ON THE DISTRIBUTION OF STORAGE PROCESSES
FROM THE CLASS $V(\varphi, \psi)$

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Abstract. Estimates for the distribution of a storage process
$Q(t) = \sup_{s \leq t} (X(t) - X(s) - (f(t) - f(s)))$
are obtained in the paper, where $(X(t), t \in T)$ is a stochastic process belonging to
the class $V(\varphi, \psi)$ and where the service output rate $f(t)$ is a continuous function. In
particular, the results hold if $(X(t), t \in T)$ is a Gaussian process. Several examples of
applications of the results obtained in the paper are given for sub-Gaussian stationary
stochastic processes.

1. Introduction

A considerable number of papers are devoted nowadays to studies of the behavior of
the data traffic in computer networks (see, for example, [1, 2, 4, 8, 10]). A special place
in this topic is occupied by the problem of estimating the reliability when transferring
the data. Several stochastic models of the data traffic have been studied over recent
years. Among those models are stochastic processes belonging to some classes describing
a specific traffic of the data, namely Markov stochastic processes, processes with long as
well as with short memory, automodel processes, Gaussian processes, especially fractional
Brownian motion, autoregressive integrated moving average processes (ARIMA), etc.

The current paper deals with the distribution of a storage process generated by a
stochastic process $(X(t), t \in T)$ belonging to a rather general class $V(\varphi, \psi)$ containing,
for example, Gaussian stochastic processes.

Let $T$ be a set of parameters.

Definition 1.1. A process $Q(t) = \{Q(t), t \in T\}$ given by
$Q(t) = \sup_{s \leq t} (X(t) - X(s) - (f(t) - f(s)))$
$s, t \in T,$
is called a storage process generated by the input $X(t) = \{X(t), t \in T\}$ and a service
output rate $f(t)$.

Assume that a work arrives to a server. If more work arrives than can be processed,
the surplus waits in the queue buffer of length $x \geq 0$. A work that arrives after the buffer
overflows is lost. An important problem of queuing theory is to estimate the overflow
probability, namely
$P\{Q(t) > x\}.$

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The same problem is stated in terms of the ruin probability for the corresponding risk process in risk theory.

We consider the case where the input process \( X(t) \) belongs to the class \( V(\varphi, \psi) \). Recall that \( V(\varphi, \psi) \) contains the class of \( \varphi \)-sub-Gaussian stochastic processes. More detail and results concerning the processes of \( V(\varphi, \psi) \) can be found in [3, 13], and other sources are included in the list of references at the end of the paper.

The paper is organized as follows. Section 2 contains necessary definitions and results concerning \( \varphi \)-sub-Gaussian random variables and stochastic processes. Section 3 is devoted to the main results proved with the help of the metric entropy method. An application of the results obtained in Section 2 to stationary sub-Gaussian stochastic processes is given in Section 3.

2. Necessary definitions and results

Let \( \{\Omega, \mathcal{B}, P\} \) be a standard probability space and let \( T \) be a space of parameters.

2.1. Orlicz N-functions, Young–Fenchel transform.

**Definition 2.1** ([3]). A function \( U =\{U(x), x \in \mathbb{R}\} \) is called an Orlicz N-function if \( U \) is continuous, even, and such that

1) \( U(0) = 0 \),
2) \( U(x) \) increases for \( x > 0 \),
3) \( U(x)/x \to 0 \) as \( x \to 0 \) and \( U(x)/x \to \infty \) as \( x \to \infty \).

**Condition Q.** We say that condition Q holds for an Orlicz N-function \( \varphi \) if

\[
\liminf_{x \to 0} \frac{\varphi(x)}{x^2} = c > 0.
\]

**Definition 2.2.** We say that an N-function \( \varphi_1 \) is subordinated to an N-function \( \varphi_2 \) and denote \( \varphi_1 \prec \varphi_2 \) if there are some constants \( c > 0 \) and \( x_0 > 0 \) such that \( \varphi_1(x) < \varphi_2(cx) \) for \( x > x_0 \).

We say that two N-functions \( \varphi_1 \) and \( \varphi_2 \) are equivalent if \( \varphi_1 \prec \varphi_2 \) and \( \varphi_2 \prec \varphi_1 \).

**Definition 2.3.** Let \( \varphi =\{\varphi(x), x \in \mathbb{R}\} \) be an N-function. Then

\[
\varphi^*(x) = \sup_{y \in \mathbb{R}} (xy - \varphi(y))
\]

is called the Young–Fenchel transform of the function \( \varphi \).

2.2. \( \varphi \)-sub-Gaussian random variables and stochastic processes.

**Definition 2.4** ([3]). Let \( \varphi \) be an N-function satisfying condition Q. We say that a random variable \( \xi \) belongs to the space \( \text{Sub}_\varphi(\Omega) \) if

1) \( \mathbb{E} \xi = 0 \),
2) \( \mathbb{E} \exp\{\lambda \xi\} \) is finite for all \( \lambda \in \mathbb{R} \), and
3) there exists a constant \( a > 0 \) such that

\[
\mathbb{E} \exp\{\lambda \xi\} \leq \exp\{\varphi(\lambda a)\}
\]

for all \( \lambda \in \mathbb{R} \).

**Theorem 2.1** ([3]). The space \( \text{Sub}_\varphi(\Omega) \) is a Banach space with respect to the norm

\[
\tau_\varphi(\xi) = \sup_{\lambda > 0} \frac{\varphi^{(-1)}(\log \mathbb{E} \exp\{\lambda \xi\})}{\lambda},
\]
where \( \varphi^{-1} \) is the inverse function to \( \varphi \). In addition,

\[
E \exp\{\lambda \xi\} \leq \exp\{\varphi(\lambda \tau_\varphi(\xi))\}
\]

for all \( \lambda \in \mathbb{R} \).

