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# MODELLING LOG GAUSSIAN COX PROCESSES WITH A GIVEN RELIABILITY AND ACCURACY

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Abstract. We consider stochastic Cox processes governed by a random intensity. Namely we consider the case where the logarithm of intensity is a separable stationary Gaussian stochastic process. We construct models approximating log Gaussian Cox processes with a given reliability and accuracy.

## 1. Introduction

The paper is devoted to the construction of models for the so-called doubly stochastic Poisson processes or, in other words, Cox processes governed by a random intensity. In particular, we consider the case where the intensity is a log Gaussian stochastic process. Log Gaussian Cox processes and their models are studied in the papers [1, 2] for the case where the intensity is a random field. In contrast to the papers [1, 2], we propose an approach to model stochastic processes with a given reliability and accuracy.

Below we consider necessary definitions and the setting of the modelling problem. Section 2 is devoted to the problem of choosing a partition used in the model. The model itself is constructed in Section 3. Sufficient conditions for the approximation with a given reliability and accuracy are also given in Section 3.

Let $\{\Omega, \mathfrak{F}, \mathsf{P}\}$ be a standard probability space, $\mathfrak{B}$ the $\sigma$-algebra of Borel sets in

$$\mathbf{T} = [0, T], \qquad T \in \mathbf{R},$$

$\{Y(t), t \in \mathbf{T}\}$ a stationary mean square continuous Gaussian stochastic process such that $\mathsf{E}\, Y(t) = 0$ and $\mathsf{E}\, Y(t)Y(s) = B(t-s)$.

**Definition 1.1.** We say that $\{\nu(B), B \in \mathfrak{B}\}$ is a log Gaussian Cox process or a Cox process governed by a log Gaussian process $\exp\{Y(t)\}$ if

1) the random variables $\nu(B_1)$ and $\nu(B_2)$ are independent for all $B_1, B_2 \in \mathfrak{B}$ such that $B_1 \cap B_2 = \varnothing$;
2) $\mathsf{P}\{\nu(B) = k \,/\, Y(t), t \in \mathbf{T}\} = \exp\{-\mu(B)\}\,(\mu(B))^k/k!$, $k = 0, 1, 2, \dots$, where
$$\mu(B) = \int_B \exp\{Y(t, \cdot)\}\, dt,$$
and $Y(t, \cdot)$, $t \in \mathbf{T}$, is a realization of the process $\{Y(t), t \in \mathbf{T}\}$.

A model for the log Gaussian Cox process is constructed in the following way: consider a partition of the interval $\mathbf{T} = [0, T]$ substituted by $k$ subintervals of length $d = T/k$. Let $0 = t_0 < t_1 < \dots < t_k = T$ and $t_{i+1} - t_i = d$ for all $i = 0, \dots, k-1$. Put $B_i = [t_i, t_{i+1}]$

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and denote by $\widetilde{Y}(t)$ a model of the process $Y(t)$. Let $\widetilde{\mu}(B_i) = \int_{B_i} \exp\{\widetilde{Y}(t)\}\,dt$. For every $i = 0, \dots, k-1$, we construct a model of the log Gaussian Cox process $\widetilde{\nu}(B_i)$, that is, a model for a Poisson random variable with the expectation $\widetilde{\mu}(B_i)$.

Since $\widetilde{\nu}(B_i)$ means the number of points belonging to the set $B_i$, we choose these points in the interval $B_i$ in an arbitrary way. However if $\widetilde{\nu}(B_i) = 1$, then we choose the middle point of the interval.

We treat a model as acceptable if the conditional probabilities

$$p_{kY}(B_i) = \mathsf{P}\{\nu(B_i) = k \,/\, Y(t), t \in \mathbf{T}\}$$

and

$$\widetilde{p}_{kY}(B_i) = \mathsf{P}\big\{\widetilde{\nu}(B_i) = k \,/\, \widetilde{Y}(t), t \in \mathbf{T}\big\}$$

are close to each other and the probability of the event that the number of points $\nu(B_i)$ (as well as $\widetilde{\nu}(B_i)$) exceeds 1 is small.

Thus the problem of modelling for a log Gaussian Cox process consists of two steps. Namely, a partition of the interval $\mathbf{T}$ is chosen at the first step, while a model for the process $Y(t)$ is constructed at the second step.

## 2. The partition of the domain $\mathbf{T}$

A partition of the domain $\mathbf{T}$ (in other words, a number $d$ or, equivalently, a number $k$) should be such that

$$\mathsf{P}\{\nu(B_i) > 1\} < \delta, \tag{1}$$

where $\delta$ is a given number (say, $\delta = 0.01$).

**Theorem 2.1.** *The inequality* $\mathsf{P}\{\nu(B_i) > 1\} < \delta$ *holds if the number* $d = T/k$ *is chosen such that*

$$d \le \left[2\delta \exp\left\{-2B(0)\right\}\right]^{1/2}. \tag{2}$$

*Proof.* Since

$$\mathsf{P}\left\{\nu(B_i) > 1\right\} = \mathsf{E}\left(1 - \exp\left\{-\mu(B_i)\right\} - \mu\left(B_i\right)\exp\left\{-\mu(B_i)\right\}\right),$$

a partition should be such that

$$\mathsf{E}\left(1 - \exp\left\{-\mu(B_i)\right\} - \mu\left(B_i\right)\exp\left\{-\mu(B_i)\right\}\right) < \delta.$$

Using the inequality $1 - \exp\{-x\}(1+x) \le x^2/2$ for all $x > 0$, the latter bound follows from

$$\mathsf{E}[\mu(B_i)]^2 < 2\delta. \tag{3}$$

We know for Gaussian random variables $\xi \simeq N(0, \sigma^2)$ that

$$\mathsf{E}\exp\{\lambda\xi\} = \exp\left\{\frac{\lambda^2\sigma^2}{2}\right\},$$

whence

$$\begin{aligned}
\mathsf{E}[\mu(B_i)]^2 &= \mathsf{E}\left[\int_{B_i} \exp\{Y(t)\}\,dt\right]^2 = \mathsf{E}\int_{B_i} \exp\{Y(t)\}\,dt \int_{B_i} \exp\{Y(s)\}\,ds \\
&= \iint_{B_i\times B_i} \mathsf{E}\exp\{Y(t)+Y(s)\}\,dt\,ds \\
&= \iint_{B_i\times B_i} \exp\left\{\frac{\mathsf{E}(Y(t)+Y(s))^2}{2}\right\} dt\,ds \\
&= \iint_{B_i\times B_i} \exp\left\{\frac{\mathsf{E}\,(Y(t))^2}{2} + \mathsf{E}\,Y(t)Y(s) + \frac{\mathsf{E}(Y(s))^2}{2}\right\} dt\,ds \\
&= \iint_{B_i\times B_i} \exp\{B(0)+B(t-s)\}\,dt\,ds \\
&= \exp\{B(0)\} \iint_{B_i\times B_i} \exp\{B(t-s)\}\,dt\,ds \\
&\le d^2 \exp\{2B(0)\}.
\end{aligned}$$

Therefore inequality (3) follows from assumption (2). Thus inequality (1) also follows from (2). □

## 3. Construction of a model for the process $Y(t)$

Recall that our aim is to construct a model for the process $Y(t)$ such that the conditional probabilities

$$p_{kY}(B_i) = \mathsf{P}\{\nu(B_i) = k \,/\, Y(t), t\in \mathbf{T}\} \quad \text{and} \quad \widetilde{p}_{kY}(B_i) = \mathsf{P}\big\{\widetilde{\nu}(B_i) = k \,/\, \widetilde{Y}(t), t\in \mathbf{T}\big\},$$
$$i = 0,\dots,k-1,$$

are close to each other with probability approaching one. The following definition is natural for our setting.

