ASYMPTOTIC NORMALITY OF $L_p$-ESTIMATORS IN NONLINEAR REGRESSION MODELS WITH WEAK DEPENDENCE

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ABSTRACT. A theorem on asymptotic normality is proved, and the limit distribution is found for $L_p$-estimators of a vector parameter in a nonlinear regression model with continuous time and weakly dependent random noise.

INTRODUCTION

We obtain conditions of asymptotic normality for $L_p$-estimators of an unknown parameter of nonlinear regression model with random noise satisfying a weak dependence condition.

$L_p$-estimators belong to the class of the so-called $M$-estimators. The number of papers related to $M$-estimators is rather large. Applications of $M$-estimators in linear regression models with independent observation errors are considered in the pioneering papers by Huber [22, 13]. Several asymptotic results for $M$-estimators of parameters of linear and nonlinear regression models with independent observation errors are proved by Hampel et al. [21], Chen and Wu [16], Jurečková [29], Liese and Vajda [38, 39, 40, 41, 42], Müller [33], Liese [37], Arcones [15], Wu and Zen [47], van de Geer [17], Orlovskiĭ [12], Ivanov and Orlovskiĭ [10, 27], and by many other authors.

Asymptotic properties of $M$-estimators of parameters of linear and nonlinear regression models with random noise satisfying a strong dependence condition are studied by Koul [30, 31], Koul and Mukherjee [33], Giraitis et al. [19], Koul and Surgailis [34, 35, 36], Giraitis and Koul [18], Koul et al. [32] in the case of discrete time, and by Ivanov and Leonenko [25] and Ivanov and Orlovskiĭ [11] in the case of continuous time.


The most studied in the class of $L_p$-estimators are the least squares estimator (the case of $p = 2$) and least modules estimator (the case of $p = 1$). Asymptotic properties of least squares estimators and least modules estimators of parameters of nonlinear regression models are studied by many authors. We only mention the monographs by Ivanov and Leonenko [9] and Ivanov [23], where a rather complete bibliography concerning this question can be found.

Asymptotic properties of $L_p$-estimators of parameters of linear and nonlinear regression models with independent observation errors are considered by Huber [13], Ronner [46], Ivanov [8], Bardadym and Ivanov [1, 2]; those for nonlinear models with random

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Consider the following regression model:

\[ X(t) = g(t, \theta) + \varepsilon(t), \quad t \in [0, T], \]

where \( g(t, \theta) : [0; \infty) \times \Theta^c \to \mathbb{R}^1 \) is a real continuous function, \( \Theta^c \) a closure in \( \mathbb{R}^q \) of a bounded open set \( \Theta \subset \mathbb{R}^q \), and \( \varepsilon(t), t \in \mathbb{R}^1 \), a stochastic process satisfying the following condition:

**Condition A1.** Let \( \varepsilon(t), t \in \mathbb{R}^1 \), be a real mean square continuous measurable stationary Gaussian process with zero mean and the covariance function \( B(t) = \mathbb{E} \varepsilon(0)\varepsilon(t) \), \( B(0) = 1 \).

**Definition 1.** Any random vector \( \hat{\theta}_T = \hat{\theta}_T(X(t), t \in [0, T]) \in \Theta^c \), such that

\[ Q_{p,T}(\hat{\theta}_T) = \inf_{\tau \in \Theta^c} Q_{p,T}(\tau), \quad Q_{p,T}(\tau) = \int_0^T |X(t) - g(t, \tau)|^p dt \]

is called an \( L_p \)-estimator of the unknown parameter \( \theta \in \Theta \) constructed from observations \( X(t), t \in [0, T] \), described by the model (1).

Note that the estimator \( \hat{\theta}_T \) exists for \( p > 0 \) under the conditions introduced above (see, for example, [28, 45, 14]). The case of \( p \in (1, 2] \) is the most interesting one.

Let \( \rho(x) = |x|^p, p \in (1, 2) \). Then \( \rho'(x) = \psi(x) = p|x|^{p-2} \text{sgn } x, \psi'(x) = p(p-1)|x|^{p-2}, \) and \( \psi''(x) = p(p-1)(p-2)|x|^{p-3} \text{sgn } x \) for \( x \neq 0 \). We also assume that \( \psi'(0) = \infty \) and \( |\psi''(0)| = \infty \).

The most technically complicated part in the proof of the asymptotic (as \( T \to \infty \)) properties of \( L_p \)-estimators for \( p \in (1, 2] \) is related to the observation that the derivatives \( \psi' \) and \( \psi'' \) are not bounded in a neighborhood of the origin if \( \rho(x) = |x|^p \) (this differs from the case of a number of standard \( M \)-estimators obtained when minimizing the functionals \( \int_0^T \rho(X(t) - g(t, \tau)) \) dt).

Assume that \( g(t, \tau) \) is a twice continuously differentiable function with respect to \( \tau \in \Theta^c \). Put

\[ g_i(t, \tau) = \frac{\partial}{\partial \tau_i} g(t, \tau), \quad g_{il}(t, \tau) = \frac{\partial^2}{\partial \tau_i \partial \tau_l} g(t, \tau), \quad i, l = 1, \ldots, q, \]

\[ d_i^2(\theta) = \text{diag} (d_{iT}(\theta))^q, \]

where

\[ d_i^2(\theta) = \int_0^T g_i^2(t, \theta) dt, \quad \lim_{T \to \infty} T^{-1} d_i^2(\theta) > 0, \quad T \to \infty, \quad i = 1, \ldots, q. \]

Our approach works even in the case where the latter limits are infinite. Also let

\[ d_i^2(\tau) = \int_0^T g_i^2(t, \tau) dt, \quad \tau \in \Theta^c, \quad i, l = 1, \ldots, q. \]

By the symbol \( k \) with various indices, we denote positive constants. Assume that, for all sufficiently large \( T \) (\( T > T_0 \)), the following condition holds.
Condition B1.

\[(2) \quad \sup_{t \in [0,T]} \sup_{\tau \in \Theta^r} \frac{|g_i(t, \tau)|}{d_{dT}(\theta)} \leq k_i T^{-1/2}, \quad i = 1, \ldots, q,\]

\[(3) \quad \sup_{t \in [0,T]} \sup_{\tau \in \Theta^r} \frac{|g_i(t, \tau)|}{d_{dT}(\theta)} \leq k_i T^{-1/2}, \quad i, l = 1, \ldots, q,\]

\[(4) \quad \sup_{\tau \in \Theta^r} \frac{d_{dT}(\tau)}{d_{dT}(\theta)} \leq k_i T^{-1/2}, \quad i, l = 1, \ldots, q.\]

Put

\[J_T(\theta) = (J_{dT}(\theta))_{i,l=1}^q, \quad J_{dT}(\theta) = d_{d_{dT}(\theta)}^{-1}(\theta) d_{dT}(\theta) \int_0^T g_i(t, \theta) g_l(t, \theta) dt,\]

\[A_T(\theta) = (A_{dT}(\theta))_{i,l=1}^q = J_{dT}(\theta).\]

By \(\lambda_{\min}(A)(\lambda_{\max}(A))\) we denote the minimal (maximal) eigenvalue of a positive definite matrix \(A\).