**Lemma 2.1** ([13]). Let a random variable \( \xi \in \text{Sub}_\varphi(\Omega) \) be such that \( \tau_\varphi(\xi) > 0 \) and let \( \varepsilon > 0 \). Then

\[
\begin{align*}
\mathbb{P}\{\xi > \varepsilon\} & \leq \exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}, \\
\mathbb{P}\{\xi < -\varepsilon\} & \leq \exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}, \\
\mathbb{P}\{|\xi| > \varepsilon\} & \leq 2 \exp\left\{-\varphi^*\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}.
\end{align*}
\]

**Lemma 2.2** ([13]). Let a random variable \( \xi \) belong to the space \( \text{Sub}_\varphi(\Omega) \). Then

\[
|E \xi^k| \leq E |\xi|^k \leq 2(\tau_\varphi(\xi))^k \frac{e^k}{(\varphi^{-1}(k))^k k!}
\]

for \( k = 1, 2, \ldots \).

**Example 2.1** ([3]). Let \( \xi \) be a centered Gaussian random variable, that is, \( \xi \in \mathcal{N}(0, \sigma^2) \). Then \( \xi \) belongs to the space \( \text{Sub}_{x^2/2}(\Omega) \) and \( \tau(\xi) = (E \xi^2)^{1/2} \).

**Definition 2.5.** A stochastic process \( X = \{X(t), t \in T\} \) is called \( \varphi \)-sub-Gaussian if the random variables \( X(t), t \in T \), are \( \varphi \)-sub-Gaussian. We write \( X(t) \in \text{Sub}_\varphi(\Omega) \) in this case.

If \( \varphi(x) = x^2/2 \), then the processes \( X(t) \in \text{Sub}_\varphi(\Omega) \) are called sub-Gaussian.

**Example 2.2.** Every centered Gaussian stochastic process is a sub-Gaussian process.

**2.3. Strictly \( \varphi \)-sub-Gaussian random variables and stochastic processes.**

**Definition 2.6** ([6]). A family \( \Delta \) of random variables belonging to the space \( \text{Sub}_\varphi(\Omega) \) is called strictly \( \varphi \)-sub-Gaussian if there exists a constant \( C_\Delta > 0 \) such that

\[
\tau_\varphi\left(\sum_{i \in I} \lambda_i \xi_i\right) \leq C_\Delta \left(\mathbb{E}\left(\sum_{i \in I} \lambda_i \xi_i\right)^2\right)^{1/2}
\]

for all finite sets \( I \), all \( \xi_i \in \Delta, i \in I \), and all \( \lambda_i \in \mathbb{R}^1 \).

The constant \( C_\Delta \) is called a defining constant for the family \( \Delta \).

**Theorem 2.2** ([6]). Let \( \Delta \) be a strictly \( \varphi \)-sub-Gaussian family. Then the linear closure of the family \( \Delta \) in the space \( \text{Sub}_\varphi(\Omega) \) (or in \( L_2(\Omega) \)) also is a strictly \( \varphi \)-sub-Gaussian family with the same defining constant.

The linear closure of strictly \( \text{Sub}_\varphi(\Omega) \) random variables forms the space of strictly \( \text{Sub}_\varphi(\Omega) \) random variables. This space is denoted by \( \text{SSub}_\varphi(\Omega) \).

**Definition 2.7.** A stochastic process \( X = \{X(t), t \in T\} \) is called strictly \( \varphi \)-sub-Gaussian if the random variables \( \{X(t), t \in T\} \) form a strictly \( \varphi \)-sub-Gaussian family.

The defining constant of this family is called the defining constant of the process \( X \) and is denoted by \( C_X \).
2.4. Stochastic processes belonging to the class $V(\varphi, \psi)$.

**Definition 2.8 ([13])**. Let $\varphi \prec \psi$ be two Orlicz $N$-functions. We say that a stochastic process $X = \{X(t), t \in T\}$ belongs to the class $V(\varphi, \psi)$ if the random variable $X(t)$ belongs to the space $\text{Sub}_\psi(\Omega)$ for all $t \in T$ and if the increments $X(t) - X(s)$ belong to the space $\text{Sub}_\varphi(\Omega)$ for all $s, t \in T$.

**Example 2.3 ([3])**. Every sub-Gaussian stochastic process belongs to the class $V(\varphi, \varphi)$, where $\varphi = x^2/2$.

**Example 2.4 ([13])**. Let

$$X(t) = \xi_0 + \sum_{k=1}^{\infty} \xi_k f_k(t),$$

where the random variable $\xi_0$ belongs to $\text{Sub}_\psi(\Omega)$, $\{\xi_k, k = 1, 2, \ldots\} \in \text{Sub}_\varphi(\Omega)$, and where $\varphi$ is an Orlicz $N$-function such that $\varphi(\sqrt{x})$ is even and

$$\sum_{k=1}^{\infty} \tau_\varphi(\xi_k)|f_k(t)| < \infty.$$

Then the stochastic process $X(t)$ belongs to the class $V(\varphi, \psi)$.

3. Main results

Let $(T, \rho)$ be a pseudo-metric (metric) separable space equipped with a pseudo-metric (metric) $\rho$. Recall that a pseudo-metric satisfies all the conditions for a metric except the following one: if $\rho(t, s) = 0$, then $t = s$. The latter means that the set $\{(t, s) : \rho(t, s) = 0\}$ is possibly wider than the “diagonal” $\{(t, s) : t = s\}$ (see [3]).