**Definition 3.1.** We say that a model $\widetilde{\nu}(B_i)$ of a log Gaussian Cox stochastic process $\{\widetilde{\nu}(B), B\in\mathfrak{B}\}$ approximates it with an accuracy $\alpha$, $0<\alpha<1$, and reliability $1-\gamma$, $0<\gamma<1$, if

$$\max_{B_i\in\mathfrak{B}, i=0,\dots,k-1} \mathsf{P}\left\{|p_{kY}(B_i) - \widetilde{p}_{kY}(B_i)| > \alpha\right\} < \gamma.$$

Now we estimate the difference $|p_{kY}(B_i) - \widetilde{p}_{kY}(B_i)|$ by using the mean value theorem for derivatives:

$$\begin{aligned}
|p_{kY}(B_i) - \widetilde{p}_{kY}(B_i)| &= \left|\frac{\exp\{-\mu(B_i)\}\,(\mu(B_i))^k}{k!} - \frac{\exp\{-\widetilde{\mu}(B_i)\}\,(\widetilde{\mu}(B_i))^k}{k!}\right| \\
&= |\mu(B_i) - \widetilde{\mu}(B_i)|\,\frac{1}{k!}\exp\{-\hat{\mu}(B_i)\}\,(\hat{\mu}(B_i))^{k-1}\,|k-\hat{\mu}(B_i)| \\
&= \begin{cases} |\mu(B_i) - \widetilde{\mu}(B_i)|\,\frac{1}{(k-1)!}\exp\{-\hat{\mu}(B_i)\}\,(\hat{\mu}(B_i))^{k-1} & \\ \qquad \le |\mu(B_i) - \widetilde{\mu}(B_i)|, & k\ge\hat{\mu}(B_i), \\ |\mu(B_i) - \widetilde{\mu}(B_i)|\,\frac{1}{k!}\exp\{-\hat{\mu}(B_i)\}\,(\hat{\mu}(B_i))^{k} & \\ \qquad \le |\mu(B_i) - \widetilde{\mu}(B_i)|, & k<\hat{\mu}(B_i). \end{cases}
\end{aligned}$$

If $k=0$, then

$$\begin{aligned}
|p_{0Y}(B_i) - \widetilde{p}_{0Y}(B_i)| &= |\exp\{-\mu(B_i)\} - \exp\{-\widetilde{\mu}(B_i)\}| \\
&\le |\mu(B_i) - \widetilde{\mu}(B_i)|\,|\exp\{-\hat{\mu}(B_i)\}| \le |\mu(B_i) - \widetilde{\mu}(B_i)|.
\end{aligned}$$

Therefore the difference of probabilities $|p_{kY}(B_i) - \widetilde{p}_{kY}(B_i)|$ is estimated in terms of $|\mu(B_i) - \widetilde{\mu}(B_i)|$, whence we obtain

$$\mathsf{P}\left\{|p_{kY}(B_i) - \widetilde{p}_{kY}(B_i)| > \alpha\right\} \le \mathsf{P}\left\{|\mu(B_i) - \widetilde{\mu}(B_i)| > \alpha\right\}, \qquad i = 0, \dots, k-1. \tag{4}$$

It is easy to check that

$$\begin{aligned}
&\max_{B_i \in \mathfrak{B},\, i=0,\dots,k-1} \mathsf{P}\left\{|\mu(B_i) - \widetilde{\mu}(B_i)| > \alpha\right\} \\
&\quad = \max_{B_i \in \mathfrak{B}, i=0,\dots,k-1} \mathsf{P}\left\{\left|\int_{B_i} \exp\{Y(t)\}\, dt - \int_{B_i} \exp\left\{\widetilde{Y}(t)\right\} dt\right| > \alpha\right\} \\
&\quad \le \max_{B_i \in \mathfrak{B},\, i=0,\dots,k-1} \mathsf{P}\left\{\int_{B_i} \sup_{t \in B_i} \left|\exp\{Y(t)\} - \exp\left\{\widetilde{Y}(t)\right\}\right| dt > \alpha\right\} \\
&\quad = \max_{B_i \in \mathfrak{B},\, i=0,\dots,k-1} \mathsf{P}\left\{\int_{B_i} dt \cdot \sup_{t \in B_i} \left|\exp\{Y(t)\} - \exp\left\{\widetilde{Y}(t)\right\}\right| > \alpha\right\} \\
&\quad = \mathsf{P}\left\{\sup_{t \in B_i} \left|\exp\{Y(t)\} - \exp\left\{\widetilde{Y}(t)\right\}\right| > \frac{\alpha}{d}\right\}.
\end{aligned} \tag{5}$$

Thus the problem of estimating $|\mu(B_i) - \widetilde{\mu}(B_i)|$ is reduced to that for

$$\sup_{t \in B_i} \left|\exp\{Y(t)\} - \exp\{\widetilde{Y}(t)\}\right|.$$

In what follows we use the following result.

**Theorem 3.1.** *Let $X = \{X(t), t \in \mathbf{T}\}$ be a stochastic process in the space $L_p(\Omega)$ such that $\sup_{t \in \mathbf{T}} \|X(t)\|_{L_p} < +\infty$. Assume that the space $(\mathbf{T}, \rho_X)$ as well as the pseudometric $\rho_X(t,s)$ generated by the process $X(t)$ and the process itself are separable. Let*

$$\int_0^\infty N^{1/p}(\varepsilon)\, d\varepsilon < +\infty,$$

*where $N(\varepsilon)$ is the metric capacity of the set $\mathbf{T}$ and $\varepsilon_0 = \sup_{t,s \in \mathbf{T}} \rho_X(t,s)$. Then*

$$\left(\mathsf{E}\left[\sup_{t \in \mathbf{T}} |X(t)|\right]^p\right)^{1/p} \le V_p.$$

*Moreover*

$$\mathsf{P}\left\{\sup_{t \in \mathbf{T}} |X(t)| \ge x\right\} \le \frac{V_p^p}{x^p}$$

*for all $x > 0$, where*

$$V_p = \inf_{t \in \mathbf{T}} \left(\mathsf{E}\,|X(t)|^p\right)^{1/p} + \inf_{0<\theta<1} \frac{1}{\theta(1-\theta)} \int_0^{\theta\varepsilon_0} N^{1/p}(\varepsilon)\, d\varepsilon.$$

Let $\mathbf{T} = [0,T]$ and $\sup_{t \in \mathbf{T}} \|X(t+\varepsilon) - X(t)\|_{L_p} \le \varphi(\varepsilon)$, where $\varphi(\varepsilon)$, $\varepsilon > 0$, is a nondecreasing function such that $\varphi(0) = 0$. Then

$$N(\varepsilon) \le \frac{T}{2\varphi^{(-1)}(\varepsilon)} + 1 \le \frac{T}{\varphi^{(-1)}(\varepsilon)}.$$

The proof of the latter theorem and the above inequality can be found in [3].