Condition B2. For some \(\lambda_* > 0\) and \(T > T_0\),

\[\lambda_{\min}(J_{dT}(\theta)) \geq \lambda_* .\]

We have

\[E \psi(\varepsilon(0)) = p \int_{-\infty}^{\infty} |x|^{p-1} \text{sgn} x \phi(x) dx = 0,\]

where \(\phi(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}\). On the other hand, we get for \(p \in (1, 2)\) that

\[E \psi^2(\varepsilon(0)) = \frac{p^2 p - 1}{\sqrt{\pi}} \Gamma\left( p - \frac{1}{2} \right), \quad E \psi^4(\varepsilon(0)) = \frac{p^2 p - 1}{\sqrt{\pi}} \Gamma\left( p - \frac{3}{2} \right) > 0.\]

Moreover

\[E(\psi'(\varepsilon(0)))^2 = \frac{p^2(p - 1)^2 2^{p-2}}{\sqrt{\pi}} \Gamma\left( p - \frac{3}{2} \right) < \infty\]

if \(p > 3/2\).

Therefore \(E \psi^2(\varepsilon(0)) < \infty\) and \(E(\psi'(\varepsilon(0)))^2 < \infty\) for \(p \in (3/2, 2]\). In this case, the functions \(\psi(\varepsilon(t))\) and \(\psi'(\varepsilon(t))\), \(t \in \mathbb{R}^1\), can be expanded in the series in the Hilbert space \(L_2(\mathbb{R}^1, \varphi(x) dx)\), namely

\[(5) \quad \psi(x) = \sum_{k=0}^{\infty} \frac{C_k(\psi)}{k!} H_k(x), \quad C_k(\psi) = \int_{-\infty}^{\infty} \psi(x) H_k(x) \varphi(x) dx,\]

\[\psi'(x) = \sum_{k=0}^{\infty} \frac{C_k(\psi')}{k!} H_k(x), \quad C_k(\psi') = \int_{-\infty}^{\infty} \psi'(x) H_k(x) \varphi(x) dx,\]

where \(H_k\) are the Hermite–Chebyshev polynomials

\[H_k(u) = (-1)^k e^{u^2/2} \frac{d^k}{du^k} e^{-u^2/2}, \quad k \geq 0.\]

Consider the following condition of weak dependence.

Condition A2. Let \(\varepsilon(t), t \in \mathbb{R}^1\), be a stochastic process such that

\[\alpha(r) = \sup_{A \in \sigma(-\infty, s], B \in \sigma[t+r, \infty)} |P(AB) - P(A) P(B)| = O(r^{1-\varepsilon}), \quad r \to \infty,\]

for some \(\varepsilon > 0\), where \(\sigma(I)\) denotes the \(\sigma\)-algebra generated by the random variables \(\{\varepsilon(t), t \in I\}\).
Condition A2 implies that \( B(\cdot) \in L_1(\mathbb{R}^1) \) and that the process \( \varepsilon(t) \) has a bounded continuous spectral density \( f(\lambda) \), \( \lambda \in \mathbb{R}^1 \).

It is easy to see that if \( \varepsilon(t) \) satisfies the weak dependence condition, then \( \psi(\varepsilon(t)) \) also satisfies this condition. For all Borel sets \( G \),

\[
\{ \psi(\varepsilon(t)) \in G \} = \{ \varepsilon(t) \in \psi^{-1}(G) \},
\]

and \( \psi^{-1}(G) \) is also a Borel set, whence

\[
\alpha_\psi(r) = \sup_{A \in \sigma_\psi(-\infty,t),B \in \sigma_\psi[t,r,\infty)} |P(AB) - P(A)P(B)| \leq \alpha(r),
\]

where \( \sigma_\psi(I) \) is the \( \sigma \)-algebra generated by the random variables \( \{ \psi(\varepsilon(t)), t \in I \} \).

Since the stochastic process \( \psi(\varepsilon(t)), t \in \mathbb{R}^1 \), can be expanded in the series \( (5) \) with respect to the Hermite–Chebyshev polynomials,

\[
E H_m(\varepsilon(t))H_k(\varepsilon(s)) = \delta_{m,k}! A^k(t-s),
\]

and \( E \psi(\varepsilon(0)) = 0 \), we get

\[
cov(\psi(\varepsilon(t)),\psi(\varepsilon(s))) = \sum_{k=1}^{\infty} \frac{C^2_\psi(0)}{k!} g^k(t-s).
\]

In view of \( |B(t)| \leq 1, t \in \mathbb{R}^1 \), we obtain

\[
|cov(\psi(\varepsilon(t)),\psi(\varepsilon(s)))| \leq \sum_{k=1}^{\infty} \frac{C^2_\psi(0)}{k!} |B(t-s)| = E \psi^2(\varepsilon(0))|B(t-s)|.
\]

Therefore the stationary stochastic process \( \psi(\varepsilon(t)), t \in \mathbb{R}^1 \), also has a bounded and continuous spectral density \( f_\psi(x) \).

Consider the matrix measure \( \mu_T(dx;\theta) \) on \((\mathbb{R}^1,\mathcal{B}^1)\) with the following matrix density:

\[
\mu_T^j(x;\theta) = g_T^j(x,\theta)\overline{g_T^j(x,\theta)} \left( \int_{\mathbb{R}^1} |g_T^j(x,\theta)|^2 \, dx \int_{\mathbb{R}^1} |g_T^j(x,\theta)|^2 \, dx \right)^{-1/2},
\]

\[
g_T^j(x,\theta) = \int_0^T e^{xt} g_j(t,\theta) \, dt, \quad j, l = 1, \ldots, q.
\]

Note that \( d_T^2(\theta) = (2\pi)^{-1} \int_{\mathbb{R}^1} |g_T^j(x,\theta)|^2 \, dx \).

**Condition B3.** The family of measures \( \mu_T(\cdot; \theta) \) weakly converges to the measure \( \mu(\cdot; \theta) \) as \( T \to \infty \) and

\[
\int_{\mathbb{R}^1} f_\psi(x) \, \mu(dx; \theta)
\]

is a positive definite matrix.

**Definition 2.** The matrix measure \( \mu(\cdot; \theta) \) is called the spectral measure of the regression function \( g(t, \theta) \) [3][5][20].