Consider a separable stochastic process $X = \{X(t), t \in T\}$ belonging to the class $V(\varphi, \psi)$. Assume that there exists a continuous increasing function $\sigma = \{\sigma(h), h > 0\}$ such that $\sigma(h) \to 0$ as $h \to 0$ and

$$\sup_{\rho(t, s) \leq h} \tau_\varphi(X(t) - X(s)) \leq \sigma(h).$$

(7)

Note that the function

$$\sigma(h) = \sup_{\rho(t, s) \leq h} \tau_\varphi(X(t) - X(s))$$

can be used on the right hand side of (7) if the process $X(t)$ is continuous in the norm $\tau_\varphi(\cdot)$.

Let $B$ be a compact set, $B \subset T$. In what follows we use the following notation:

1. $\gamma(u) = \tau_\psi(X(u))$;
2. $B_t = \{u \in B : u \leq t\}$;
3. $\beta > 0$ is a number such that $\beta \leq \sigma(\inf_{s \in B} \sup_{t \in B} \rho(t, s))$;
4. $\xi_\varphi(v) = \frac{v}{\psi^{-1}(v)}$;
5. $N_B(u) = N_{(B, \rho)}(u)$ is the metric capacity of the space $(B, \rho)$, that is, the minimum number of closed balls of radius $u$ covering the space $(B, \rho)$;
6. $H_B(u) = H_{(B, \rho)}(u) = \ln N_B(u)$ is the metric entropy of the space $(B, \rho)$.

Let a separable stochastic process $X = \{X(t), t \in B\}$ belong to the class $V(\varphi, \psi)$ and be defined on a compact set $B$. Assume that $\gamma(u) < \infty$ and that $f = \{f(t), t \in B\}$ is a continuous function. The following result contains conditions for the boundedness of the stochastic process

$$Q(t) = \sup_{s \leq t} (X(t) - X(s) - (f(t) - f(s))), \quad s, t \in B.$$  

(8)

Lemma [3.1] also provides the bounds for the exponential moments of this process.
Lemma 3.1. Let \( f = \{f(t), t \in B\} \) be a continuous function such that
\[
|f(u) - f(w)| \leq \delta(p(u, w)),
\]
where \( \delta = \{\delta(s), s > 0\} \) is a nonnegative increasing function. Let \( \{q_k, k = 1, 2, \ldots\} \) be a sequence such that \( q_k > 1, k \geq 1 \), and
\[
\sum_{k=1}^{\infty} \frac{1}{q_k} \leq 1.
\]
Then
\[
\mathbb{E}\exp\{\lambda Q(t)\} \leq W(\lambda, t, p) \left( \prod_{k=2}^{\infty} \left( N_{B_k} \left( \sigma^{(-1)} \left( \beta p^k \right) \right) \right)^{1/q_k} \right)
\]
\[
\times \left( \prod_{k=2}^{\infty} \exp \left\{ -\frac{\varphi}{q_k} \left( \lambda q_k \beta p^{k-1} \right) + \lambda \delta \left( \sigma^{(-1)} \left( \beta p^{k-1} \right) \right) \right\} \right)
\]
for all \( \lambda > 0 \) and \( p \in (0, 1) \), where
\[
W(\lambda, t, p) = \min \{X_{\lambda, t, p}; Y_{\lambda, t, p}\}
\]
and
\[
X_{\lambda, t, p} = \left( \sum_{j=1}^{N_{B_1} \left( \sigma^{(-1)} \left( \beta p \right) \right)} \exp \left\{ \varphi \left( q_1 \lambda \sigma \left( 2j \sigma^{(-1)} \left( \beta p \right) \right) \right) + q_1 \lambda \delta \left( 2j \sigma^{(-1)} \left( \beta p \right) \right) \right\} \right)^{1/q_1},
\]
\[
Y_{\lambda, t, p} = \left( N_{B_1} \left( \sigma^{(-1)} \left( \beta p \right) \right) \right)^{1/q_1}
\]
\[
\times \inf_{w > 1} \exp \left\{ \frac{\psi(w q_1 \lambda \gamma(t))}{w q_1} \right\}
\]
\[
\times \max_{u \in B_{1}} \left\{ \left( w - 1 \right) \psi \left( \frac{w}{w-1} q_1 \lambda \gamma(u) \right) + \lambda f(u) \right\} - \lambda f(t) \right\}.
\]

Remark 3.1. Clearly, inequality (9) makes sense if the denominators on the right hand side are finite.

Proof. Denote by \( V_{\varepsilon_k} \) the set of centers of closed balls of radius
\[
\varepsilon_k = \sigma^{(-1)} \left( \beta p^k \right), \quad p \in (0, 1), \quad k = 0, 1, 2, \ldots,
\]
that form the minimal covering of the space \((B, \rho)\). The number of sets in the set \( V_{\varepsilon_k} \) is equal to \( N_B(\varepsilon_k) \).

Note that \( X(t) \) and thus \( X_f(t) = X(t) - f(t) \) are separable processes.