We consider a centered mean square continuous and wide sense stationary stochastic process. The covariance function $B(\tau)$ of such a process can be represented as follows:

$$B(\tau) = \int_0^\infty \cos \lambda\tau\, dF(\lambda),$$

where $F(\lambda)$, $\lambda \in [0, +\infty]$, is a nondecreasing left continuous function such that $F(0) = 0$ and $F(+\infty) = B(0)$. By the Karhunen theorem the centered stationary process $Y(t)$ can be represented in the following form:

$$Y(t) = \int_0^\infty \cos \lambda t \, d\xi(\lambda) + \int_0^\infty \sin \lambda t \, d\eta(\lambda),$$

where $\xi(\lambda)$ and $\eta(\lambda)$ are centered stochastic processes with uncorrelated increments and such that

$$\mathsf{E}(\xi(\lambda_2) - \xi(\lambda_1))^2 = \mathsf{E}(\eta(\lambda_2) - \eta(\lambda_1))^2 = F(\lambda_2) - F(\lambda_1)$$

for all $\lambda_1 < \lambda_2$. Here $F(\lambda)$ is the spectral function defined in the representation of the covariance function.

We say that a finite sum

$$\widetilde{Y}(t) = \sum_{k=0}^{N-1} \big(\cos \lambda_k t \Delta_k \xi(\lambda) + \sin \lambda_k t \Delta_k \eta(\lambda)\big) \tag{6}$$

is a model of the process $\widetilde{Y}(t)$, where

$$\Delta_k \xi(\lambda) = \int_{\lambda_k}^{\lambda_{k+1}} d\xi(\lambda), \qquad \Delta_k \eta(\lambda) = \int_{\lambda_k}^{\lambda_{k+1}} d\eta(\lambda),$$

and $\lambda_k$ are the points of the partition $D_\Lambda$, that is, $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_N = \Lambda$.

*Remark* 3.1. Since the process $Y(t)$ is Gaussian, the processes $\xi(\lambda)$ and $\eta(\lambda)$ also are Gaussian (this follows from the Karhunen theorem).

**Theorem 3.2.** *Let $Y(t)$ be a separable centered mean square continuous stationary Gaussian stochastic process with the spectral function $F(\lambda)$. Let the assumptions of Theorem* 3.1 *hold for the process $Y(t)$ and assume that the spectral moment*

$$\int_0^\infty \lambda^{2b} \, dF(\lambda)$$

*exists for some $0 < b \le 1$. Consider a partition $D_\Lambda$:*

$$\lambda_{k-1} - \lambda_k = \Lambda/N, \qquad \Lambda \in \mathbf{R}, \qquad N \in \mathbf{N}.$$

*Then a model $\{\widetilde{\nu}(B), B \in \mathfrak{B}\}$ with $\widetilde{Y}(t)$ defined by* (6) *approximates the log Gaussian Cox process with accuracy $\alpha$ and reliability $1 - \gamma$ if*

$$\alpha > dM_N \exp\left\{\frac{Z}{b}\right\},$$

$$\begin{aligned} &\left(\frac{dG}{\alpha}\right)^{\frac{\ln \frac{\alpha}{M_N d}}{Z}} \left(\frac{\ln \frac{\alpha}{M_N d}}{Z}\right)^{\frac{\ln \frac{\alpha}{M_N d}}{2Z}} \exp\left\{\frac{v_2 B(0) \ln^2 \frac{\alpha}{M_N d}}{2Z^2}\right\} \\ &\quad + D\left(\frac{\ln \frac{\alpha}{M_N d}}{Z}\right)^{\frac{\ln \frac{\alpha}{M_N d}}{Z} + \frac{1}{b}} \exp\left\{-\frac{\ln^2 \frac{\alpha}{M_N d}}{2Z}\right\} \left(1 - \frac{Z}{b \ln \frac{\alpha}{M_N d}}\right)^{-\frac{\ln \frac{\alpha}{M_N d}}{Z}} < \gamma, \end{aligned} \tag{7}$$

*where*

$$G = 2\left(B(0) - F(\Lambda)\right)^{1/2}(v_1+1)^{1/2}, \qquad Z = r_2(f_1 A_N + f_2 B(0)),$$

$$M_N = 2d^b(\max(r_1,s_1,s_2)+1)K_{N,b}, \qquad D = b^{1/b}\exp\left\{\frac{1}{b}+\frac{1}{b^2}\right\},$$

$$K_{N,b} = k_{N,b}^{1/2} + \left(2^{3-2b}\Lambda^{2b}F(\Lambda)A_N\right)^{1/2},$$

$$A_N = B(0) - F(\Lambda) + 2^{2-2b}d^{2b}\left(\frac{\Lambda}{N}\right)^{2b}F(\Lambda),$$

$$k_{N,b} = 2^{5-4b}\left(\left(\frac{\Lambda}{N}\right)^b + \left(\frac{\Lambda}{N}\right)^a\left(\frac{d}{2}\right)^a(2\Lambda)^b\right)^2 F(\Lambda) + 2^{3-2b}C_b,$$

$$C_b = \int_\Lambda^\infty \lambda^{2b}\,dF(\lambda), \qquad 0<a\le 1,\ 0<b\le 1.$$

*Here the numbers $v_1$, $v_2$, $f_1$, $f_2$, $s_1$, $s_2$, $s_3$, $r_1$, and $r_2$ are such that*

$$\frac{1}{v_1}+\frac{1}{v_2}=1, \qquad \frac{1}{f_1}+\frac{1}{f_2}=1, \qquad \frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}=1, \qquad \frac{1}{r_1}+\frac{1}{r_2}=1, \qquad r_2=s_3$$

*and $B(t)$ is the covariance function of the process $Y(t)$.*

*Proof.* To estimate the moment of the difference $\left(\mathsf{E}\left|\exp\{Y(t)\}-\exp\{\widetilde{Y}(t)\}\right|^p\right)^{1/p}$ we use the inequality $|\exp\{x\}-\exp\{y\}|\le|x-y|\exp\{\max(x,y)\}$ and then the Hölder inequality. As a result we obtain

$$\begin{aligned}
&\left(\mathsf{E}\left|\exp\{Y(t)\}-\exp\left\{\widetilde{Y}(t)\right\}\right|^p\right)^{1/p}\\
&\quad\le\left(\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^p\exp\left\{p\max\left(Y(t),\widetilde{Y}(t)\right)\right\}\right)^{1/p}\\
&\quad\le\left(\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^{pv_1}\right)^{1/(pv_1)}\left(\exp\left\{pv_2\max\left(Y(t),\widetilde{Y}(t)\right)\right\}\right)^{1/(pv_2)},
\end{aligned}\tag{8}$$

where $v_1$ and $v_2$ are real numbers such that $v_1^{-1}+v_2^{-1}=1$.