Conditions B2 and B3 imply that

\[
\int_{\mathbb{R}^1} \mu(dx; \theta) = \left( \int_{\mathbb{R}^1} \mu^{jl}(dx; \theta) \right)_{j,l=1}^q
\]

is a nonsingular matrix.

Let

\[
\sigma(\theta) = 2\pi^{2/2} \left( \int_{\mathbb{R}^1} \mu(dx; \theta) \right)^{-1} \left( \int_{\mathbb{R}^1} f_\psi(x) \mu(dx; \theta) \right) \left( \int_{\mathbb{R}^1} \mu(dx; \theta) \right)^{-1},
\]
where

\[
\gamma = \frac{1}{E\psi'(\varepsilon(0))}.
\]

Consider the following condition for the asymptotic uniqueness of a solution of the system of “normal” equations defining the \( L_p \)-estimator.

**Condition C.** For all \( \varepsilon > 0 \) and \( R > 0 \), the probability that the system of equations (10) with \( T > T_0 \) has a unique solution in the ball \( v^c(R) \) is not less than \( 1 - \varepsilon \).

Sufficient conditions for Condition C can be found in [24].

Now we state the main result of the paper.

**Theorem.** Let Conditions A1, A2, B1 - B3, and C hold. If \( p \in \left( \frac{1}{2}, 2 \right) \), then the distribution of the normalized \( L_p \)-estimator

\[
\hat{\theta}_T = \hat{\theta}_T(\theta) = d_T(\theta)(\hat{\theta}_T - \theta)
\]

converges as \( T \to \infty \) to the Gaussian \( N(0, \sigma(\theta)) \) distribution.

### 2. Auxiliary results

Consider the change of variables \( u = d_T(\theta)(\tau - \theta) \) corresponding to the normalization \( [5] \). Applying this change to the regression function and its derivatives, we get

\[
g(t, \tau) = g(t, \theta + d_T^{-1}(\theta)u) = h(t, u),
\]

\[
g_i(t, \tau) = g_i(t, \theta + d_T^{-1}(\theta)u) = h_i(t, u), \quad i = 1, \ldots, q,
\]

\[
g_{il}(t, \tau) = g_{il}(t, \theta + d_T^{-1}(\theta)u) = h_{il}(t, u), \quad i, l = 1, \ldots, q.
\]

We also use the notation

\[
H(t; u_1, u_2) = h(t, u_1) - h(t, u_2),
\]

\[
H_i(t; u_1, u_2) = h_i(t, u_1) - h_i(t, u_2), \quad i = 1, \ldots, q.
\]

Consider the vectors

\[
M_T(u) = (M_T(u))_i = \left( \gamma \int_0^T \psi(X(t) - h(t, u)) \frac{h_i(t, u)}{d_T(\theta)} dt \right)_{i=1}^q
\]

and

\[
\Psi_T(u) = (\Psi_T(u))_i = \left( \gamma \int_0^T \psi(\varepsilon(t)) \frac{h_i(t, u)}{d_T(\theta)} dt + \int_0^T H(t; 0, u) \frac{h_i(t, u)}{d_T(\theta)} dt \right)_{i=1}^q,
\]

where \( \gamma \) is defined by (8). The vectors \( M_T(u) \) and \( \Psi_T(u) \) are defined for \( u \in U_T^\theta(\theta) \),

\[
U_T(\theta) = d_T(\theta)(\Theta - \theta).
\]

Our assumptions mean that the sets \( U_T(\theta) \) are extending to \( \mathbb{R}^q \) as \( T \to \infty \). Then, for all \( R > 0 \),

\[
v^c(R) = \{ u \in \mathbb{R}^q : \|u\| \leq R \} \subset U_T(\theta)
\]

for \( T > T_0(R) \).

The statistical meaning of the vectors \( M_T(u) \) and \( \Psi_T(u) \) is clear. Consider the functional \( \gamma Q_T(\theta + d_T^{-1}(\theta)u) \). Then the normalized \( L_p \)-estimator \( \hat{\theta}_T \) satisfies the following system of equations:

\[
M_T(u) = 0.
\]

Let

\[
\eta(t) = \gamma \psi(\varepsilon(t)), \quad t \in \mathbb{R}^1,
\]
and let the observations be of the following form:

\begin{equation}
Y(t) = g(t; \theta) + \eta(t), \quad t \in [0, T].
\end{equation}

Then \( \Psi_T(u) = 0 \) is the system of normal equations determining the least squares estimator

\[
\tilde{u}_T = \bar{u}_T(\theta) = d_T(\theta)(\bar{\theta}_T - \theta)
\]

of the unknown parameter \( \theta \) of the auxiliary nonlinear regression model \([12]\).

**Lemma 1.** Let Conditions \( A1, A2, \) and \( B1 \) hold. Then, for arbitrary \( R > 0 \) and \( r > 0 \),

\begin{equation}
P \left\{ \sup_{u \in v^c(R)} \|M_T(u) - \Psi_T(u)\| > r \right\} \rightarrow 0.
\end{equation}

**Proof.** For a fixed \( i \), consider the difference

\[
M^i_T(u) - \Psi^i_T(u)
\]

\begin{equation}
= \gamma \int_0^T h_i(t, u) \frac{\partial}{\partial T} \left( \psi(\varepsilon(t) + H(t; 0, u)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t)) H(t; 0, u) \right) dt
\end{equation}

\[+ \gamma \int_0^T H(t; 0, u) \frac{\partial}{\partial T} \zeta(t) dt = I_1(u) + I_2(u), \]

\[
\zeta(t) = \psi'(\varepsilon(t)) - \mathbb{E} \psi'(\varepsilon(t)), \quad t \in \mathbb{R}^1.
\]

One needs to prove that \( I_1(u) \) and \( I_2(u) \) converge to zero in probability uniformly with respect to \( u \in v^c(R) \). Let \( u \in v^c(R) \) be fixed. Then

\begin{equation}
\mathbb{E} I^2_2(u) = \gamma^2 \int_0^T \int_0^T H(t; 0, u) H(s; 0, u) \frac{h_i(t, u)}{d_T(\theta)} \frac{h_i(s, u)}{d_T(\theta)} \text{cov}(\zeta(t), \zeta(s)) dt ds.
\end{equation}

