ANALYTICAL PROBLEMS OF THE ASYMPTOTIC BEHAVIOR OF MARKOV FUNCTIONALS. II

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Abstract. The asymptotic behavior of Markov functionals of a homogeneous ergodic Markov process is studied in this paper.

This paper is a continuation of [1]. We number the displayed formula continuously after the last number in [1].

Let us turn to the proof of the theorem. First we show that

\[ h'_\varepsilon(x, t \cdot s) \xrightarrow{\varepsilon \to 0} \delta^{\|e_{\varepsilon,x}\|}(\pi, \varphi). \]

We recall that \((\pi, \varphi) = \int_E \pi(dy)\varphi(y)\) and the convergence is uniform with respect to \(x \in D\) and \(t \in [\sigma, T]\) for all \(T > \sigma > 0\).

To prove the latter relation we use a result similar to the Markov renewal theorem. Let a family of nonnegative semihomogeneous kernels \(G_\varepsilon(x, dy \times dt)\) depending on a small parameter \(\varepsilon > 0\) converge as \(\varepsilon \to 0\) to a nonnegative stochastic kernel \(G(x, dy \times dt)\) in the sense that

\[ \sup_{A \in A} \left\| \int_0^\infty G_\varepsilon(\cdot, A \times dt)\varphi(t) - \int_0^\infty G(\cdot, A \times dt)\varphi(t) \right\| \xrightarrow{\varepsilon \to 0} 0 \]

for all continuous bounded functions \(\varphi(t), t \geq 0\), where

\[ \|f\| = \inf\{c: |f(x)| \leq ch(x)\} \quad \pi\text{-almost everywhere.} \]

Denote by \(G_\varepsilon(x, dy)\) and \(G(x, dy)\) the bases of the kernels \(G_\varepsilon(x, dy \times dt)\) and \(G(x, dy \times dt)\), respectively. Let \(U_\varepsilon(x, dy \times dt)\) denote the potential of the kernel \(G_\varepsilon(x, dy \times dt)\).

We assume that the kernel \(G(x, A)\) does not depend on \(x\), that is,

\[ G(x, A) = \pi(A) \quad \text{and} \quad h(x) = 1, \]

where \(\pi(A)\) is a probability distribution on \((E, B)\).

We denote by \(\lambda_\varepsilon\) the spectral radius of the kernel \(G_\varepsilon(x, A) = G_\varepsilon(x, A \times [0, \infty))\) and by \(\pi_\varepsilon\) the eigenprobability measure of the kernel \(G_\varepsilon(x, A)\) corresponding to the eigenvalue \(\lambda_\varepsilon\). Put \(\gamma_\varepsilon = (1 - \lambda_\varepsilon)/m\).

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Theorem 1. Assume that conditions (49) and (50) hold. We also assume that

a) the kernel $G(x, dy \times dt)$ is nonlattice and

\begin{align}
(51) \quad & \sup_{\varepsilon > 0} \sup_{x \in E} \int_0^\infty G_\varepsilon(x, E \times dt)t < \infty, \\
(52) \quad & \sup_{\varepsilon > 0} \int_E \int_T \pi(dx)G_\varepsilon(x, E \times dt)t \to 0;
\end{align}

b) a family of $\mathcal{B} \times \mathcal{B}_{t_+}$-measurable functions $g_\varepsilon(x, t)$ is such that $g_\varepsilon(x, t) \geq 0$ and

\begin{align}
(53) \quad & \sup_{\varepsilon > 0} \sup_{x \in E} \int_0^\infty g_\varepsilon(x, t) dt < \infty, \\
(54) \quad & \sup_{\varepsilon > 0} \sup_{x \in E, t \geq 0} U_\varepsilon \ast g_\varepsilon(x, t) < \infty, \\
(55) \quad & \lim_{N \to \infty} \sup_{\varepsilon > 0} \sup_{x \in E} \sum_{k=N}^{\infty} \sup_{0 \leq t \leq k+1} g_\varepsilon(x, t) = 0,
\end{align}