If a random variable $\xi$ is Gaussian with parameters 0 and $\sigma^2$, then its moments are such that

$$\mathsf{E}\,|\xi|^p = c_p\left(\sigma^2\right)^{p/2}\tag{9}$$

for $c_p = 2^{p/2}\pi^{-1/2}\Gamma((p+1)/2)$. Using Stirling's formula we get

$$\begin{aligned}
c_p &\le \frac{2^{p/2}}{\sqrt{\pi}}\sqrt{\frac{2\pi}{\frac{p+1}{2}}}\left(\frac{p+1}{2e}\right)^{(p+1)/2}\exp\left\{\frac{1}{12\frac{p+1}{2}}\right\}\\
&\le 2^{(p+1)/2}\left(\frac{p+1}{2}\right)^{p/2}\frac{\exp\{1/12\}}{\exp\{p/2\}}\le\sqrt{2}\,(p+1)^{p/2}.
\end{aligned}\tag{10}$$

Considering equality (9) we obtain

$$\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^{pv_1} = \left(\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^2\right)^{pv_1/2}c_{pv_1}.$$

Since $\mathsf{E}(\xi(t))^2 = B(0)$ and $\mathsf{E}\left(\widetilde{\xi}(t)\right)^2 = F(\Lambda)$ for centered stationary Gaussian stochastic processes, we have

$$\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^2 = B(0)+F(\Lambda)-2\,\mathsf{E}\,Y(t)\widetilde{Y}(t).$$

Furthermore,

$$\begin{aligned}
\mathsf{E}\, Y(t)\widetilde{Y}(t) &= \mathsf{E}\Bigg(\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \cos\lambda t\, d\xi(\lambda) + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \sin\lambda t\, d\eta(\lambda) \\
&\qquad + \int_{\Lambda}^{\infty} \cos\lambda t\, d\xi(\lambda) + \int_{\Lambda}^{\infty} \sin\lambda t\, d\eta(\lambda)\Bigg) \\
&\quad \times \left(\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \cos\lambda_k t\, d\xi(\lambda) + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \sin\lambda_k t\, d\eta(\lambda)\right) \\
&= \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \cos\lambda t \cos\lambda_k t\, dF(\lambda) + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \sin\lambda t \sin\lambda_k t\, dF(\lambda) \\
&= \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \cos t(\lambda-\lambda_k)\, dF(\lambda), \\
\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^2 &= B(0) - F(\Lambda) + 2F(\Lambda) - 2\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \cos t\,(\lambda-\lambda_k)\; dF(\lambda) \\
&= B(0) - F(\Lambda) + 2\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 2\sin^2\left(\frac{t\,(\lambda-\lambda_k)}{2}\right) dF(\lambda) \\
&\le B(0) - F(\Lambda) + 4\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \frac{t^{2b}\,(\lambda-\lambda_k)^{2b}}{2^{2b}}\, dF(\lambda).
\end{aligned}$$

Since $\lambda - \lambda_k \le \lambda_{k+1} - \lambda_k = \Lambda/N$ for $k = 0, \dots, N-1$, we get

$$\begin{gathered}
\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^2 \le A_N, \\
A_N = B(0) - F(\Lambda) + 2^{2-2b} d^{2b}\left(\frac{\Lambda}{N}\right)^{2b} F(\Lambda).
\end{gathered} \tag{11}$$

Thus

$$\left(\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^{pv_1}\right)^{1/(pv_1)} \le A_N^{1/2} c_{pv_1}^{1/(pv_1)}. \tag{12}$$

Now we estimate the expectation of the exponent $\exp\{pv_2 \max(Y(t), \widetilde{Y}(t))\}$. Note that

$$\mathsf{E}\exp\{\lambda\xi\} = \exp\{\lambda^2\sigma^2/2\}$$

for a random variable $\xi \simeq N(0, \sigma^2)$. Hence

$$\begin{aligned}
\mathsf{E}\exp\left\{pv_2 \max\left(Y(t), \widetilde{Y}(t)\right)\right\} &\le \mathsf{E}\exp\{pv_2 Y(t)\} + \mathsf{E}\exp\left\{pv_2\widetilde{Y}(t)\right\} \\
&= \exp\left\{\frac{(pv_2)^2}{2}B(0)\right\} + \exp\left\{\frac{(pv_2)^2}{2}F(\Lambda)\right\} \\
&\le 2\exp\left\{\frac{(pv_2)^2}{2}B(0)\right\}.
\end{aligned}$$

The latter inequality and (12) imply that

$$
\begin{aligned}
&\left(\mathsf{E}\left|\exp\{Y(t)\}-\exp\left\{\widetilde{Y}(t)\right\}\right|^p\right)^{1/p}\\
&\quad\le A_N^{1/2}c_{pv_1}^{1/(pv_1)}2^{1/(pv_2)}\exp\left\{\frac{pv_2}{2}B(0)\right\}\\
&\quad= A_N^{1/2}\left(\sqrt{2}\,(pv_1+1)^{pv_1/2}\right)^{1/(pv_1)}2^{1/(pv_2)}\exp\left\{\frac{pv_2}{2}B(0)\right\}\\
&\quad\le 2^{1/p}A_N^{1/2}\,(pv_1+1)^{1/2}\exp\left\{\frac{pv_2}{2}B(0)\right\}
\end{aligned}
\tag{13}
$$

by bound (8).

Then

$$
\begin{aligned}
&\left(\mathsf{E}\left|\exp\{Y(t+h)\}-\exp\left\{\widetilde{Y}(t+h)\right\}-\left(\exp\{Y(t)\}-\exp\left\{\widetilde{Y}(t)\right\}\right)\right|^p\right)^{1/p}\\
&\quad=\Bigg(\mathsf{E}\Bigg|\left(e^{Y(t+h)-\widetilde{Y}(t+h)}-e^{Y(t)-\widetilde{Y}(t)}\right)e^{\widetilde{Y}(t+h)}\\
&\qquad\qquad+\left(e^{\widetilde{Y}(t+h)}-e^{\widetilde{Y}(t)}\right)\left(e^{Y(t)-\widetilde{Y}(t)}-1\right)\Bigg|^p\Bigg)^{1/p}\\
&\quad\le\left(\mathsf{E}\left|\left(e^{Y(t+h)-\widetilde{Y}(t+h)}-e^{Y(t)-\widetilde{Y}(t)}\right)e^{\widetilde{Y}(t+h)}\right|^p\right)^{1/p}\\
&\qquad+\left(\mathsf{E}\left|\left(e^{\widetilde{Y}(t+h)}-e^{\widetilde{Y}(t)}\right)\left(e^{Y(t)-\widetilde{Y}(t)}-1\right)\right|^p\right)^{1/p}\\
&\quad\le\left(\mathsf{E}\left|\Delta_1(Y)V_1\right|^p\right)^{1/p}+\left(\mathsf{E}\left|\Delta_2(Y)\Delta_3(Y)V_2\right|^p\right)^{1/p},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1(Y)&=\left|Y(t+h)-\widetilde{Y}(t+h)-Y(t)+\widetilde{Y}(t)\right|,\\
V_1&=\exp\left\{\max\left(Y(t+h)-\widetilde{Y}(t+h),Y(t)-\widetilde{Y}(t)\right)\right\}\exp\left\{\widetilde{Y}(t+h)\right\},\\
\Delta_2(Y)&=\left|\widetilde{Y}\,(t+h)-\widetilde{Y}(t)\right|,\\
\Delta_3(Y)&=\left|Y(t)-\widetilde{Y}\,(t)\right|,\\
V_2&=\exp\left\{\max\left(\widetilde{Y}(t+h),\widetilde{Y}(t)\right)\right\}\exp\left\{\left|Y(t)-\widetilde{Y}(t)\right|\right\}.
\end{aligned}
$$