We have

\[
\sup_{t \in [0, T]} |H(t; 0, u)| = \sup_{t \in [0, T]} \left| \sum_{i=1}^q \frac{h_i(t, u^*_i)}{d_T(\theta)} u_i \right| \leq \|u\| \sup_{t \in [0, T]} \left( \sum_{i=1}^q \left| \frac{h_i(t, u^*_i)}{d_T(\theta)} \right|^2 \right)^{1/2},
\]

where \( \|u^*_i\| \leq \|u\| \). Using \([2]\), we obtain

\begin{equation}
\sup_{t \in [0, T]} |H(t; 0, u)| \leq T^{-1/2} \|k\| \cdot \|u\|,
\end{equation}

where \( k = (k^1, \ldots, k^q) \) are the constants involved in inequality \([2]\). Applying once more inequalities \([16]\) and \([2]\) to integral \([14]\), we get

\[
\mathbb{E} I^2_2(u) \leq \gamma^2 \|k\|^2 (k^i)^2 R^2 \frac{1}{T^2} \int_0^T \int_0^T |\text{cov}(\zeta(t), \zeta(s))| dt ds.
\]

We show that

\begin{equation}
\frac{1}{T^2} \int_0^T \int_0^T |\text{cov}(\zeta(t), \zeta(s))| dt ds \rightarrow 0
\end{equation}

as \( T \rightarrow \infty \). Similarly to \([6]\),

\[
|\text{cov}(\psi'(\varepsilon(t)), \psi'(\varepsilon(s)))| \leq \text{Var} \psi'(\varepsilon(0)) |B(t - s)|
\]
and
\[
\frac{1}{T^2} \int_0^T \int_0^T |\text{cov}(\zeta(t), \zeta(s))| \, dt \, ds \leq \frac{\text{Var} \, \psi'(\varepsilon(0))}{T^2} \int_0^T \int_0^T |B(t-s)| \, dt \, ds
\]
\[
\leq \frac{2 \text{Var} \, \psi'(\varepsilon(0))}{T} \int_0^T |B(u)| \, du \to 0.
\]

This implies that \( I_2(u) \xrightarrow{p} 0 \) pointwise for \( u \in v^c(R) \).

For \( u_1, u_2 \in v^c(R) \), consider the difference
\[
I_2(u_1) - I_2(u_2) = \gamma \int_0^T H(t; 0, u_1) \frac{H_1(t; u_1, u_2)}{d_t(\theta)} \zeta(t) \, dt - \gamma \int_0^T H(t; u_1, u_2) \frac{h_1(t, u_2)}{d_t(\theta)} \zeta(t) \, dt
\]
\[
= I_3(u_1, u_2) + I_4(u_1, u_2).
\]

For all \( h > 0 \) and \( r > 0 \), we write
\[
P \left\{ \sup_{\|u_1-u_2\| \leq h} |I_3(u_1, u_2)| > r \right\} \leq r^{-1} E \sup_{\|u_1-u_2\| \leq h} |I_3(u_1, u_2)|
\]
\[
\leq 2r^{-1} T \sup_{t \in [0, T]} |H(t; 0, u)| \sup_{\|u_1-u_2\| \leq h} \sup_{t \in [0, T]} |H_1(t; u_1, u_2)| \sup_{d_t(\theta)} \frac{h_1(t, u_2)}{d_t(\theta)}
\]
\[
\sup_{\|u_1-u_2\| \leq h} \sup_{t \in [0, T]} H_1(t; u_1, u_2)
\]
\[
\leq h \sup_{t \in [0, T]} \left[ \sum_{l=1}^q \left( \sup_{u \in v^c(R)} |h_{il}(t, u)| \frac{d_{il}(\theta)}{d_t(\theta)} \right) \frac{d_{il}(\theta)}{d_t(\theta)} \right]
\]
\[
\leq \sum_{l=1}^q k_i l_i k_i T h^{-1}
\]
by using conditions (3) and (4). Applying (16) and (19) in (18), we get
\[
P \left\{ \sup_{\|u_1-u_2\| \leq h} |I_3(u_1, u_2)| > r \right\} \leq k_1 r^{-1} T^{-1/2} h,
\]
where \( k_1 = 2R \|k\| \left( \sum_{l=1}^q k_i l_i k_i \right) \). Similarly, we use inequality (2) and obtain
\[
P \left\{ \sup_{\|u_1-u_2\| \leq h} |I_4(u_1, u_2)| > r \right\} \leq r^{-1} E \sup_{\|u_1-u_2\| \leq h} |I_4(u_1, u_2)|
\]
\[
\leq 2r^{-1} T \sup_{t \in [0, T]} |h_1(t, u)| \sup_{\|u_1-u_2\| \leq h} \sup_{t \in [0, T]} |H(t; u_1, u_2)| \leq k_2 r^{-1} h,
\]
where \( k_2 = 2k_i \|k\| \). It follows from bounds (20) and (21) that
\[
P \left\{ \sup_{\|u_1-u_2\| \leq h} |I_2(u_1) - I_2(u_2)| > r \right\} \leq 2r^{-1} h \left( k_1 T^{-1/2} + k_2 \right) \leq k_3 r^{-1} h.
\]

Denote by \( N_h \) a finite \( h \)-net of the ball \( v^c(R) \). Then
\[
\sup_{u \in v^c(R)} |I_2(u)| \leq \sup_{\|u_1-u_2\| \leq h} |I_2(u_1) - I_2(u_2)| + \max_{u \in N_h} |I_2(u)|.
\]
From (22) and (23) we obtain for all \( r > 0 \) that
\[
P \left\{ \sup_{u \in v^c(R)} |I_2(u)| > r \right\} \leq 2k_3 r^{-1} h + P \left\{ \max_{u \in N_h} |I_2(u)| > \frac{r}{2} \right\}.
\]
Let $\varepsilon > 0$ and $h = \varepsilon r/(4k_3)$. In view of the pointwise convergence in probability of $I_2(u)$ to zero we prove that

$$P\left\{ \max_{u \in N_{\varepsilon r/(4k_3)}} |I_2(u)| > r/2 \right\} \leq \frac{\varepsilon}{2}$$

for $T > T_0$, whence

$$P\left\{ \sup_{u \in v^c(R)} |I_2(u)| > r \right\} \leq \varepsilon.$$

On the other hand, if $u^*_T \in v^c(R)$ is a random variable, then

$$\text{(24) } \sup_{u \in v^c(R)} |I_1(u)|$$

\begin{align*}
&\leq \gamma \int_0^T \left| \frac{h_i(t, u^*_T)}{d_i(T(\theta))} \right| \left| (\psi(\varepsilon(t) + H(t; 0, u^*_T)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t; 0, u^*_T)) \right| dt.
\end{align*}