Lemma 2.4 and assumption (7) imply that
\[
P\{|X(t) - X(s)| > \varepsilon\} \leq 2 \exp \left\{ -\varphi^* \left( \frac{\varepsilon}{\tau_p(X(t) - X(s))} \right) \right\}
\]
\[
\leq 2 \exp \left\{ -\varphi^* \left( \frac{\varepsilon}{\sigma(\rho(t, s))} \right) \right\}
\]
for all \( \varepsilon > 0 \). Hence the process \( X \) and thus \( X_f \), too, are continuous in probability. If a separable stochastic process defined on \((B, \rho)\) is continuous in probability, then every finite and everywhere dense set with respect to \( \rho \) is the set of separability for this
process. Thus $V = \bigcup_{k=1}^{\infty} V_{\varepsilon_k}$ is a set of $\rho$-separability of the process $X_f$. Moreover, with probability one,

$$Q(t) = \sup_{s \leq t, s \in B} (X_f(t) - X_f(s)) = \sup_{s \leq t, s \in V} (X_f(t) - X_f(s)).$$

Consider the mapping $\alpha_n = \{\alpha_n(s), n = 0, 1, \ldots\}$ of the set $V$ to $V_{\varepsilon_n}$ such that $\alpha_n(s)$ is a point of the set $V_{\varepsilon_n}$ for which $\rho(s, \alpha_n(s)) < \varepsilon_n$. If $s \in V_{\varepsilon_n}$, then $\alpha_n(s) = s$. If there are several points in the set $V_{\varepsilon_n}$ such that $\rho(s, \alpha_n(s)) < \varepsilon_n$, then we choose an arbitrary point among them and denote it by $\alpha_n(s)$.

Chebyshev’s inequality, Lemma 2.1, and assumption (7) imply that

$$P \left\{ |X(s) - X(\alpha_n(s))| > p^{n/2} \right\} \leq \frac{E(X(s) - X(\alpha_n(s)))^2}{p^n} \leq \frac{c^2 \sigma^2(X(s) - X(\alpha_n(s)))}{p^n} \leq \frac{c^2 \sigma^2(\varepsilon_n)}{p^n} = \frac{c^2 \beta^2 p^n}{\varphi(-1)(2)},$$

where

$$c = \frac{2e}{\varphi(-1)(2)}.$$

The latter inequality means that

$$\sum_{n=1}^{\infty} P \left\{ |X(s) - X(\alpha_n(s))| > p^{n/2} \right\} < \infty.$$

Now the Borel–Cantelli lemma implies that $X(s) - X(\alpha_n(s)) \to 0$ as $n \to \infty$ with probability one. Since the function $f$ is continuous,

$$X_f(s) - X_f(\alpha_n(s)) \to 0 \quad \text{as } n \to \infty$$

with probability one. The set $V$ is countable; hence $X(s) - X(\alpha_n(s)) \to 0$ with probability one as $n \to \infty$ uniformly in $s$.

Let $s$ be an arbitrary point of the set $V$. Let, for all $m \geq 1$,

$$s_m = \alpha_m(s), \quad s_{m-1} = \alpha_{m-1}(s_m), \quad \ldots, \quad s_1 = \alpha_1(s_2).$$

Then

$$X_f(s) = X_f(s_1) + \sum_{k=2}^{m} (X_f(s_k) - X_f(s_{k-1})) + X_f(s) - X_f(\alpha_m(s))$$

for all $m \geq 2$ and similarly

$$-X_f(s) \leq - \min_{u \in V_{\varepsilon_1}} X_f(u) - \sum_{k=2}^{m} \min_{u \in V_{\varepsilon_{k}}} (X_f(u) - X_f(\alpha_{k-1}(u))) - X_f(s) + X_f(\alpha_m(s)).$$

Relations (8) and (13) imply that

$$Q(t) = \sup_{s \leq t, s \in V} (X_f(t) - X_f(s))$$

$$\leq \lim_{m \to \infty} \inf \left( X_f(t) - \min_{u \leq t, u \in V_{\varepsilon_1}} X_f(u) - \sum_{k=2}^{m} \min_{u \leq t, u \in V_{\varepsilon_{k}}} (X_f(u) - X_f(\alpha_{k-1}(u))) - X_f(s) + X_f(\alpha_m(s)) \right)$$

with probability one.
Now Hölder’s inequality, \([12], [14]\), and Fatou’s lemma yield

\[
\begin{align*}
\mathbb{E}\exp \{\lambda Q(t)\} &\leq \mathbb{E} \liminf_{m \to \infty} \exp \left\{ \lambda \left( X_f(t) - \min_{u \leq t, u \in V_{\epsilon_1}} X_f(u) - \sum_{k=2}^{m} \min_{u \leq t, u \in V_{\epsilon_k}} (X_f(u) - X_f(\alpha_{k-1}(u))) \right) \right\} \\
&\leq \liminf_{m \to \infty} \mathbb{E} \exp \left\{ \lambda \left( X_f(t) - \min_{u \leq t, u \in V_{\epsilon_1}} X_f(u) - \sum_{k=2}^{m} \min_{u \leq t, u \in V_{\epsilon_k}} (X_f(u) - X_f(\alpha_{k-1}(u))) \right) \right\} \\
&\leq \liminf_{m \to \infty} \left( \mathbb{E} \exp \left\{ q_1 \lambda \left( X_f(t) - \min_{u \leq t, u \in V_{\epsilon_1}} X_f(u) \right) \right\} \right)^{1/q_1} \times \prod_{k=2}^{m} \left( \mathbb{E} \exp \left\{ q_k \lambda \max_{u \leq t, u \in V_{\epsilon_k}} (X_f(\alpha_{k-1}(u)) - X(u)) \right\} \right)^{1/q_k} \\
&\leq \left( \mathbb{E} \exp \left\{ q_1 \lambda \left( X_f(t) - \min_{u \leq t, u \in V_{\epsilon_1}} X_f(u) \right) \right\} \right)^{1/q_1} \times \prod_{k=2}^{\infty} \left( \mathbb{E} \exp \left\{ q_k \lambda \max_{u \leq t, u \in V_{\epsilon_k}} (X_f(\alpha_{k-1}(u)) - X(u)) \right\} \right)^{1/q_k} \\
&= I_1 \cdot \prod_{k=2}^{\infty} I_k
\end{align*}
\]

for all \(\lambda > 0\).