Let

$$
r_1^{-1}+r_2^{-1}=1\quad\text{and}\quad s_1^{-1}+s_2^{-1}+s_3^{-1}=1.
$$

An application of the Hölder inequality yields

$$
\begin{aligned}
\mathsf{E}\left|\Delta_1(Y)V_1\right|^p&\le\left(\mathsf{E}\left|\Delta_1(Y)\right|^{pr_1}\right)^{1/r_1}\left(\mathsf{E}\left|V_1\right|^{pr_2}\right)^{1/r_2},\\
\mathsf{E}\left|\Delta_2(Y)\Delta_3\left(Y\right)V_2\right|^p&\le\left(\mathsf{E}\left|\Delta_2\left(Y\right)\right|^{ps_1}\right)^{1/s_1}\left(\mathsf{E}\left|\Delta_3(Y)\right|^{ps_2}\right)^{1/s_2}\left(\mathsf{E}\left|V_2\right|^{ps_3}\right)^{1/s_3}.
\end{aligned}
\tag{14}
$$

Note that

$$
\left(\mathsf{E}\left|\Delta_1(Y)\right|^{pr_1}\right)=\left(\sigma^2\right)^{pr_1/2}c_{pr_1}
$$

by (9), where $\sigma^2=\mathsf{E}\left|\Delta_1(Y)\right|^2$.

Now we estimate $\sigma^2$:

$$
\begin{aligned}
\sigma^2 &= \mathsf{E}\left|Y(t+h)-\widetilde{Y}(t+h)-Y(t)+\widetilde{Y}(t)\right|^2 \\
&= \mathsf{E}\Bigg|\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\big(\cos\lambda(t+h)-\cos\lambda_k(t+h)-\cos\lambda t+\cos\lambda_k t\big)\,d\xi(\lambda) \\
&\qquad+\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\big(\sin\lambda(t+h)-\sin\lambda_k(t+h)-\sin\lambda t+\sin\lambda_k t\big)\,d\eta(\lambda) \\
&\qquad+\int_{\Lambda}^{\infty}(\cos\lambda(t+h)-\cos\lambda t)\,d\xi(\lambda)+\int_{\Lambda}^{\infty}(\sin\lambda(t+h)-\sin\lambda t)\,d\eta(\lambda)\Bigg|^2 \\
&= \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\big(\cos\lambda(t+h)-\cos\lambda_k(t+h)-\cos\lambda t+\cos\lambda_k t\big)^2\,dF(\lambda) \\
&\quad+\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\big(\sin\lambda(t+h)-\sin\lambda_k(t+h)-\sin\lambda t+\sin\lambda_k t\big)^2\,dF(\lambda) \\
&\quad+\int_{\Lambda}^{\infty}(\cos\lambda(t+h)-\cos\lambda t)^2\,dF(\lambda)+\int_{\Lambda}^{\infty}(\sin\lambda(t+h)-\sin\lambda t)^2\,dF(\lambda) \\
&= \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}4\Bigg|\sin\frac{(t+h)\,(\lambda+\lambda_k)}{2}\sin\frac{(t+h)\,(\lambda-\lambda_k)}{2} \\
&\qquad\qquad-\sin\frac{(\lambda+\lambda_k)\,t}{2}\sin\frac{(\lambda-\lambda_k)\,t}{2}\Bigg|^2\,dF\,(\lambda) \\
&\quad+\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}4\Bigg|\cos\frac{(t+h)\,(\lambda+\lambda_k)}{2}\sin\frac{(t+h)\,(\lambda-\lambda_k)}{2} \\
&\qquad\qquad-\cos\frac{(\lambda+\lambda_k)\,t}{2}\sin\frac{(\lambda-\lambda_k)\,t}{2}\Bigg|^2\,dF\,(\lambda) \\
&\quad+\int_{\Lambda}^{\infty}4\left|\sin\frac{\lambda\,(2t+h)}{2}\right|^2\left|\sin\frac{\lambda h}{2}\right|^2\,dF\,(\lambda)+\int_{\Lambda}^{\infty}4\left|\cos\frac{\lambda\,(2t+h)}{2}\right|^2\left|\sin\frac{\lambda h}{2}\right|^2\,dF(\lambda) \\
&\le \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}4\Bigg(\left|\sin\frac{(t+h)(\lambda+\lambda_k)}{2}\right|\left|\sin\frac{(t+h)\,(\lambda-\lambda_k)}{2}-\sin\frac{(\lambda-\lambda_k)\,t}{2}\right| \\
&\qquad\qquad+\left|\sin\frac{(\lambda-\lambda_k)\,t}{2}\right|\left|\sin\frac{(t+h)\,(\lambda+\lambda_k)}{2}-\sin\frac{(\lambda+\lambda_k)\,t}{2}\right|\Bigg)^2\,dF(\lambda) \\
&\quad+\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}4\Bigg(\left|\sin\frac{(t+h)\,(\lambda-\lambda_k)}{2}\right|\left|\cos\frac{(t+h)\,(\lambda+\lambda_k)}{2}-\cos\frac{(\lambda+\lambda_k)t}{2}\right| \\
&\qquad\qquad+\left|\cos\frac{(\lambda+\lambda_k)\,t}{2}\right|\left|\sin\frac{(t+h)\,(\lambda-\lambda_k)}{2}-\sin\frac{(\lambda-\lambda_k)\,t}{2}\right|\Bigg)^2\,dF(\lambda) \\
&\quad+8\int_{\Lambda}^{\infty}\left(\frac{\lambda h}{2}\right)^{2b}\,dF(\lambda)
\end{aligned}
$$