Let $\chi_T(t)$ be the indicator of the random event $\{ |\varepsilon(t)| \leq 2||k||RT^{-1/2} \}$ and let

$$\overline{\chi}_T(t) = 1 - \chi_T(t).$$

Using inequality (2) again, we continue the estimation in (24) as follows:

$$\text{sup}_{u \in v^c(R)} |I_1(u)|$$

\begin{align*}
&\leq \gamma k^i T^{-1/2} \int_0^T \left| \psi(\varepsilon(t) + H(t; 0, u^*_T)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t; 0, u^*_T) \right| \chi_T(t) dt \\
&\quad + \gamma k^i T^{-1/2} \int_0^T \left| \psi(\varepsilon(t) + H(t; 0, u^*_T)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t; 0, u^*_T) \right| \overline{\chi}_T(t) dt \\
&= \Delta_1(T) + \Delta_2(T).
\end{align*}

Bound (10) implies that

$$E \Delta_1(T) \leq \gamma k^i p (3p^{-1} + 2p^{-2}) \left( ||k||R \right)^{p-1} T^{-p/2} \int_0^T \left( \int_{\{ |x| \leq 2 \|k\||RT^{-1/2} \}} \varphi(x) dx \right) dt$$

\begin{align*}
&\quad + \gamma k^i p (p - 1) \|k\|R \int_0^T \left( \int_{\{ |x| \leq 2 \|k\||RT^{-1/2} \}} \|x\|^{p-2} \varphi(x) dx \right) dt \\
&\leq k_4 T^{(1-p)/2},
\end{align*}

where $k_4 = 4\gamma k^i (2\pi)^{-1/2} p (||k||R)^p (3p^{-1} + 2p^{-1} + 2p^{-2})$. Therefore $E \Delta_1(T) \rightarrow 0$ as $T \rightarrow \infty$.

Now we estimate $E \Delta_2(T)$. Note that

$$|\psi(\varepsilon(t) + H(t; 0, u^*_T)) - \psi(\varepsilon(t)) - \psi'(\varepsilon(t))H(t; 0, u^*_T)|$$

\begin{align*}
&= \frac{1}{2} |\psi''(\varepsilon(t) + \delta_1 H(t; 0, u^*_T))| H^2(t; 0, u^*_T) \\
&\leq \frac{1}{2} p (p - 1) (2 - p) \frac{1}{|\varepsilon(t) + \delta_1 H(t; 0, u^*_T)|^3} \|k\|^2 R^2 T^{-1}
\end{align*}
for some \( \delta_1 \in (0,1) \). Let \( k_5 = \frac{1}{2} \gamma k^i (p-1)(2-p)\|k\|^2R^2 \). Taking into account (25), we obtain

\[
E \Delta_2(T) \leq k_5 T^{-3/2} \int_0^T \left( \int_{\{x:|x| \geq 2\|R\|RT^{-1/2}\}} \frac{\varphi(x) \, dx}{(x - \|k\|RT^{-1/2})^{3-p}} \right) dt
\]

(26)

\[
\leq \frac{2k_5}{\sqrt{2\pi}} \frac{T^{-1/2}}{2\|k\|RT^{-1/2}} \int_0^\infty \frac{dx}{(x - \|k\|RT^{-1/2})^{3-p}} = k_6 T^{(1-p)/2},
\]

where \( k_6 = \gamma k^i (2\pi)^{-1/2} p(p-1)\|k\|^p R^p \), that is, \( E \Delta_2(T) \to 0 \).

Bounds (25) and (26) yield

\[
\sup_{u \in v^i(R)} |I_1(u)| \overset{p}{\to} 0 \text{ as } T \to \infty.
\]

Lemma 1 is proved. \( \square \)

Consider the random vector

\[(27) \quad L_T(u) = (L_T(u))_1^q = \left( T \left( \eta(t) - \sum_{i=1}^q g_i(t,\theta)u_i \right) \frac{g_i(t,\theta)}{d_i T(\theta)} dt \right)_{i=1}^q\]

corresponding to the auxiliary linear regression model

\[Z(t) = \sum_{i=1}^q g_i(t,\theta) \beta_i + \eta(t), \quad t \in [0, T],\]

where \( \eta(t) \) is defined by equality (11).

The system of normal equations

\[(28) \quad L_T(u) = 0\]

determines a normalized linear least squares estimator \( \tilde{\beta}_T \) of the parameter \( \beta \in \mathbb{R}^q \); namely,

\[(29) \quad \tilde{u}_T = \tilde{u}_T(\theta) = d_T(\theta) (\tilde{\beta}_T - \beta).\]

**Lemma 2.** If Conditions A1, A2, and B1 hold, then

\[(30) \quad \mathcal{P} \left\{ \sup_{u \in v^i(R)} \|\Psi_T(u) - L_T(u)\| > r \right\} \to 0 \text{ as } T \to \infty\]

for all \( R > 0 \) and \( r > 0 \).

**Proof.** For an arbitrary \( i \in \{1, \ldots, q\} \),

\[
\Psi_{T,i}(u) - L_{T,i}(u) = \int_0^T \eta(t) \frac{h_i(t, u)}{d_i T(\theta)} \, dt + \int_0^T H(t; 0, u) \frac{h_i(t, u)}{d_i T(\theta)} \, dt - \int_0^T \eta(t) \frac{g_i(t, \theta)}{d_i T(\theta)} \, dt
\]

\[
+ \int_0^T g_i(t, \theta) \sum_{i=1}^q \frac{g_i(t, \theta)}{d_i T(\theta)} u_i \, dt
\]

\[
= \int_0^T \eta(t) \frac{H_i(t; u, 0)}{d_i T(\theta)} \, dt + \int_0^T H(t; 0, u) \frac{H_i(t; u, 0)}{d_i T(\theta)} \, dt
\]

\[
+ \int_0^T g_i(t, \theta) \left[ H(t; 0, u) + \sum_{i=1}^q \frac{g_i(t, \theta)}{d_i T(\theta)} u_i \right] \, dt
\]

\[
= I_5(u) + I_6(u) + I_7(u).
\]
Let \( u \in v^r(R) \) be fixed. Using inequalities (6) and (19), we obtain
\[
\mathbb{E} I_5^2(u) = \int_0^T \int_0^T \text{cov}(\eta(t),\eta(s)) \frac{H_s(t;u,0)}{d_s(t)} \frac{H_s(s;u,0)}{d_s(s)} \, dt \, ds
\]
\[
\leq \left( \sum_{j=1}^q k^j \right)^2 R^2 \frac{1}{T^2} \int_0^T \int_0^T |\text{cov}(\eta(t),\eta(s))| \, dt \, ds \to 0 \quad \text{as} \quad T \to \infty;
\]
that is, \( I_5(u) \overset{p}{\to} 0 \) as \( T \to \infty \) pointwise for \( u \in v^r(R) \). On the other hand,
\[
\mathbb{E} \sup_{\|u_1-u_2\| \leq h} |I_5(u_1) - I_5(u_2)| \leq |\gamma| \mathbb{E} |\psi(\varepsilon(0))| \left( \sum_{j=1}^q k^j \right) h
\]
in view of (19). Similarly to the proof of the convergence of \( I_2(u) \) in Lemma 1, one can show that \( I_5(u) \) converges uniformly to zero in probability with respect to \( u \in v^r(R) \).

