\varepsilon)\frac{1 - s^{-\varepsilon}}{\varepsilon} - \varepsilon < \frac{1}{f(t_2,t_1)} \log \frac{U(t_2 s) - U(t_1)}{U(t_2) - U(t_1)} < (1+\varepsilon)\frac{s^\varepsilon - 1}{\varepsilon} + \varepsilon \tag{2.6}$$

provided that $\gamma \geq 0$, *and*

$$1-(1+\varepsilon)s^{\gamma+\varepsilon} < \frac{1}{f(t_2,t_1)}\log\frac{U(t_2s)-U(t_1)}{U(t_2)-U(t_1)} < 1-(1-\varepsilon)s^{\gamma-\varepsilon} \tag{2.7}$$

provided that $\gamma < 0$.

Remark 2.1. In Lemma 2.1 we can take the function $f(t_2,t_1)$ to satisfy

$$f(t_2,t_1) \sim \frac{a(t_2)}{U(t_2)-U(t_1)} \sim \begin{cases} \gamma, & \gamma > 0, \\ \log^{-1}(t_2/t_1), & \gamma = 0, \\ (-\gamma)(t_2/t_1)^{\gamma}, & \gamma < 0, \end{cases}$$

where $a(t)$ is the auxiliary function of $U(t)$. In particular, $f(t_2,t_1) > 0$ and

$$f(t_2,t_1) \to \max(\gamma, 0)$$

as $t_1 \to \infty$ and $t_2/t_1 \to \infty$.

Proof. We prove (2.5) for the three cases $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$.

(i) Consider $\gamma > 0$. For $0 < s \leq 1$,

$$U(t_2s) - U(t_2) \leq 0. \tag{2.8}$$

As t_1, $t_2/t_1 \to \infty$ and for all $0 < \varepsilon < 1$, there exists N_1 such that $t_1 > N_1$, $t_1 s \geq N_1$, $t_1/t_2 < \varepsilon$ and

$$U(t_2) - U(t_2\varepsilon) < U(t_2) - U(t_1) < U(t_2). \tag{2.9}$$

Combining (2.8) and (2.9),

$$\frac{U(t_2s)-U(t_2)}{U(t_2)-U(t_2\varepsilon)} < \frac{U(t_2s)-U(t_2)}{U(t_2)-U(t_1)} < \frac{U(t_2s)-U(t_2)}{U(t_2)}. \tag{2.10}$$

Letting $t_2 \to \infty$ in (2.10),

$$\frac{s^{\gamma}-1}{1-\varepsilon^{\gamma}} \leq \liminf \frac{U(t_2s)-U(t_2)}{U(t_2)-U(t_1)} \leq \limsup \frac{U(t_2s)-U(t_2)}{U(t_2)-U(t_1)} \leq s^{\gamma}-1. \tag{2.11}$$

Letting $\varepsilon \downarrow 0$ in (2.11),

$$\frac{U(t_2s)-U(t_2)}{U(t_2)-U(t_1)} \to s^{\gamma}-1.$$

Thus,

$$\log\frac{U(t_2s)-U(t_1)}{U(t_2)-U(t_1)} = \log\left(\frac{U(t_2s)-U(t_2)}{U(t_2)-U(t_1)}+1\right) \to \log(s^{\gamma}-1+1) = \gamma\log s.$$

The proof is similar for $s > 1$.

(ii) Consider $\gamma = 0$. Note that

$$\frac{a(t_2)}{U(t_2)-U(t_1)} = \frac{1}{(U(t_2)-U(t_1))/a(t_2)} \sim \log^{-1}\frac{t_2}{t_1} \to 0,$$

since $\log(1+x) \sim x$ as $x \to 0$, and

$$\frac{U(t_2s)-U(t_2)}{a(t_2)} \to \log s$$

as t_2, $t_2/t_1 \to \infty$. The result follows by rewriting

$$\log\frac{U(t_2s)-U(t_1)}{U(t_2)-U(t_1)} = \log\left(\frac{U(t_2s)-U(t_2)}{a(t_2)}\frac{a(t_2)}{U(t_2)-U(t_1)}+1\right).$$