c) there exists a $\mathcal{B}_{t_+}$-measurable function $g(t)$ such that

\begin{align}
(56) \quad & \int_0^\infty \left| g(t) - \int_E \pi(dx)g_\varepsilon(x, t) \right| dt \to 0;
\end{align}

d) there exists a number $\delta > 0$ such that

\begin{align}
(57) \quad & \inf_{x \in E} \inf_{\varepsilon > 0} G_\varepsilon(x, E \times [\delta, \infty)) \geq \delta, \\
& \int_E \pi(dx)G_\varepsilon(x, A \times [\delta, \infty)) \geq \delta \pi_t(A).
\end{align}

Then

\begin{align}
\lim_{\varepsilon \to 0} U_\varepsilon \times g_\varepsilon(x, t) = e^{-c} \frac{1}{m} \int_0^\infty g(s) ds
\end{align}

uniformly with respect to $x \in E$, where

\begin{align}
m = \int_E \int_0^\infty \pi(dx)G(x, E \times dt)t < \infty.
\end{align}

Conditions (12)–(15) imply that the kernels $Q_\varepsilon^t$ satisfy all the assumptions of Theorem 2. Now we show that the functions $g_\varepsilon^t(x, t)$ satisfy conditions (55)–(57). Without loss of generality, we assume that $0 \leq \varphi(x) \leq 1$. Then

\begin{align}
\sup_{k \leq t \leq k+1} g_\varepsilon^t(x, t) &= \sup_{k \leq t \leq k+1} \mathbb{P}_{x, t}\{\varphi(X(t)), \xi_t(t) = l, t \leq \tau\} \leq \mathbb{P}_x\{k < \tau\} \\
& \leq \int_{k-1}^\infty \mathbb{P}_x\{\tau > t\} dt.
\end{align}

Thus

\begin{align}
\sup_{x \in D} \sum_{k=n+1}^\infty \sup_{k \leq t \leq k+1} g_\varepsilon^t(x, t) &= \sup_{x \in D} \sum_{k=n+1}^\infty \int_{k-1}^\infty \mathbb{P}_x\{\tau > t\} dt  \\
& = \sup_{x \in D} \int_{n}^\infty \mathbb{P}_x\{\tau > t\} dt  \\
& \leq \sup_{x \in D} [\mathbb{P}_x\{\tau > N\} + \mathbb{P}_x\{\tau, \tau > N\}].
\end{align}
In view of (14), the latter expression approaches zero as $N \to \infty$, and this proves the uniform convergence of series (55) with respect to $n \geq 1$ and $x \in D$.

Now we check condition (56). It follows from (16) that the process $\{\varphi(X(t)), \xi_\varepsilon(t)\}$ is stochastically continuous uniformly in $\varepsilon > 0$ and $t \in [0,T]$ for an arbitrary initial distribution of the process $X(t)$. In particular,

$$\sup_{a \leq \varepsilon \leq t \leq b} \mathbb{P}_{x,i}[\varphi(X(t)) - \varphi(X(s))] \quad \longrightarrow \quad 0,$$

and

$$\sup_{a \leq \varepsilon \leq t \leq b} \mathbb{P}_{x,i}[\xi_\varepsilon(t) = l, \xi_\varepsilon(s) \neq l] \quad \longrightarrow \quad 0$$

for all $T > 0$, $i,l \in I$, and $x \in E$.