Taking into account inequalities (16) and (19) we get
\[
\sup_{u \in v^r(R)} |I_6(u)| \leq \|k\| \left( \sum_{j=1}^q k^j \right) R^2 T^{-1/2} \to 0 \quad \text{as} \quad T \to \infty.
\]
The term \( I_7(u) \) can be represented as follows:
\[
I_7(u) = -\frac{1}{2} \sum_{j,l=1}^q \left( \int_0^T \frac{h_{jl}(t,u_t)}{d_T(t)} g_j(t,\theta) \right) u_j u_l
\]
for some \( u_t \in v(R) \). Under Condition B1 we have
\[
|I_7(u)| \leq \frac{k_i}{2} \sum_{j,l=1}^q \left( k^j \right) |u_j||u_l| T^{-1/2} \leq \frac{qk_i}{2} \max_{j,l=1,...,q} \left[ k^j \right] \|u\|^2 T^{-1/2},
\]
whence \( \sup_{u \in v^r(R)} |I_7(u)| \to 0. \) Lemma 2 is proved.

Relations (19) and (30) imply the following corollary.

**Corollary.** Let Conditions A1, A2, and B1 hold. Then
\[
\mathbb{P} \left\{ \sup_{u \in v^r(R)} \|M_T(u) - L_T(u)\| > r \right\} \to 0 \quad \text{as} \quad T \to \infty
\]
for all \( R > 0 \) and \( r > 0 \).

Using (27) and (28) one can evaluate \( \bar{u}_T \) in an explicit form (see (29)) if Condition B2 holds; namely,
\[
\bar{u}_T = \Lambda_T(\theta) \int_0^T \eta(t) d_T^{-1}(\theta) \nabla g(t,\theta) \, dt,
\]
where \( \Lambda_T(\theta) = J_T^{-1}(\theta) \) and \( \nabla g(t,\theta) \) is the gradient of the function \( g(t,\theta) \):
\[
\nabla g(t,\theta) = \begin{bmatrix} g_1(t,\theta) \\ \vdots \\ g_q(t,\theta) \end{bmatrix}.
\]
Note that the covariance matrix of the vector \( \bar{u}_T \) is given by
\[
\sigma_T(\theta) = 2\pi \gamma^2 \left( \int_{\mathbb{R}^1} \mu_T(dx;\theta) \right)^{-1} \left( \int_{\mathbb{R}^1} f_\psi(x) \mu_T(dx;\theta) \right) \left( \int_{\mathbb{R}^1} \mu_T(dx;\theta) \right)^{-1}.
\]
Consider the random event
\begin{equation}
A_T = \{ \tilde{u}_T \in \nu^c(R - r) \}.
\end{equation}

**Lemma 3.** Let Conditions **B2** and  hold. Then, for arbitrary \( \varepsilon > 0 \) and \( r > 0 \), there exists \( R > r \) such that \( \mathbb{P}\{A_T\} \leq \varepsilon \) for \( T > T_0 \).

**Proof.** Fix \( \varepsilon > 0 \). Analogously to the proof of convergence (17) we derive that
\[
\mathbb{P}\{A_T\} = \mathbb{P}\{||\tilde{u}_T|| > R - r \} \leq \frac{\mathbb{E}||\tilde{u}_T||^2}{(R - r)^2}
\]
\[
\leq \frac{1}{\lambda^2(R - r)^2} \sum_{i=1}^{q} \int_{0}^{T} \int_{0}^{T} |\text{cov}(\eta(t), \eta(s))| \frac{g_i(t, \theta) g_i(s, \theta)}{d\theta R T(\theta)} \, dt \, ds
\]
\[
\leq \frac{||k||^2}{T \lambda^2(R - r)^2} \int_{0}^{T} \int_{0}^{T} |\text{cov}(\eta(t), \eta(s)))| \, dt \, ds \leq \frac{k_6}{(R - r)^2},
\]
where \( k_6 = 2(\lambda_*)^{-2} \gamma^2 ||k||^2 \mathbb{E}(\mathbb{E}(\varepsilon(0)))^2 \int_{0}^{\infty} |B(u)| \, du.
\]
Setting \( R = r + \sqrt{k_6/\varepsilon} \) we get the desired result. \( \square \)

We need the following results for the proof of the theorem.

**Proposition 1.** Let \( F: \nu^c(R) \to \nu^c(R) \) be a continuous mapping. Then there exists \( x_0 \in \nu^c(R) \) such that \( F(x_0) = x_0 \).

This proposition is a special case of the Brouwer fixed point theorem (see, for example, [3]).

**Proposition 2.** Let Conditions **A1**, **A2**, **B2**, and **B3** hold. Then the random vector \( \tilde{u}_T \) defined by (32) is asymptotically normal with parameters \( 0 \) and \( \sigma(\theta) \), where the covariance function \( \sigma(\theta) \) is defined by (12).

The latter result is a multivariate central limit theorem for the integral of a weighted stationary stochastic process satisfying the weak dependence condition. A more general result of this kind as well as its proof can be found in [3] (Theorem 1.7.5).

Let \( \mathcal{B}^q \) be the \( \sigma \)-algebra of Borel sets of \( \mathbb{R}^q \). For \( C \in \mathcal{B}^q \) and \( \varepsilon > 0 \), let
\[
C_\varepsilon = \{ x : x \in \mathbb{R}^q, d(x, C) < \varepsilon \},
\]
where \( d(x, C) = \inf_{y \in C} ||x - y|| \) and \( C_{-\varepsilon} = \mathbb{R}^q \setminus (\mathbb{R}^q \setminus C)_{\varepsilon} \).

**Proposition 3.** Let \( \nu \geq 0 \) be a differentiable function defined on \([0, \infty)\) and such that
\[
b = \int_{0}^{\infty} |\nu'(\lambda)| \lambda^{q-1} \, d\lambda < \infty, \quad \lim_{\lambda \to \infty} \nu(\lambda) = 0.
\]
Then
\[
\int_{C_\varepsilon \setminus C_{-\delta}} \nu(||x||) \, dx \leq b \left( \frac{2\pi^{q/2}}{\Gamma(q/2)} \right) (\varepsilon + \delta)
\]
for an arbitrary convex set \( C \in \mathcal{B}^q \) and for all \( \varepsilon, \delta > 0 \).