$$
\begin{aligned}
&\le \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 4\Bigg(\left|2\cos\frac{(2t+h)(\lambda-\lambda_k)}{4}\sin\frac{(\lambda-\lambda_k)h}{4}\right| \\
&\qquad\qquad + \left|\sin\frac{(\lambda-\lambda_k)t}{2}\right|\left|2\cos\frac{(2t+h)(\lambda+\lambda_k)}{4}\sin\frac{(\lambda+\lambda_k)h}{4}\right|\Bigg)^2 dF(\lambda) \\
&\quad + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} \Bigg(\left|\sin\frac{(t+h)(\lambda-\lambda_k)}{2}\right|\left|2\sin\frac{(2t+h)(\lambda+\lambda_k)}{4}\sin\frac{(\lambda+\lambda_k)h}{4}\right| \\
&\qquad\qquad + \left|2\cos\frac{(2t+h)(\lambda-\lambda_k)}{4}\sin\frac{(\lambda-\lambda_k)h}{4}\right|\Bigg)^2 dF(\lambda) \\
&\quad + h^{2b}2^{3-2b}\int_{\Lambda}^{\infty}\lambda^{2b}\,dF(\lambda) \\
&\le \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 16\left(\frac{(\lambda-\lambda_k)^b h^b}{4^b} + \frac{(\lambda-\lambda_k)^a t^a}{2^a}\frac{(\lambda+\lambda_k)^b h^b}{4^b}\right)^2 dF(\lambda) \\
&\quad + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 16\left(\frac{(t+h)^a(\lambda-\lambda_k)^a}{2^a}\frac{(\lambda+\lambda_k)^b h^b}{4^b} + \frac{(\lambda-\lambda_k)^b h^b}{4^b}\right)^2 dF(\lambda) \\
&\quad + h^{2b}2^{3-2b}C_b \\
&= 32\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\left(\left(\frac{\Lambda}{N}\right)^b\frac{h^b}{4^b} + \left(\frac{\Lambda}{N}\right)^a\frac{d^a}{2^a}(2\Lambda)^b\frac{h^b}{4^b}\right)^2 dF(\lambda) + h^{2b}2^{3-2b}C_b \\
&= h^{2b}\left(2^{5-4b}\left(\frac{\Lambda}{N}\right)^b + \left(\frac{\Lambda}{N}\right)^a\frac{d^a}{2^a}(2\Lambda)^b\right)^2 F(\Lambda) + h^{2b}2^{3-2b}C_b,
\end{aligned}
$$

where $C_b = \int_\Lambda^\infty \lambda^{2b}\,dF(\lambda)$, $0 < a \le 1$, and $0 < b \le 1$. Thus

$$\mathsf{E}\,|\Delta_1(Y)|^2 \le k_{N,b}h^{2b}$$

for

$$k_{N,b} = \left(2^{5-4b}\left(\frac{\Lambda}{N}\right)^b + \left(\frac{\Lambda}{N}\right)^a\frac{d^a}{2^a}(2\Lambda)^b\right)^2 F(\Lambda) + 2^{3-2b}C_b.$$

Therefore

$$\mathsf{E}\,|\Delta_1(Y)|^{pr_1} \le h^{pr_1 b}k_{N,b}^{pr_1/2}c_{pr_1}. \tag{15}$$

Now we estimate $\mathsf{E}\,|V_1|^{pr_2}$. Let $f_1^{-1} + f_2^{-1} = 1$. It follows from the Hölder inequality that

$$
\begin{aligned}
\mathsf{E}\,|V_1|^{pr_2} &= \mathsf{E}\left|\exp\left\{\max\left(Y(t+h)-\widetilde{Y}(t+h), Y(t)-\widetilde{Y}(t)\right)\right\}\exp\left\{\widetilde{Y}(t+h)\right\}\right|^{pr_2} \\
&\le \left(\mathsf{E}\exp\left\{pr_2 f_1\max\left(Y(t+h)-\widetilde{Y}(t+h), Y(t)-\widetilde{Y}(t)\right)\right\}\right)^{1/f_1} \\
&\quad\times\left(\mathsf{E}\exp\left\{pr_2 f_2\widetilde{Y}(t+h)\right\}\right)^{1/f_2}.
\end{aligned}
\tag{16}
$$

We have

$$
\begin{aligned}
&\mathsf{E}\exp\left\{pr_2 f_1\max\left(Y(t+h)-\widetilde{Y}(t+h), Y(t)-\widetilde{Y}(t)\right)\right\} \\
&\quad\le \exp\left\{\frac{(pr_2 f_1)^2}{2}\mathsf{E}\left|Y(t+h)-\widetilde{Y}(t+h)\right|^2\right\} + \exp\left\{\frac{(pr_2 f_1)^2}{2}\mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^2\right\}.
\end{aligned}
$$

Applying the same argument as that used in the proof of formula (11) we easily see that $\mathsf{E}\left|Y(t+h)-\widetilde{Y}(t+h)\right|^2 \le A_N$. Thus

$$\left(\mathsf{E}\exp\left\{pr_2 f_1 \max\left(Y(t+h)-\widetilde{Y}(t+h), Y(t)-\widetilde{Y}(t)\right)\right\}\right)^{1/f_1} \le 2^{1/f_1}\exp\left\{\frac{(pr_2)^2 f_1}{2}A_N\right\},$$

$$\left(\mathsf{E}\exp\left\{pr_2 f_2 \widetilde{Y}(t+h)\right\}\right)^{1/f_2} \le \left(\exp\left\{\frac{(pr_2 f_2)^2}{2}F(\lambda)\right\}\right)^{1/f_2} = \exp\left\{\frac{(pr_2)^2 f_2}{2}F(\lambda)\right\}.$$

Combining the two preceding inequalities we derive from (16) that

$$\mathsf{E}\left|V_1\right|^{pr_2} \le 2^{1/f_1}\exp\left\{\frac{(pr_2)^2}{2}\left(f_1 A_N + f_2 B(0)\right)\right\}. \tag{17}$$

Now we estimate $\mathsf{E}\left|\Delta_2(Y)\right|^{ps_1}$. By equality (9),

$$\left|\Delta_2(Y)\right|^{ps_1} = \left(\mathsf{E}\left|\Delta_2(Y)\right|^2\right)^{ps_1/2} c_{ps_1}.$$

Then

$$\begin{aligned}
\mathsf{E}\left|\Delta_2(Y)\right|^2 &= \mathsf{E}\left|\widetilde{Y}(t+h)-\widetilde{Y}(t)\right|^2 \\
&= \mathsf{E}\Bigg(\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\cos\lambda_k(t+h)\,d\xi(\lambda) + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\sin\lambda_k(t+h)\,d\eta(\lambda) \\
&\qquad - \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\cos\lambda_k t\,d\xi(\lambda) - \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\sin\lambda_k t\,d\eta(\lambda)\Bigg)^2 \\
&= \mathsf{E}\Bigg(\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} -2\sin\frac{\lambda_k(2t+h)}{2}\sin\frac{\lambda_k h}{2}\,d\xi(\lambda) \\
&\qquad + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 2\sin\frac{\lambda_k h}{2}\cos\frac{\lambda_k(2t+h)}{2}\,d\eta(\lambda)\Bigg)^2 \\
&= \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 4\sin^2\frac{\lambda_k(2t+h)}{2}\sin^2\frac{\lambda_k h}{2}\,dF(\lambda) \\
&\qquad + \sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}} 4\sin^2\frac{\lambda_k h}{2}\cos^2\frac{\lambda_k(2t+h)}{2}\,dF(\lambda) \\
&\le 8\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\sin^2\frac{\lambda_k h}{2}\,dF(\lambda) \le 8\sum_{k=0}^{N-1}\int_{\lambda_k}^{\lambda_{k+1}}\left(\frac{\lambda_k h}{2}\right)^{2b}dF(\lambda) \\
&\le 2^{3-2b}\Lambda^{2b}h^{2b}F(\Lambda), \qquad 0<b\le 1.
\end{aligned}$$

Therefore

$$\mathsf{E}\left|\Delta_2(Y)\right|^{ps_1} \le \left(2^{3-2b}\Lambda^{2b}h^{2b}F(\Lambda)\right)^{ps_1/2} c_{ps_1}. \tag{18}$$