Next we apply the inequality

$$\sup_{a \leq \varepsilon \leq t \leq b} g^i(x,t) - \inf_{a \leq \varepsilon \leq t \leq b} g^i(x,t) \leq \sup_{a \leq \varepsilon \leq t \leq b} \mathbb{P}_{x,i}[\varphi(X(t)) - \varphi(X(s))] + \sup_{a \leq \varepsilon \leq t \leq b} \mathbb{P}_{x,i}[\xi_\varepsilon(t) = l, \xi_\varepsilon(s) \neq l] + \mathbb{P}_{x,i}[a \leq \tau \leq b].$$

The further reasoning is the same as that used in the proof of Theorem 3, Chapter 2, Section 2 in [2] and this proves condition (56). Since

$$\lim_{\varepsilon \to 0} \mathbb{P}_{x,i}[\xi_\varepsilon(t) \neq i] = 0,$$

relation (57) follows from (13) for all $x \in E$, $i \in I$, $t \geq 0$, and for the function

$$g(t) = \delta^i \mathbb{P}_{x,i}[\varphi(X(t)), t < \tau].$$

Note that

$$\frac{1}{m} \int_0^\infty g(t) \, dt = \delta^i \langle \pi, \varphi \rangle,$$

whence (48) follows.

Consider a sequence of functions $p^i_k(t)$, $k \geq 0$, defined by the following recurrence relation:

$$p^i_0(t) = \delta^i e^{c_{ii} t},$$

$$p^i_{k+1}(t) = \int_0^t e^{c_{ii} s} \sum_{j \neq i} c_{ij} p^j_k(t - s).$$

Note that the sum of the series

$$\sum_{k=0}^\infty p^i_k(t) = p^i(t)$$

is equal to the entry $(i,l)$ of the matrix $\exp\{tC\}$, where

$$C = \|c_{ij}\|_{i,j=1}^\infty.$$

We prove by induction in $k \geq 0$ that

$$\lim_{\varepsilon \to 0} h^i_{\varepsilon,k}(x,t) = p^i_k(u, \varphi)$$

for all $u \geq 0$ and $i \in I$ uniformly in $x \in D$. Recall that the functions $h^i_{\varepsilon,k}(x,t)$ are defined in (19).
Equality (61) coincides with (48) for \( k = 0 \). We show that (48) holds for \( k = 1 \) uniformly in \( 0 \leq t \leq T \) and \( x \in D \) for all \( T > 0 \). We have

\[
(62) \quad h^1_{\varepsilon,1}(x,t) = \sum_{j \neq i} \int_D \int_0^1 R^i_j(x, dy \times t ds) h^1_{\varepsilon}(y,t(1-s)).
\]

Using (48) we get

\[
(63) \quad \sup_{y \in D} \sup_{0 \leq \varepsilon \leq T} \sup_{0 \leq s \leq T - \sigma} |r^i_\varepsilon(y,1-s)| \to 0.
\]

The term corresponding to an index \( j \) on the right hand side of (62) can be represented as follows:

\[
(64) \quad \int_0^1 R^i_j(x, D \times t ds)p^i_\varepsilon(u(1-s))\langle \pi, \varphi \rangle + \int_D \int_0^{1-\sigma} R^i_j(x, dy \times t ds)r^i_\varepsilon(y,1-s) + \int_D \int_{1-\sigma}^1 R^i_j(x, dy \times t ds)r^i_\varepsilon(y,1-s).
\]

According to Lemma 2, the first term of (64) converges as \( \varepsilon \to 0 \) to \( \langle \pi, \varphi \rangle p^i\varepsilon(u) \) uniformly in \( x \in D \) and \( 0 \leq u \leq T \). We apply Lemma 2 to the second term and deduce from (63) that it converges to zero uniformly in \( x \in D \) and \( 0 \leq u \leq T \). If \( \sigma > 0 \) is sufficiently small, then the third term also is small uniformly in \( x \in D \) and \( 0 \leq u \leq T \).

The proof of the induction step from \( k \) to \( k + 1 \) for \( k \geq 1 \) is the same as above, and thus we omit it.

It follows from (20), (21), (61), (17), and (48) that

\[
\lim_{\varepsilon \to 0} f^i_\varepsilon(x,t) = p^i(u)\langle \pi, \varphi \rangle
\]

for all \( x \in E \) and \( u > 0 \). Thus the theorem is proved if (18) holds.