Every term on the right hand side of (15) is considered separately. Theorem 2.1 and assumption (7) imply that

\[
\mathbb{E} \exp \{q_1 \lambda (X(t) - X(u))\} \leq \exp \{\varphi(q_1 \lambda \tau_{\varphi}(X(t) - X(u)))\} \leq \exp \{\varphi(q_1 \lambda \sigma(\rho(t, u)))\}.
\]

Then the condition \(|f(u) - f(w)| \leq \delta(\rho(u, w))\) yields

\[
(I_1)^{q_1} \leq \sum_{u \leq t, u \in V_{\epsilon_1}} \mathbb{E} \exp \{q_1 \lambda (X(t) - X(u))\} \exp \{-q_1 \lambda (f(t) - f(u))\} \\
\leq \sum_{u \leq t, u \in V_{\epsilon_1}} \exp \{\varphi(q_1 \lambda \sigma(\rho(t, u)))\} + q_1 \lambda \delta(\rho(t, u))
\]

(16)

\[
N_{\delta_{\epsilon_1}}(\epsilon_1) \leq \sum_{j=1}^{N_{\delta_{\epsilon_1}}(\epsilon_1)} \exp \{\varphi(q_1 \lambda \sigma(2j\epsilon_1)) + q_1 \lambda \delta(2j\epsilon_1)\}.
\]
On the other hand, the Hölder inequality and (4) imply that, for all \( w > 1 \),
(17)
\[
(I_1)^q_1 \leq \sum_{u \leq t, \ u \in V_1} \left( \mathbb{E} \exp \left\{ \psi(w_1 \lambda \gamma(t)) \right\} \right) \left( \mathbb{E} \left\{ \psi \left( \frac{\lambda \gamma(u)}{w} \right) \right\} \right) \leq \sum_{u \leq t, \ u \in V_1} \left[ \exp \left\{ -q_1 \lambda f(t) - f(u) \right\} \right]
\]
\[
\leq \sum_{u \leq t, \ u \in V_1} \left[ \exp \left\{ -q_1 \lambda f(t) + \frac{(w - 1)\psi \left( \frac{w_1 \lambda \gamma(u)}{w} \right) + q_1 \lambda f(u)}{w} \right\} \right]
\]
\[
\leq N_{B_t}(\varepsilon_1) \exp \left\{ \frac{\psi(w_1 \lambda \gamma(t))}{w} - q_1 \lambda f(t) \right\}
\]
\[
\times \exp \left\{ \max_{u \in B_t} \left( \frac{(w - 1)\psi \left( \frac{w_1 \lambda \gamma(u)}{w} \right)}{w} + q_1 \lambda f(u) \right) \right\}.
\]
Further, Theorem 2.1 and (7) imply that
\[
\mathbb{E} \exp \left\{ q_k \lambda (X(u) - X(\alpha_k - 1(u))) \right\} \leq \exp \left\{ \varphi(q_k \lambda \sigma(\varepsilon_k - 1)) \right\}.
\]
Then
(18)
\[
(I_k)^q_k \leq N_{B_t}(\varepsilon_k) \max_{u \leq t, \ u \in V_k} \exp \left\{ q_k \lambda [f(u) - f(\alpha_k - 1(u))] \right\}
\]
\[
\leq N_{B_t}(\varepsilon_k) \max_{u \leq t, \ u \in V_k} \exp \left\{ \varphi(q_k \lambda \sigma(\varepsilon_k - 1)) + q_k \lambda \delta(\rho(u, \alpha_k - 1(u))) \right\}
\]
\[
\leq N_{B_t}(\varepsilon_k) \exp \left\{ \varphi \left( q_k \lambda \rho^{-1} \right) + q_k \lambda \delta \left( \rho^{-1} \left( \beta \rho^{-1} \right) \right) \right\}.
\]
Inequalities (15)–(18) complete the proof of Lemma 3.1.

\begin{theorem}
Let all the assumptions of Lemma 3.1 hold. Then
(19)
\[
P \{ Q(t) > x \} \leq Z(p, t, x)
\]
for all \( p \in (0, 1) \) and \( x > 0 \), where
\[
Z(p, t, x) = \inf_{\lambda > 0} W(\lambda, t, p) \left( \prod_{k=2}^{\infty} \left( N_{B_k} \left( \sigma^{-1} \left( \beta \rho^{-1} \right) \right) \right)^{\frac{1}{n_k}} \right)
\]
\[
\times \exp \left\{ \sum_{k=2}^{\infty} \left( \frac{1}{q_k} \varphi \left( q_k \lambda \rho^{-1} \right) + \lambda \delta \left( \rho^{-1} \left( \beta \rho^{-1} \right) \right) \right) - \lambda x \right\}
\]
and where \( W(\lambda, t, p) \) is defined in equation (10).
\end{theorem}

\begin{proof}
The proof of Theorem 3.1 follows directly from Lemma 3.1 and Chebyshev’s inequality.
\end{proof}

Choosing a specific sequence \( q_k \) in Lemma 3.1, we obtain the following result.