The proof of the latter result can be found in §3 of the book [3].
3. Proof of the theorem

One needs to prove that the distribution function $G_T(y, \theta)$ of the random vector $\hat{u}_T$ defined by (29) converges as $T \to \infty$ to the Gaussian distribution function $\Phi_{0,\sigma^2(\theta)}(y)$. Note that $\sigma(\theta)$ is a positive definite matrix \[ \text{in view of Conditions B2 and B3}. \]

We show that
\[ \Delta_T(r) = P\{\|\hat{u}_T - \bar{u}_T\| > r\} \to 0, \quad T \to \infty, \]
for all $r > 0$.

Let $A_T$ be the random event defined by equality (33) and let $R$ be such that $P\{\bar{X}_T\} \leq \frac{\varepsilon}{3}$ (this inequality holds by Lemma 3), where $\varepsilon > 0$ is a fixed but small number.

We also consider the following random events:
\[ B_T = \left\{ \sup_{u \in v^c(R)} \|\Lambda_T(\theta)(M_T(u) - L_T(u))\| \leq r \right\}, \]
\[ C_T = \{ \text{system of equations (11) has a unique solution in the ball } v^c(R) \}. \]

Conditions B2 and C together with the corollary to Lemmas 1 and 2 imply for $T > T_0$ that
\[ P\{B_T\} \leq P\left\{ \sup_{u \in v^c(R)} \|M_T(u) - L_T(u)\| > \lambda_3 r \right\} \leq \frac{\varepsilon}{3}, \]
whence
\[ P\{A_T \cap B_T \cap C_T\} > 1 - \varepsilon \]
for $T > T_0$.

Taking into account relations (27) and (32) we get $\Lambda_T(\theta)L_T(u) = \bar{u}_T - u$. If the event $A_T \cap B_T \cap C_T$ occurs, then
\[ \|u + \Lambda_T(\theta)M_T(u)\| \leq \|\Lambda_T(\theta)(M_T(u) - L_T(u))\| + \|\bar{u}_T\| \leq r + (R - r) = R \]
for $u \in v^c(R)$; that is, $F_T(u) = u + \Lambda_T(\theta)M_T(u)$ is a continuous mapping of $v^c(R)$ to $v^c(R)$. To prove (34) we apply the Brouwer fixed point theorem (see Proposition 1).

Applying this theorem to $F_T(u)$, we prove that there exists a point $u^0_T \in v^c(R)$ such that $F_T(u^0_T) = u^0_T$. Since $\Lambda_T(\theta)$ is nonsingular, we also have $M_T(u^0_T) = 0$. According to Condition C, the normalized $L_p$-estimator $\hat{u}_T$ is a unique solution of the system of equations $M_T(u) = 0$. Therefore $A_T \cap B_T \cap C_T \subset \{\hat{u}_T \in v^c(R)\}$ and hence
\[ P\{\hat{u}_T \in v^c(R)\} > 1 - \varepsilon. \]

Note also that inequality (35) yields
\[ 1 - \varepsilon < P\{\hat{u}_T \in v^c(R)\} \leq P\left\{ \|\hat{u}_T \in v^c(R)\| \cap B_T \right\} \leq P\left\{ \|\Lambda_T(\theta)(M_T(u) - L_T(u))\| \leq r \right\} = P\{\|\hat{u}_T - \bar{u}_T\| \leq r\}. \]

Relation (36) for all $\varepsilon > 0$ is equivalent to (34).

Put $\Pi(-\infty, y, \pm \varepsilon) = (-\infty, y_1, \pm \varepsilon) \times \cdots \times (-\infty, y_d, \pm \varepsilon)$, $\varepsilon \geq 0$. Considering relation (34), we prove for the distribution function $G_T(y, \theta) = P\{\hat{u}_T \in \Pi(-\infty, y)\}$ that
\[ -\Delta_T(\varepsilon) + P\{\hat{u}_T \in \Pi(-\infty, y, -\varepsilon)\} \leq G_T(y, \theta) \leq P\{\hat{u}_T \in \Pi(-\infty, y + \varepsilon)\} + \Delta_T(\varepsilon). \]

Proposition 2 implies that
\[ P\{\hat{u}_T \in \Pi(-\infty, y, \pm \varepsilon)\} - \Phi_{0,\sigma(\theta)}(y, \pm \varepsilon) \to 0 \]
as $T \to \infty$.  

Let \( \varphi(y, \theta) \) be the Gaussian density corresponding to the distribution function
\[
\Phi_{0, \sigma(\theta)}(y).
\]
Since \( \lambda_{\min}(\sigma(\theta)) = \lambda > 0 \) and \( \lambda_{\max}(\sigma(\theta)) = \bar{\lambda} < \infty \), we get
\[
\varphi(y, \theta) \leq (2\pi \lambda)^{-\nu/2} \exp\{-\|y\|^2/2\bar{\lambda}\} = \nu(\|y\|).
\]
If \( A = \Pi(-\infty, y) \), then \( A_{-\varepsilon} = \Pi(-\infty, y - \varepsilon) \) and \( (\Pi(-\infty, y + \varepsilon))_{-\varepsilon} = \Pi(-\infty, y) = A^c \).
Applying Proposition 3 to the function \( \nu(\|y\|) \) we conclude that
\[
|\Phi_{0, \sigma(\theta)}(y) - \Phi_{0, \sigma(\theta)}(y + \phi)| = \int_{\Pi} \varphi(y, \theta) \, dy \leq b \left( \frac{2\pi \nu/2}{(\nu/2)^2} \right) |\phi|,
\]
(38)
\[
\Pi = \begin{cases} 
\Pi(-\infty, y + \phi) \setminus A^c & \text{if } \phi > 0, \\
\Pi(A \setminus A_{\phi}) & \text{if } \phi < 0,
\end{cases}
\]
for all \( \phi \neq 0 \).

Further
\[
G_T(y, \theta) - \Phi_{0, \sigma(\theta)}(y) \leq \Delta_T(\varepsilon) + |P\{\bar{\omega}_T \in \Pi(-\infty, y + \varepsilon)\} - \Phi_{0, \sigma(\theta)}(y + \varepsilon)|
\]
\[
+ |\Phi_{0, \sigma(\theta)}(y + \varepsilon) - \Phi_{0, \sigma(\theta)}(y)|,
\]
(39)
\[
\Phi_{0, \sigma(\theta)}(y) - G_T(y, \theta) \leq |\Phi_{0, \sigma(\theta)}(y) - \Phi_{0, \sigma(\theta)}(y - \varepsilon)|
\]
\[
+ |\Phi_{0, \sigma(\theta)}(y - \varepsilon) - P\{\bar{\omega}_T \in \Pi(-\infty, y - \varepsilon)\}| + \Delta_T(\varepsilon)
\]
(40)
for all \( y \in \mathbb{R}^d \) and \( \varepsilon > 0 \). Applying (37) and (38) to (39), (40) we obtain the desired result.