(iii) Consider $\gamma < 0$. Given $N > 1$, there exists N_1 such that $t_1 > N_1$, $t_2/t_1 > N$ and

$$U(t_1N) - U(t_1) \leq U(t_2) - U(t_1) \leq U(\infty) - U(t_1).$$

If $0 < s \leq 1$, then we get from (2.8) that

$$\frac{U(t_2 s) - U(t_2)}{U(t_1 N) - U(t_1)} \leq \frac{U(t_2 s) - U(t_2)}{U(t_2) - U(t_1)} \leq \frac{U(t_2 s) - U(t_2)}{U(\infty) - U(t_1)}, \tag{2.12}$$

which can be rewritten as

$$\begin{aligned} -\frac{1}{\gamma}\frac{(V(t_2 s) - V(t_2))/V(t_2)}{(V(t_1 N) - V(t_1))/V(t_1)} &\leq \frac{1}{-\gamma V(t_2)/V(t_1)}\frac{U(t_2 s) - U(t_2)}{U(t_2) - U(t_1)} \\ &\leq \frac{V(t_2 s) - V(t_2)}{\gamma V(t_2)}. \end{aligned} \tag{2.13}$$

Let $V(t) = U(\infty) - U(t)$ so that $V(t) \in RV_\gamma$. Letting $t_1 \to \infty$ and $t_2 \to \infty$ in (2.13) we obtain that

$$\begin{aligned} -\frac{1}{\gamma}\frac{s^\gamma - 1}{N^\gamma - 1} &\leq \liminf \frac{1}{-\gamma V(t_2)/V(t_1)}\frac{U(t_2 s) - U(t_2)}{U(t_2) - U(t_1)} \\ &\leq \limsup \frac{1}{-\gamma V(t_2)/V(t_1)}\frac{U(t_2 s) - U(t_2)}{U(t_2) - U(t_1)} \leq \frac{s^\gamma - 1}{\gamma}. \end{aligned} \tag{2.14}$$

Letting $N \to \infty$ in (2.14),

$$\frac{a(t_2)}{U(t_2) - U(t_1)} = \frac{a(t_2)}{V(t_1) - V(t_2)} \sim -\gamma\frac{V(t_2)}{V(t_1)} \to 0 \tag{2.15}$$

as $t_2/t_1 \to \infty$, which proves the result. The proof is similar for $s > 1$.

The other results of the lemma follow from Geluk and de Haan [12], page 27. □

Lemma 2.2. *Let $0 < k = k(n) \leq n$ and $k(n) \to \infty$ as $n \to \infty$.*

(i) *If $F(x) = x^\alpha$, $0 < x < 1$, for some $\alpha > 0$, then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{i,n}}{X_{k(n)+1,n}} = \frac{\alpha}{\alpha + 1}$$

in probability.

(ii) *If $F(x) = 1 - x^{-\alpha}$, $x > 1$, for some $\alpha > 1$, then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{n-i-1,n}}{X_{n-k(n),n}} = \frac{\alpha}{\alpha - 1}$$

in probability.

The proof follows from Lemma 2.4 in Dekkers et al. [6].

Lemma 2.3. *Let $0 < k = k(n) \leq n$ and $k(n)/(\log n)^\delta \to \infty$ as $n \to \infty$ for some $\delta > 0$.*

(i) *If $F(x) = x^\alpha$, $0 < x < 1$, for some $\alpha > 0$, then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{i,n}}{X_{k(n)+1,n}} = \frac{\alpha}{\alpha + 1}$$

almost surely.

(ii) *If $F(x) = 1 - x^{-\alpha}$, $x > 1$, for some $\alpha > 2(1+\delta)/\delta$, then*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} \frac{X_{n-i-1,n}}{X_{n-k(n),n}} = \frac{\alpha}{\alpha - 1}$$

almost surely.

The proof follows from Lemma 2.3 in Dekkers et al. [6].

Proof of Theorem 2.1. We prove (2.3) for the three cases $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$.

(i) Consider $\gamma > 0$. One can write

$$\begin{aligned}\hat{\gamma}_{n,j}(k_0,k) &\stackrel{d}{=} \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left[\log\frac{U(Y_{n-i,n})-U(Y_{n-k,n})}{U(Y_{n-k_0,n})-U(Y_{n-k,n})}\right]^j \\ &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left[\log\frac{(Y_{n-i,n}/Y_{n-k,n})^\gamma-1}{(Y_{n-k_0,n}/Y_{n-k,n})^\gamma-1}(1+o_p(1))\right]^j \\ &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left[\log Y_{k_0-i,k_0}^\gamma\frac{1-Y_{k-i,k}^{-\gamma}}{1-Y_{k-k_0,k}^{-\gamma}}(1+o_p(1))\right]^j \\ &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left[\gamma E_i(1+o_p(1))\right]^j,\end{aligned}$$

which implies that

$$\hat{\gamma}_{n,1}(k_0,k) \xrightarrow{\mathsf{P}} \gamma$$

and

$$\hat{\gamma}_{n,2}(k_0,k) \xrightarrow{\mathsf{P}} 2\gamma^2$$

and thus (2.3) is proved.