Since

$$\mathsf{E}\left|\Delta_3(Y)\right|^{ps_2} = \left(\mathsf{E}\left|\Delta_3(Y)\right|^2\right)^{ps_2/2} c_{ps_2},$$

$$\mathsf{E}\left|\Delta_3(Y)\right|^2 = \mathsf{E}\left|Y(t)-\widetilde{Y}(t)\right|^2 \le A_N,$$

we have

$$\mathsf{E}\,|\Delta_3(Y)|^{ps_2} = A_N^{ps_2/2} c_{ps_2}. \tag{19}$$

To estimate $\mathsf{E}\,|V_2|^{ps_3}$, let $e_1^{-1} + e_2^{-1} = 1$. We use the Hölder inequality again and find that

$$\begin{aligned}\mathsf{E}\,|V_2|^{ps_3} &= \mathsf{E}\left|\exp\left\{\max\left(\widetilde{Y}(t+h), \widetilde{Y}(t)\right)\right\}\exp\left\{\left|Y(t) - \widetilde{Y}(t)\right|\right\}\right|^{ps_3} \\ &\le \left(\mathsf{E}\exp\left\{ps_3 e_1 \max\left(\widetilde{Y}(t+h), \widetilde{Y}(t)\right)\right\}\right)^{1/e_1} \\ &\quad\times \left(\mathsf{E}\exp\left\{ps_3 e_2 \left|Y(t) - \widetilde{Y}(t)\right|\right\}\right)^{1/e_2}.\end{aligned}$$

Hence

$$\mathsf{E}\exp\left\{ps_3 e_1 \max\left(\widetilde{Y}\,(t+h)\,, \widetilde{Y}(t)\right)\right\} \le 2\exp\left\{\frac{(ps_3 e_1)^2}{2} F\,(\Lambda)\right\}.$$

It is known for a Gaussian random variable $\xi$ with parameters 0 and $\sigma^2$ that

$$\mathsf{E}\exp\{b\,|\xi|\} \le \mathsf{E}\exp\{b\xi\} + \mathsf{E}\exp\{-b\xi\} = 2\exp\left\{\frac{b^2\sigma^2}{2}\right\},$$

whence

$$\mathsf{E}\exp\left\{ps_3 e_2 \left|Y(t) - \widetilde{Y}(t)\right|\right\} \le 2\exp\left\{\frac{(ps_3 e_2)^2}{2} A_N\right\}.$$

Therefore

$$\mathsf{E}\,|V_2|^{ps_3} \le 2\exp\left\{\frac{(ps_3)^2}{2}\,(e_2 A_N + e_1 B(0))\right\}. \tag{20}$$

Applying (15), (17)–(20), and (10), we obtain

$$\begin{aligned}&\left(\mathsf{E}\left|\exp\{Y(t+h)\} - \exp\left\{\widetilde{Y}(t+h)\right\} - \left(\exp\{Y(t)\} - \exp\left\{\widetilde{Y}(t)\right\}\right)\right|^p\right)^{1/p} \\ &\quad\le (\mathsf{E}\,|\Delta_1(Y)|^{pr_1})^{1/(pr_1)}\,(\mathsf{E}\,|V_1|^{pr_2})^{1/(pr_2)} \\ &\qquad + (\mathsf{E}\,|\Delta_2(Y)|^{ps_1})^{1/(ps_1)}\,(\mathsf{E}\,|\Delta_3(Y)|^{ps_2})^{1/(ps_2)}\,(\mathsf{E}\,|V_2|^{ps_3})^{1/(ps_3)} \\ &\quad\le h^b k_{N,b}^{1/2} c_{pr_1}^{1/(pr_1)} 2^{1/(pr_2 f_1)} \exp\left\{\frac{pr_2}{2}\,(f_1 A_N + f_2 B(0))\right\} \\ &\qquad + \left(2^{3-2b}\Lambda^{2b} h^{2b} F(\Lambda)\right)^{1/2} c_{ps_1}^{1/(ps_1)} A_N^{1/2} c_{ps_2}^{1/(ps_2)} 2^{1/(ps_3)} \exp\left\{\frac{ps_3}{2}\,(e_2 A_N + e_1 B(0))\right\} \\ &\quad= h^b\Big[k_{N,b}^{1/2} 2^{1/(2pr_1)}\,(pr_1+1)^{1/2}\,2^{1/(pr_2 f_1)} \exp\left\{\frac{pr_2}{2}\,(f_1 A_N + f_2 B(0))\right\} \\ &\qquad\qquad + \left(2^{3-2b}\Lambda^{2b} F(\Lambda) A_N\right)^{1/2} 2^{1/(2ps_1)}(ps_1+1)^{1/2} 2^{1/(2ps_2)}(ps_2+1)^{1/2} 2^{1/(ps_3)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times \exp\left\{\frac{ps_3}{2}\,(e_2 A_N + e_1 B(0))\right\}\Big].\end{aligned}$$

Put $f_1 = e_2$, $f_2 = e_1$, and $r_2 = s_3$. Taking into account the following bounds,

$$2^{1/(2pr_1)} 2^{1/(pr_2 f_1)} \le 2^{1/p}, \qquad 2^{1/(2ps_1)} 2^{1/(2ps_2)} 2^{1/(ps_3 e_1)} \le 2^{1/p},$$

we get

$$\begin{aligned}
&\left(\mathsf{E}\left|\exp\{Y(t+h)\}-\exp\left\{\widetilde{Y}(t+h)\right\}-\left(\exp\{Y(t)\}-\exp\left\{\widetilde{Y}(t)\right\}\right)\right|^p\right)^{1/p}\\
&\quad\le h^b\Big[k_{N,b}^{1/2}2^{1/p}p^{1/2}\left(r_1+1\right)^{1/2}\exp\left\{\frac{pr_2}{2}\left(f_1A_N+f_2B(0)\right)\right\}\\
&\qquad+\left(2^{3-2b}\Lambda^{2b}F\left(\Lambda\right)A_N\right)^{1/2}2^{1/p}p\left(s_1+1\right)^{1/2}\left(s_2+1\right)^{1/2}\\
&\qquad\qquad\times\exp\left\{\frac{pr_2}{2}\left(f_1A_N+f_2B(0)\right)\right\}\Big]\\
&\quad\le h^bK_{N,b,p},
\end{aligned}$$

where

$$\begin{aligned}
K_{N,b,p}&=2^{1/p}\left(\max\left(r_1,s_1,s_2\right)+1\right)p\exp\left\{\frac{pr_2}{2}\left(f_1A_N+f_2B(0)\right)\right\}K_{N,b},\\
K_{N,b}&=k_{N,b}^{1/2}+\left(2^{3-2b}\Lambda^{2b}F(\Lambda)A_N\right)^{1/2}.
\end{aligned}\tag{21}$$

We evaluate the integral used in the definition of the constant $V_p$ in Theorem 3.1:

$$\int_0^{\theta\varepsilon_0}\left(\frac{d}{\varphi^{(-1)}\left(h\right)}\right)^{1/p}dh=\int_0^{\theta\varepsilon_0}\frac{d^{1/p}K_{N,b,p}^{1/(pb)}}{h^{1/(pb)}}\,dh=\frac{d^bK_{N,b,p}\theta^{1-1/(pb)}}{1-\frac{1}{pb}},\qquad 1-\frac{1}{pb}>0.$$