To complete the proof of the theorem we need to check condition (18). It is easy to see that the functions

\[
\tilde{f}^i_\varepsilon(x,t) = e^{-\varepsilon t} f^i_\varepsilon(x,t)
\]

satisfy the system of equations similar to (16); namely,

\[
(65) \quad \tilde{f}^i_\varepsilon(x,t) = \tilde{g}^i_\varepsilon(x,t) + \tilde{Q}^i_\varepsilon(x,t) + \sum_{j \neq i} \tilde{Q}^{ij}_\varepsilon(x,t),
\]

where

\[
\tilde{g}^i_\varepsilon(x,t) = e^{-\varepsilon t} g^i_\varepsilon(x,t),
\]

\[
\tilde{Q}^i_\varepsilon(x,dy \times dt) = e^{-\varepsilon t} Q^i_\varepsilon(x,dy \times dt),
\]

\[
\tilde{Q}^{ij}_\varepsilon(x,dy \times dt) = e^{-\varepsilon t} Q^{ij}_\varepsilon(x,dy \times dt).
\]

Following the method used in the first part of the paper to derive (17) from (16) we obtain from representation (64) that

\[
\tilde{f}^i_\varepsilon(x,t) = \tilde{H}^i_\varepsilon * g^i_\varepsilon(x,t) + \sum_{j \neq i} \tilde{R}^{ij}_\varepsilon * \tilde{f}^j_\varepsilon(x,t),
\]

where \( \tilde{H}^i_\varepsilon \) is the potential of the kernel \( Q^i_\varepsilon \),

\[
\tilde{R}^{ij}_\varepsilon = \tilde{H}^i_\varepsilon * \tilde{Q}^{ij}_\varepsilon.
\]
Now we check that the kernels $\tilde{R}^{ij}_\varepsilon(x, A \times [0, \infty))$ satisfy condition (18). Recalling the notation for $\hat{R}^{ij}_\varepsilon(x, A \times [0, \infty))$, we obtain

$$
\tilde{R}^{ij}_\varepsilon(x, A \times [0, \infty)) = \tilde{R}^{ij}_\varepsilon(x, A), \quad A \in \mathcal{B}.
$$

This together with (41) and Lemma 1 yields

$$
\sup_{x \in D} \sum_{j \neq i} \tilde{R}^{ij}_\varepsilon(x, D \times [0, \infty)) \leq -\frac{c_{ii}}{1 - c_{ii}}
$$

for all sufficiently small $\varepsilon > 0$. Putting

$$
r = \sup_i \frac{-u_{c_{ii}}}{1 - u_{c_{ii}}}
$$

we prove condition (18) with $\tilde{R}^{ij}_\varepsilon$ instead of $R^{ij}_\varepsilon$.

Assuming that conditions (17)–(19) and (47) hold, we get

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon m} \left[ \mathbb{P}_{\pi, D,i} \{ e^{-\varepsilon \tau}, \xi_\varepsilon(\tau) = i \} - 1 \right] = c_{ii} - 1.
$$

Applying the part of Theorem 1 already proved to the function $\tilde{f}_\varepsilon(x, t)$, we find that

$$
\lim_{\varepsilon \to 0} t \to \infty \varepsilon t \to u \tilde{f}_\varepsilon(x, t) = p^{il}(u) \langle \pi, \varphi \rangle
$$

if relations (17)–(19) and (47) hold, where

$$
\|p^{il}(u)\|_{l=1}^\infty = \exp\{u(C - I)\} = e^{-u} \exp\{uC\}.
$$

On the other hand,

$$
\tilde{f}_\varepsilon(x, t) = e^{-\varepsilon t} f^\varepsilon(x, t).
$$

Comparing this result with equality (66) we complete the proof of the theorem in the general case.

The proof of Theorem 4 follows the same lines; thus we omit it. In the proof one should apply the following result.