(ii) Consider $\gamma = 0$. Since $k = k(n)$, $k_0 = k_0(k)$ are intermediate integer sequences and $U(t) \in GRV_\gamma$,

$$\begin{aligned}\log\frac{U(Y_{n-i,n})-U(Y_{n-k,n})}{U(Y_{n-k_0,n})-U(Y_{n-k,n})} &= \log\left(1+\frac{U(Y_{n-i,n})-U(Y_{n-k_0,n})}{U(Y_{n-k_0,n})-U(Y_{n-k,n})}\right) \\ &= \frac{\log(Y_{n-i,n}/Y_{n-k_0,n})}{\log(Y_{n-k_0,n}/Y_{n-k,n})}(1+o_p(1)) \\ &= \frac{\log Y_{k_0-i,k_0}}{\log Y_{k-k_0,k}}(1+o_p(1))\end{aligned}$$

and

$$\begin{aligned}\hat{\gamma}_{n,j}(k_0,k) &= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left[\log\frac{U(Y_{n-i,n})-U(Y_{n-k,n})}{U(Y_{n-k_0,n})-U(Y_{n-k,n})}\right]^j \\ &= \frac{1}{(\log Y_{k-k_0,k})^j}\frac{1}{k_0}\sum_{i=0}^{k_0-1}(\log Y_{k_0-i,k_0})^j(1+o_p(1)).\end{aligned}$$

Thus,

$$\log Y_{k-k_0,k}\hat{\gamma}_{n,1}(k_0,k) \xrightarrow{\mathsf{P}} 1$$

and

$$\log^2 Y_{k-k_0,k}\hat{\gamma}_{n,2}(k_0,k) \xrightarrow{\mathsf{P}} 2.$$

The result in (2.3) follows by noting that $\log Y_{k-k_0,k} \xrightarrow{\mathsf{P}} \infty$.

(iii) Consider $\gamma < 0$. Since $k = k(n)$, $k_0 = k_0(k)$ are intermediate integer sequences and $U(t) \in GRV_\gamma$,

$$\begin{aligned}\log \frac{U(Y_{n-i,n}) - U(Y_{n-k,n})}{U(Y_{n-k_0,n}) - U(Y_{n-k,n})} &= \log\left(1 + \frac{U(Y_{n-i,n}) - U(Y_{n-k_0,n})}{U(Y_{n-k_0,n}) - U(Y_{n-k,n})}\right) \\ &= \frac{(Y_{n-i,n}/Y_{n-k_0,n})^\gamma - 1}{1 - (Y_{n-k_0,n}/Y_{n-k,n})^{-\gamma}}(1 + o_p(1)) \\ &= \frac{Y_{k_0-i,k_0}^\gamma - 1}{1 - Y_{k-k_0,k}^{-\gamma}}(1 + o_p(1))\end{aligned}$$

and so

$$-\big(1 - Y_{k-k_0,k}^{-\gamma}\big)\hat{\gamma}_{n,1}(k_0,k) \xrightarrow{\mathsf{P}} \frac{-\gamma}{1-\gamma}$$

and

$$\big(1 - Y_{k-k_0,k}^{-\gamma}\big)^2\hat{\gamma}_{n,2}(k_0,k) \xrightarrow{\mathsf{P}} \frac{2\gamma^2}{(1-\gamma)(1-2\gamma)}.$$

The result in (2.3) follows by noting that $1 - Y_{k-k_0,k}^{-\gamma} \xrightarrow{\mathsf{P}} -\infty$. □

Proof of Theorem 2.2. We prove (2.4) for the two cases $\gamma \geq 0$ and $\gamma < 0$ by establishing the limiting behavior of $\hat{\gamma}_{n,j}(k_0,k)$.