\begin{lemma}
Let \( f = \{ f(t), t \in B \} \) be a continuous function such that
\[
|f(u) - f(w)| \leq \delta(\rho(u, w)),
\]

where \( \delta = \{ \delta(s), s > 0 \} \) is an increasing nonnegative function. Further let
\[
(20) \quad \int_0^\beta \zeta_{\varphi} \left( H_{B_t} \left( \sigma^{-1}(u) \right) \right) \, du < \infty.
\]
Then
\[
E \exp \{ \lambda Q(t) \} \leq W_1(\lambda, t, p) \times \exp \left\{ \varphi \left( \frac{\lambda \beta}{1-p} \right) p + \frac{2\lambda}{p(1-p)} \int_0^{\beta p^2} \zeta_{\varphi} \left( H_{B_t} \left( \sigma^{-1}(u) \right) \right) \, du \right\}
\]
\[
\times \exp \left\{ \lambda \sum_{k=1}^\infty \delta \left( \sigma^{-1} \left( \beta p^{k-1} \right) \right) \right\}
\]
for all \( \lambda > 0 \) and \( p \in (0, 1) \), where
\[
(21) \quad W_1(\lambda, t, p) = \inf_{v \geq \frac{1}{1-p}} \min \left\{ X_{v,\lambda,t,p}, Y_{v,\lambda,t,p} \right\}
\]
and
\[
X_{v,\lambda,t,p} = \left( \int_1^{N_{B_t}(\sigma^{-1}(\beta p))} \varphi \left( v \lambda \sigma \left( 2x\sigma^{-1}(\beta p) \right) \right) + v \lambda \delta \left( 2x\sigma^{-1}(\beta p) \right) \right) dx
\]
\[
\times \inf_{w>1} \exp \left\{ \psi(wv\lambda\gamma(t)) \right\}
\]
\[
\times \max_{u \in B_t} \left\{ \frac{(w-1)\psi \left( \frac{wv}{w-1} \lambda \gamma(u) \right) + \lambda f(u)}{wv} - \lambda f(t) \right\}
\]
\[
Y_{v,\lambda,t,p} = \left( N_{B_t}(\sigma^{-1}(\beta p)) \right)^\frac{1}{v}
\]
\[
\times \inf_{w>1} \exp \left\{ \psi(wv\lambda\gamma(t)) \right\}
\]
\[
\times \max_{u \in B_t} \left\{ \frac{(w-1)\psi \left( \frac{wv}{w-1} \lambda \gamma(u) \right) + \lambda f(u)}{wv} - \lambda f(t) \right\}
\]
\[
(22) \quad W_1(\lambda, t, p) = \inf_{v \geq \frac{1}{1-p}} \min \left\{ X_{v,\lambda,t,p}, Y_{v,\lambda,t,p} \right\}
\]
\[
(23) \quad E \exp \{ \lambda Q(t) \} \leq W(\lambda, t, p)
\]
\[
\times \exp \left\{ \lambda \sum_{k=2}^\infty \delta \left( \sigma^{-1} \left( \beta p^{k-1} \right) \right) \right\}
\]
\[
\times \exp \left\{ \sum_{k=2}^\infty H_{B_t}(\varepsilon_k) + \varphi \left( \lambda q_k \beta p^{k-1} \right) \right\}
\]
for all \( q_k, k \geq 1 \), such that
\[
(24) \quad q_k > 1, \quad k \geq 1, \quad \sum_{k=1}^\infty \frac{1}{q_k} \leq 1,
\]
and for all \( \lambda > 0 \), where \( \varepsilon_k = \sigma^{-1}(\beta p^k) \).

Let \( v \) be a number such that \( v \geq (1-p)^{-1} \) and put \( q_1 = v \). Note that \( q_1 > 1 \). Further let, for \( k = 2, 3, \ldots \),
\[
(25) \quad q_k = \frac{1}{\lambda \beta p^{k-1}} \varphi^{-1} \left( \varphi \left( \frac{\lambda \beta}{1-p} \right) + H_{B_t}(\varepsilon_k) \right).
\]
Then
\[
q_k > \frac{1}{p^{k-1}(1-p)} > 1, \quad k = 2, 3, \ldots
\]
Since
\[ \frac{1}{q_k} \leq \frac{\lambda \beta p^{k-1}}{\varphi(-1) \left( \varphi \left( \frac{\lambda \beta}{1-p} \right) \right)} = p^{k-1}(1 - p), \quad k \geq 2, \]
we have
\[ \sum_{k=1}^{\infty} \frac{1}{q_k} \leq \sum_{k=1}^{\infty} p^{k-1}(1 - p) = 1. \]
Thus conditions (24) hold.

Now we consider
\[ \tilde{Z} = \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k) + \varphi(\lambda q_k \beta p^{k-1})}{q_k}. \]
For the sequence \( q_k \) defined by (25), we get
\[
\begin{align*}
\tilde{Z} &= \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k) + \varphi(\lambda q_k \beta p^{k-1})}{q_k} \\
&= \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k) + \varphi(\lambda q_k \beta p^{k-1})}{q_k} + \varphi \left( \frac{\lambda \beta}{1-p} \right) \sum_{k=2}^{\infty} \frac{1}{q_k} \\
&\leq 2 \sum_{k=2}^{\infty} \frac{H_{B_t}(\varepsilon_k)}{q_k} + \varphi \left( \frac{\lambda \beta}{1-p} \right) \sum_{k=2}^{\infty} p^{k-1}(1 - p) \\
&= \varphi \left( \frac{\lambda \beta}{1-p} \right) p + 2\lambda \sum_{k=2}^{\infty} \zeta \varphi \left( H_{B_t} \left( \sigma^{(-1)}(\beta p^k) \right) \right) \beta p^{k-1}.
\end{align*}
\]
The function \( \varphi(x)/x \) increases for \( x > 0 \). Thus the function \( \zeta \varphi(x) = x/\varphi^{(-1)}(x) \) increases for \( x > 0 \), too. Then
\[ \int_{\beta p^{k+1}}^{\beta p^k} \zeta \varphi \left( H_{B_t} \left( \sigma^{(-1)}(u) \right) \right) du \geq \zeta \varphi \left( H_{B_t} \left( \sigma^{(-1)}(\beta p^k) \right) \right) \beta p^{k}(1 - p). \]