4. Example

Consider the regression model
\[
X(t) = A_0 \cos \omega t + B_0 \sin \omega t + \varepsilon(t), \quad t \in (0, \infty),
\]
where \( \varepsilon(t), t \in \mathbb{R}^1 \), is a stochastic process satisfying Conditions A1 and A2 and having the covariance function \( B(t) = e^{-\pi|t|} \). The vector of parameters
\[
\theta = (\theta_1, \theta_2, \theta_3) = (A_0, B_0, \omega)
\]
belong to an open bounded set and moreover \( A_0^2 + B_0^2 > 0 \) and \( 0 < \omega < \omega_0 < \omega < \infty \).

Consider the loss function \( \rho(x) = |x|^p, p \in \left(\frac{1}{2}, 2\right) \).

It is easy to check that
\[
d_{1T}^2(\theta) = \frac{T}{2} + \frac{1}{4\omega} \sin 2\omega t,
\]
\[
d_{2T}^2(\theta) = \frac{T^2}{6} + O\left(T^2\right),
\]
\[
d_{3T}^2(\theta) = d_{1,1,1}^2(\theta) = d_{12,1}^2(\theta) = d_{21,1}^2(\theta) = d_{22,1}^2(\theta) = 0,
\]
\[
d_{13,1}^2(\theta) = d_{31,1}^2(\theta) = \frac{T^3}{6} + O\left(T^2\right),
\]
\[
d_{23,1}^2(\theta) = d_{32,1}^2(\theta) = \frac{T^3}{6} + O\left(T^2\right),
\]
\[
d_{33,1}^2(\theta) = \frac{A_0^2 + B_0^2}{10} T^5 + O\left(T^4\right).
\]
Condition B1 is easy to check, too. Further
\[
\lim_{T \to \infty} J_T(\theta) = J(\theta) = \begin{pmatrix} 1 & 0 & -\frac{B_0}{\sqrt{4(A_0^2 + B_0^2)}/3} \\ 0 & 1 & -\frac{A_0}{\sqrt{4(A_0^2 + B_0^2)}/3} \\ -\frac{B_0}{\sqrt{4(A_0^2 + B_0^2)}/3} & -\frac{A_0}{\sqrt{4(A_0^2 + B_0^2)}/3} & 1 \end{pmatrix}.
\]
The above matrix \( J(\theta) \) is positive definite; that is, Condition B2 holds.
It is known that the spectral measure for the regression function of the model (41) is given by

\[
\mu(dx, \theta) = \begin{pmatrix}
\delta_{\omega_0}(dx) & i\rho_{\omega_0}(dx) & \frac{B_0\delta_{\omega_0}(dx) - iA_0\rho_{\omega_0}(dx)}{\sqrt{4(A^2_0 + B^2_0)}} \\
-i\rho_{\omega_0}(dx) & \delta_{\omega_0}(dx) & \frac{-A_0\delta_{\omega_0}(dx) - iB_0\rho_{\omega_0}(dx)}{\sqrt{4(A^2_0 + B^2_0)}} \\
\frac{B_0\delta_{\omega_0}(dx) + iA_0\rho_{\omega_0}(dx)}{\sqrt{4(A^2_0 + B^2_0)}} & \frac{-A_0\delta_{\omega_0}(dx) + iB_0\rho_{\omega_0}(dx)}{\sqrt{4(A^2_0 + B^2_0)}} & \delta_{\omega_0}(dx)
\end{pmatrix}
\]

(see, for example, [7]) where the measure \(\delta_{\omega_0}\) and the charge \(\rho_{\omega_0}\) are concentrated at the points \(\pm \omega_0\) and

\[
\delta_{\omega_0}(\{\pm \omega_0\}) = \frac{1}{2}, \quad \rho_{\omega_0}(\{\pm \omega_0\}) = \pm \frac{1}{2}.
\]

This means that

\[
\int_{\mathbb{R}} f_\psi(x) \mu(dx) = f_\psi(\omega_0)J(\theta).
\]

If \(f_\psi(\omega_0) \neq 0\), then \(\int_{\mathbb{R}} f_\psi(x) \mu(dx)\) is a positive definite matrix and Condition B3 holds.

It remains to note that the assumptions of the paper [24] are satisfied for model (41) and that they imply Condition C. Now one can apply our theorem. Using the above evaluation of \(d_T(\theta)\), one can adjust the representation of the normalized \(L_p\)-estimator as follows:

\[
\hat{u}_T = \begin{pmatrix}
\sqrt{\frac{T}{2}}(A - A_0) \\
\sqrt{\frac{T}{2}}(B - B_0) \\
\sqrt{A^2_0 + B^2_0}T^{3/2}(\omega - \omega_0)
\end{pmatrix}
\]

(see [7] for more details). According to this normalization, the covariance function of the asymptotic normal distribution becomes of the following form:

\[
\sigma(\theta) = \frac{4\pi \gamma^2 f_\psi(\omega_0)}{A^2_0 + B^2_0} \begin{pmatrix}
A^2_0 + 4B^2_0 & -3A_0B_0 & -6B_0 \\
-3A_0B_0 & 4A^2_0 + B^2_0 & 6A_0 \\
-6B_0 & 6A_0 & 12
\end{pmatrix},
\]

where \(\gamma\) is defined by (33).

Considering the expansion of the function \(\psi(\varepsilon(t))\) in the series (35) with respect to the Hermite–Chebyshev polynomials, we obtain the following representation of the spectral density:

\[
f_\psi(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} B_\psi(t) dt = \sum_{n=1}^{\infty} \frac{C_n^2(\psi)}{n!} f_n(\lambda),
\]

where \(f_n(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\lambda t} B^n(t) dt\). By assumption,

\[
f_n(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda t} e^{-n|x|} dt = \frac{n\pi}{\pi(n^2\pi^2 + \lambda^2)}.
\]

Since \(\psi\) is an odd function, \(C_{2k}(\psi) = 0\); that is,

\[
\sigma(\theta) = \frac{4\pi \gamma^2}{A^2_0 + B^2_0} \left( \sum_{k=0}^{\infty} \frac{C_{2k+1}^2(\psi)}{(2k+1)!((2k+1)^2\pi^2 + \omega^2_0)} \right) \begin{pmatrix}
A^2_0 + 4B^2_0 & -3A_0B_0 & -6B_0 \\
-3A_0B_0 & 4A^2_0 + B^2_0 & 6A_0 \\
-6B_0 & 6A_0 & 12
\end{pmatrix}.
\]

The coefficients \(C_{2k+1}(\psi)\) are the values of known integrals (see, for example, relation 7.376.3 in the book [2]).
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