(i) Consider $\gamma \geq 0$. Since (2.6) holds, given $\varepsilon > 0$, one has by Lemma 2.1 that

$$\begin{aligned}&\frac{\hat{\gamma}_{n,j}(k_0,k)}{[f(Y_{n-k_0,n}, Y_{n-k,n})]^j} \\ &\quad= \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\frac{1}{f(Y_{n-k_0,n}, Y_{n-k,n})}\log\frac{U\left(\frac{Y_{n-i,n}}{Y_{n-k_0,n}}Y_{n-k_0,n}\right) - U(Y_{n-k,n})}{U(Y_{n-k_0,n}) - U(Y_{n-k,n})}\right\}^j \\ &\quad\overset{\text{a.s.}}{<} \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\varepsilon + (1+\varepsilon)\frac{\frac{Y_{n-i,n}^\varepsilon}{Y_{n-k_0,n}^\varepsilon} - 1}{\varepsilon}\right\}^j\end{aligned}$$

almost surely for all sufficiently large n and for $j = 1, 2$. First consider $j = 1$. Since $Y_{n-i,n}^\varepsilon$ is the $(n-i)$th order statistic from the d.f. $1 - 1/x^{1/\varepsilon}$ $(x > 1)$, we can apply Lemma 2.3 for $\varepsilon < \min\{\delta_1/(2(1+\delta_1)), \delta_2/(2(1+\delta_2))\}$ and find

$$\limsup_{n\to\infty}\frac{\hat{\gamma}_{n,1}(k_0,k)}{f(Y_{n-k_0,n}, Y_{n-k,n})} \leq \varepsilon + (1+\varepsilon)\frac{\left\{\frac{\varepsilon^{-1}}{\varepsilon^{-1}-1} - 1\right\}}{\varepsilon}$$

almost surely. This, together with a similar lower inequality, gives

$$\lim_{n\to\infty}\frac{\hat{\gamma}_{n,1}(k_0,k)}{f(Y_{n-k_0,n}, Y_{n-k,n})} = 1$$

almost surely. Next, because $kY_{n-k,n}/n \to 1$ almost surely and $k_0Y_{n-k_0,n}/n \to 1$ almost surely, it follows from the property of $f(t_2, t_1)$ that

$$\frac{1}{f(n/k_0, n/k)}f\left(\frac{Y_{n-k_0,n}}{n/k_0}\frac{n}{k_0}, \frac{Y_{n-k,n}}{n/k}\frac{n}{k}\right) = 1$$

almost surely. The case $j = 2$ is similar. Thus, for $j = 1, 2$,

$$\lim_{n\to\infty}\frac{\hat{\gamma}_{n,j}(k_0,k)}{[f(Y_{n-k_0,n}, Y_{n-k,n})]^j} = j!$$

almost surely.

(ii) Consider $\gamma < 0$. Given $\varepsilon > 0$ one finds as in part (i), now using (2.7) in Lemma 2.1, that

$$\begin{aligned}
&\frac{\hat{\gamma}_{n,j}(k_0,k)}{[f(Y_{n-k_0,n},Y_{n-k,n})]^j} \\
&\quad = \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{\frac{1}{f(Y_{n-k_0,n},Y_{n-k,n})}\log\frac{U\left(\frac{Y_{n-i,n}}{Y_{n-k_0,n}}Y_{n-k_0,n}\right)-U\left(Y_{n-k,n}\right)}{U(Y_{n-k_0,n})-U(Y_{n-k,n})}\right\}^j \\
&\quad \overset{\text{a.s.}}{<} \frac{1}{k_0}\sum_{i=0}^{k_0-1}\left\{1-(1-\varepsilon)\frac{Y_{n-i,n}^{\gamma-\varepsilon}}{Y_{n-k_0,n}^{\gamma-\varepsilon}}\right\}^j
\end{aligned}$$

almost surely for all sufficiently large n and for $j = 1, 2$. Since $Y_{n-i,n}^{\gamma-\varepsilon}$ is the $(i+1)$st order statistic from the d.f. $x^{1/(-\gamma+\varepsilon)}$ $(0 < x < 1)$, one can apply Lemma 2.3 and find that

$$\limsup_{n\to\infty}\frac{\hat{\gamma}_{n,1}(k_0,k)}{[f(Y_{n-k_0,n},Y_{n-k,n})]} \le 1-(1-\varepsilon)\frac{(\varepsilon-\gamma)^{-1}}{(\varepsilon-\gamma)^{-1}+1}$$

almost surely. This, together with a similar lower inequality, gives

$$\lim_{n\to\infty}\frac{\hat{\gamma}_{n,1}(k_0,k)}{[f(Y_{n-k_0,n},Y_{n-k,n})]} \to \frac{-\gamma}{1-\gamma}$$

almost surely. Following arguments similar to those in part (i), one can now show that

$$\lim_{n\to\infty}\frac{\hat{\gamma}_{n,j}(k_0,k)}{[f(Y_{n-k_0,n},Y_{n-k,n})]^j} = \begin{cases} -\gamma/(1-\gamma), & j=1, \\ 2\gamma^2/\{(1-\gamma)(1-2\gamma)\}, & j=2 \end{cases}$$

almost surely. □

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