Relations (26) and (27) imply that
\[ \tilde{Z} \leq \varphi \left( \frac{\lambda \beta}{1-p} \right) p + \frac{2\lambda}{p(1-p)} \int_{0}^{\beta p^2} \zeta \varphi \left( H_{B_t} \left( \sigma^{(-1)}(u) \right) \right) du. \]
Therefore inequality (21) follows from (9) and (28). □

Choosing the sequence \( q_k \) defined in Lemma 3.2, we obtain the following result.

**Theorem 3.2.** Let a stochastic process \( X(t) = \{X(t), t \in B\} \) belong to the class \( V(\varphi, \psi) \) and let \( \varphi < \psi \). Assume that condition (7) holds. Let \( f = \{f(t), t \in B\} \) be a continuous function such that
\[ |f(u) - f(w)| \leq \delta(\rho(u, w)), \]
where \( \delta = \{\delta(s), s > 0\} \) is an increasing nonnegative function. Further let
\[ r = \{r(u): u \geq 1\} \]
be a continuous function such that \( r(1) = 0, r(u) > 0 \) for \( u > 1 \), and the function \( s(t) = r(\exp\{t\}), t \geq 0, \) is convex. If
\[ \int_{0}^{\beta} \frac{r \left( N_{B_t} \left( \sigma^{(-1)}(u) \right) \right)}{p^{(-1)} \left( H_{B_t} \left( \sigma^{(-1)}(u) \right) \right)} du < \infty, \]

for the process $Q(t)$ defined by (5), then
\[
P \{ Q(t) > x \} \leq Z_r(p, t, x)
\]
for all $p \in (0, 1)$ and $x > 0$, where
\[
Z_r(p, t, x) = \inf_{\lambda > 0} W_1(\lambda, t, p) \exp \left\{ p\varphi \left( \frac{\lambda \beta}{1 - p} \right) + \lambda \left( \sum_{k=2}^{\infty} \delta \left( \sigma^{(-1)} \left( \beta p^{k-1} \right) \right) - x \right) \right\} 
\times \left( r^{(-1)} \left( \frac{\lambda}{p(1-p)} \int_0^{\beta p} \frac{r \left( N_{B_t} \left( \sigma^{(-1)}(u) \right) \right)}{\varphi^{(-1)} \left( H_{B_t} \left( \sigma^{(-1)}(u) \right) \right)} \, du \right) \right)^2
\]
and where $W_1(\lambda, t, p)$ is defined by (22).

Theorem 3.2 follows from Lemmas 3.1 and 3.2. The proof is similar to that of Theorem 3.2 of the paper [12] or to that of Theorem 3.4 in [7].

Using the sequence $q_k = 1/((1 - p)p^{k-1})$ in inequalities (11) we get the following result. An advantage of this result is that its conditions are easier to check than those of Theorem 3.2.

**Theorem 3.3.** Let $f = \{ f(t), t \in B \}$ be a continuous function such that
\[
|f(u) - f(w)| \leq \delta(\rho(u, w)),
\]
where $\delta = \{ \delta(s), s > 0 \}$ is a nonnegative increasing function and let $r_1 = \{ r_1(u), u \geq 1 \}$ be a continuous function such that $r_1(u) > 0$ for $u > 1$ and $s(t) = r_1(\exp\{t\})$, $t \geq 0$, is a convex function. If
\[
(30) \int_0^{\beta} r_1 \left( N_{B_t} \left( \sigma^{(-1)}(u) \right) \right) \, du < \infty
\]
for the process $Q(t)$ defined by (5), then
\[
P \{ Q(t) > x \} \leq Z_{r_1}(p, t, x)
\]
for all $p \in (0, 1)$ and $x > 0$, where
\[
Z_{r_1}(p, t, x) = \inf_{\lambda > 0} W_2(\lambda, t, p) \exp \left\{ p\varphi \left( \frac{\lambda \beta}{1 - p} \right) + \lambda \left( \sum_{k=2}^{\infty} \delta \left( \sigma^{(-1)} \left( \beta p^{k-1} \right) \right) - x \right) \right\} 
\times \left( r^{(-1)} \left( \frac{1}{\beta p} \int_0^{\beta p} r_1 \left( N_{B_t} \left( \sigma^{(-1)}(u) \right) \right) \, du \right) \right)
\]
and
\[
(32) W_2(\lambda, t, p) = \min \{ X_{\lambda, t, p}; Y_{\lambda, t, p} \},
\]
\[
X_{\lambda, t, p} = \left( \int_1^{N_{B_t} \left( \sigma^{(-1)}(\beta p) \right) + 1} \varphi \left( \frac{\lambda \sigma \left( 2x \sigma^{(-1)}(\beta p) \right)}{1 - p} + \lambda \delta \left( 2x \sigma^{(-1)}(\beta p) \right) \right) \, dx \right)^{1-p},
\]
\[
Y_{\lambda, t, p} = \left( N_{B_t} \left( \sigma^{(-1)}(\beta p) \right) \right)^{1-p} \inf_{v > 1} \exp \left\{ \frac{(1-p)\psi \left( \frac{v \lambda \gamma(t)}{1-p} \right)}{v} \right\} - \lambda f(t) + \max_{u \in B_t} \left( \frac{(v-1)(1-p)\psi \left( \frac{v \lambda \gamma(t)}{(v-1)(1-p)} \right)}{v} + \lambda f(u) \right) \right\}.
\]