**Theorem 2.** Assume that conditions (49)–(51) hold and

a) the step of the kernel $G(x, dy \times dt)$ is equal to one and

$$
G_\varepsilon(x, E \times [0, \infty)) = \sum_{k=0}^\infty G_\varepsilon(x, E \times \{k\})
$$

for all $x \in E$ and $n \geq 1$;

b) $\mathcal{B}$-measurable functions $g_\varepsilon(x, k), \varepsilon > 0, k \geq 1$, are such that

$$
\sup_{\varepsilon > 0} \sup_{x \in E} \sum_{k \geq N} g_\varepsilon(x, k) \to 0, \quad N \to \infty
$$

if $g_\varepsilon(x, k)$ are such that

$$
\sum_{k \geq 0} \left| g_k - \int_E \pi(dy)g_\varepsilon(x, k) \right| \to 0.
$$

Then

$$
\lim_{\varepsilon \to 0} \lim_{k \to \infty} U_\varepsilon * g_\varepsilon(x, k) = e^{-c} \frac{1}{m} \sum_{k \geq 0} g_k.
$$
Theorem 3. Assume conditions (1), (2), (11), and (17)–(19). Let the parameter $t$ vary in the set $\{0, 1, \ldots \}$. Then

$$\left\{ \mathbb{P}_{x,i}[\varphi(X(t)), \xi_\varepsilon(t) = j] - p_{ij}(u) \int_E \pi(dy) \varphi(y) \right\} \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty$$

for all $u \geq 0$, $i, j \in I$, $x \in E$, and all $\mathcal{B}$-measurable bounded functions $\varphi(y)$, where $p_{ij}(u)$ is the entry $(i, j)$ of the matrix $e^{uC}$, $C = \|c_{ij}\|_{i,j=1}^\infty$.

3. Concluding remarks

Remark 1. Condition (16) holds if

$$\sup_{\varepsilon > 0} \sup_{x \in E} \mathbb{P}_{x,i}\{\xi_\varepsilon(t) \neq i\} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0$$

for all $i \in I$.

Remark 2. We assume condition (16) in order to apply the theorem to the functions $g_i^\varepsilon(x, t)$. Let $\Psi(y, j)$ be a bounded function of two arguments $y \in E$ and $j \in I$ that is continuous in $y \in E$ for all $i \in I$. Then the functions

$$g_i^\varepsilon(x, t) = \mathbb{P}_{x,i}\{\Psi(X(t), \xi_\varepsilon(t)), t < \tau\} = \mathbb{P}_{x,i}\{\Psi(X(t), \xi_\varepsilon(t)), t < (\tau \wedge \zeta_\varepsilon)\}$$

satisfy the assumptions of the theorem if the integer-valued stochastic process

$$\Psi(X(t), \xi_\varepsilon(t)), \quad t \geq 0,$$

is stochastically continuous uniformly in $\varepsilon > 0$, that is, if

$$(67) \quad \sup_{\varepsilon > 0} \mathbb{P}_{x,i}|\Psi(X(t), \xi_\varepsilon(t)) - \Psi(X(s), \xi_\varepsilon(s))| \rightarrow 0 \quad \text{as} \quad s \rightarrow t$$

for all $x \in E$, $i \in I$, and $t \geq 0$.

This remark implies the following result.

Corollary 1. If conditions (2) and (67) hold, then

$$\lim_{\varepsilon \rightarrow 0} \left\{ \mathbb{P}_{x,i}\Psi(X(t), \xi_\varepsilon(t)) - \sum_{j=1}^\infty p_{ij}(u) \int_E \Psi(y, j) \pi(dy) \right\} = 0$$

for all $x \in E$, $i \in I$, and $u \geq 0$.

Setting $\Psi(y, j) \equiv 1$ we obtain the following result.

Corollary 2. If condition (2) holds, then

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}_{x,i}\{\varepsilon \zeta_\varepsilon \geq u\} - \sum_{j=1}^\infty p_{ij}(u) = 0$$

for all $x \in E$, $i \in I$, and $u > 0$.

Bibliography


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