The function

$$f(\theta)=\frac{\theta^{1-1/(pb)}}{\theta(1-\theta)}$$

attains its minimal value

$$f\left(\frac{1}{1+pb}\right)=\frac{(1+pb)^{1+1/(pb)}}{pb}$$

at the point $\theta_0=(1+pb)^{-1}$. One can easily derive from inequality (13) that

$$\begin{aligned}
&\inf_{t\in B_i}\left(\mathsf{E}\left|\exp\{Y(t)\}-\exp\left\{\widetilde{Y}(t)\right\}\right|^p\right)^{1/p}\\
&\qquad\le\left(B(0)-F(\Lambda)\right)^{1/2}2^{1/p}p^{1/2}(v_1+1)^{1/2}\exp\left\{\frac{pv_2}{2}B(0)\right\}.
\end{aligned}$$

Using the notation of Theorem 3.1 we get

$$\frac{V_p^p}{x^p}\le\frac{\left[\left(B(0)-F\left(\Lambda\right)\right)^{1/2}2^{1/p}p^{1/2}\left(v_1+1\right)^{1/2}\exp\left\{\frac{pv_2}{2}B(0)\right\}+d^bK_{N,b,p}\frac{(1+pb)^{1+1/(pb)}}{pb-1}\right]^p}{x^p}.$$

Applying the elementary inequality $(a+b)^p\le 2^{p-1}\left(a^p+b^p\right)$ we obtain

$$\begin{aligned}
\frac{V_p^p}{x^p}&\le\frac{2^p\left(B(0)-F\left(\Lambda\right)\right)^{p/2}p^{p/2}(v_1+1)^{p/2}\exp\left\{\frac{p^2v_2}{2}B(0)\right\}}{x^p}\\
&\quad+\frac{2^{p-1}d^{pb}K_{N,b,p}^p\frac{(1+pb)^{1+1/(pb)}}{pb-1}}{x^p}.
\end{aligned}\tag{22}$$

Now we use the inequality

$$\frac{(1+pb)^{p+1/b}}{(pb-1)^p}\le\frac{p^{1/b}b^{1/b}\exp\left\{\frac{1}{b}+\frac{1}{b^2}\right\}}{\left(1-\frac{1}{pb}\right)^p}$$

Table 1

| $\alpha$ | $1-\gamma$ | $c_1$ | $c_2$ | $k$ | $d$ | $N$ |
|---|---|---|---|---|---|---|
| 0.01 | 0.99 | 1 | 6 | 12 | 0.0591766 | 114504 |
| 0.01 | 0.99 | 1 | 4 | 20 | 0.0554157 | 59898 |
| 0.01 | 0.99 | 1 | 1 | 30 | 0.0519309 | 16433 |

Table 2

| $\alpha$ | $1-\gamma$ | $c$ | $k$ | $d$ | $N$ |
|---|---|---|---|---|---|
| 0.01 | 0.99 | 2 | 4 | 0.0659933 | 13851 |
| 0.01 | 0.99 | 4 | 8 | 0.064061 | 8912 |
| 0.01 | 0.99 | 10 | 20 | 0.0593889 | 1353 |

and definitions (21) to estimate from above the second term of the expression on the right hand side of (22):

$$\frac{2^{p-1}d^{pb}K_{N,b,p}^{p}\frac{(1+pb)^{1+1/(pb)}}{pb-1}}{x^p} \le \frac{DM_N^p p^{p+1/p}\exp\left\{\frac{p^2 r_2}{2}\left(f_1A_N+f_2B(0)\right)\right\}}{x^p\left(1-\frac{1}{pb}\right)^p}, \tag{23}$$

where

$$D = b^{1/b}\exp\left\{\frac{1}{b}+\frac{1}{b^2}\right\},$$

$$M_N = 2d^b\left(\max\left(r_1,s_1,s_2\right)+1\right)K_{N,b}.$$

Let $r_2(f_1A_N+f_2B(0)) = Z$ and $2(B(0)-F(\Lambda))^{1/2}(v_1+1)^{1/2} = G$. Taking into account (23) we evaluate the right hand side of (22) at the point

$$p_0 = \frac{\ln\frac{x}{M_N}}{Z},$$

which is close to the point of minimum of the function

$$\frac{DM_N^p p^{p+1/p}\exp\left\{\frac{p^2}{2}Z\right\}}{x^p\left(1-\frac{1}{pb}\right)^p}.$$

Since $1-(p_0b)^{-1} > 0$, we apply Theorem 3.1 to obtain bounds (7), whence Theorem 3.2 follows. □

**Example 3.1.** Let a stochastic process $Y(t)$ satisfy the assumptions of Theorem 3.2 and let its spectral density be $f(\lambda) = c_1/(1+c_2\lambda^k)$. Table 1 contains the values of $d$ and $N$ for this process, given the accuracy $\alpha$ and reliability $1-\gamma$.

Table 2 contains the values of $d$ and $N$ evaluated for a stochastic process with the spectral density $f(\lambda) = \exp\{-c\lambda^k\}$.

*Remark* 3.2. One can use the following condition,

$$\mathsf{P}\left\{\max_{B_i\in\mathfrak{B},\, i=0,\dots,k-1}\left|p_{kY}(B_i)-\widetilde{p}_{kY}(B_i)\right|>\alpha\right\}<\gamma,$$

in Definition 3.1 instead of

$$\max_{B_i\in\mathfrak{B},\, i=0,\dots,k-1}\mathsf{P}\left\{\left|p_{kY}(B_i)-\widetilde{p}_{kY}(B_i)\right|>\alpha\right\}<\gamma.$$

Then we obtain

$$\begin{aligned}\mathsf{P}\left\{\max_{B_i\in\mathfrak{B},\, i=0,\dots,k-1}|\mu(B_i)-\widetilde{\mu}(B_i)|>\alpha\right\} &\le \sum_{i=1}^{k}\mathsf{P}\left\{|\mu\left(B_i\right)-\widetilde{\mu}(B_i)|>\alpha\right\}\\ &\le k\cdot\max_{B_i\in\mathfrak{B},\, i=0,\dots,k-1}\mathsf{P}\left\{|\mu(B_i)-\widetilde{\mu}(B_i)|>\alpha\right\}\\ &\le k\cdot\mathsf{P}\left\{\sup_{t\in B_i}\left|\exp\{Y(t)\}-\exp\left\{\widetilde{Y}(t)\right\}\right|>\frac{\alpha}{d}\right\}\end{aligned}$$

using estimates (4) and (5). This result is analogous to Theorem 3.2, where $\gamma/k$ is substituted for $\gamma$ in formula (7).

The latter setting is more involved, since the model approximates the process only for sufficiently large $N$.

## 4. Concluding remarks

A method for modelling a log Gaussian Cox stochastic process with a given accuracy and reliability is presented in this paper for the case where the logarithm of the intensity is a separable stationary Gaussian stochastic process. The main result of the paper, Theorem 3.2, contains sufficient conditions that a log Gaussian Cox stochastic process is approximated by its model.

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