Theorem 3.2 follows from Lemma 3.1. The proof is similar to that of Theorem 3.1 of [12] or that of Theorem 3.5 of [7].
4. An estimate for the distribution of a stationary \( \varphi \)-sub-Gaussian storage process

Consider a stationary \( \varphi \)-sub-Gaussian stochastic process \( X = \{ X(t), t \in B \} \) defined on a compact set \( B \subset T \), where \( T \) is a set of parameters. Recall (see [13]) that a \( \varphi \)-sub-Gaussian process \( X \) is called stationary if

1) its norm \( \tau_{\varphi} (X(t)) = c_{\varphi} = \gamma \) is constant for all \( t, s \in B \), and if
2) \( \tau_{\varphi} (X(t) - X(s)) = \sigma_{\varphi} (t - s) \).

Assume that \( X \) satisfies conditions (7). Let a continuous function \( f = \{ f(t), t \in B \} \) be such that

\[
|f(u) - f(w)| \leq \delta (\rho(u, w)),
\]

where \( \delta = \{ \delta(s), s > 0 \} \) is an increasing nonnegative function.

Example 4.1. We apply Lemma 3.2 to \( X \). Then we estimate both terms on the right hand side of (22) defining the minimum of the function \( W_1(\lambda, t, p) \).

First we consider the case of

\[
v = \frac{w - 1}{\lambda \gamma w} \varphi^{-1} \left( \varphi \left( \frac{\lambda \gamma w}{(1 - p)(w - 1)} \right) + H_{B_i} \left( \sigma^{-1}(\beta p) \right) \right).
\]

Then

\[
v \geq \frac{w - 1}{\lambda \gamma w} \varphi^{-1} \left( \varphi \left( \frac{\lambda \gamma w}{(1 - p)(w - 1)} \right) \right) = \frac{1}{1 - p}.
\]

In this case,

\[
W_1(\lambda, t, p)
\]

\[
\leq \left( N_{B_i} \left( \sigma^{-1}(\beta p) \right) \right)^{\frac{1}{\beta}} \times \inf_{w > 1} \exp \left\{ \varphi \left[ \frac{w - 1}{\lambda \gamma w} \varphi^{-1} \left( \varphi \left( \frac{\lambda \gamma w}{(1 - p)(w - 1)} \right) + H_{B_i} \left( \sigma^{-1}(\beta p) \right) \right) \right] \right\}
\]

\[
= \left( N_{B_i} \left( \sigma^{-1}(\beta p) \right) \right)^{\frac{1}{\beta}} \times \inf_{w > 1} \exp \left\{ \varphi \left[ (w - 1) \varphi^{-1} \left( \varphi \left( \frac{\lambda \gamma w}{(1 - p)(w - 1)} \right) + H_{B_i} \left( \sigma^{-1}(\beta p) \right) \right) \right] \right\}
\]

Now let

\[
v = \frac{1}{\lambda} \varphi^{-1} \left( \varphi \left( \frac{\lambda}{1 - p} \right) + H_{B_i} \left( \sigma^{-1}(\beta p) \right) \right).
\]

Then

\[
v \geq \frac{1}{\lambda} \varphi^{-1} \left( \varphi \left( \frac{\lambda}{1 - p} \right) \right) = \frac{1}{1 - p},
\]
whence

\[ W_1(\lambda, t, p) \leq \left( \int_1^{N_{B_i}(\sigma(-1)(\beta p)) + 1} \exp \left\{ \frac{1}{\lambda} \varphi(-1) \left( \varphi \left( \frac{\lambda}{1-p} \right) + J(t, p) \lambda \sigma(K(x, p)) \right) + \frac{1}{\lambda} \varphi(-1) \left( \varphi \left( \frac{\lambda}{1-p} \right) + J(t, p) \lambda \delta(K(x, p)) \right) \right\} \, dx \right)^{\frac{1}{\beta}} \]

\[ = \left( \int_1^{N_{B_i}(\sigma(-1)(\beta p)) + 1} \exp \left\{ \varphi \left( \frac{\lambda}{1-p} \right) + J(t, p) \sigma(K(x, p)) \right\} + \varphi(-1) \left( \varphi \left( \frac{\lambda}{1-p} \right) + J(t, p) \delta(K(x, p)) \right) \, dx \right)^{\frac{1}{\beta}}, \]

where

\[ J(t, p) = H_{B_i} \left( \sigma(-1)(\beta p) \right), \]

\[ K(x, p) = 2x^2 \sigma(-1)(\beta p). \]

**Example 4.2.** Let \( X \) be a stationary sub-Gaussian stochastic process defined in the interval \( B = [a, b], -\infty < a < b < \infty \), and let \( \rho(u, v) = |u - v| \). Then

\[ \inf_{u \in B_i} \sup_{v \in B_i} \rho(u, v) = \frac{t - a}{2} \]

and

\[ \beta \leq \sigma \left( \frac{b - a}{2} \right). \]

Thus

\[ N_{B_i}(u) \leq \frac{t - a}{2u} + 1, \]

whence

\[ N_{B_i}(u) \leq \frac{t - a}{u} \]

if

\[ u \leq \frac{t - a}{2}. \]

As in Example 4.1, we apply Lemma 3.2 to the process \( X \). Since \( \psi(u) = \varphi(u) = u^2/2 \), the expression in (22) simplifies:

\[ W_1(\lambda, t, p) = \inf_{v \geq \frac{1}{1-p}} \min \left\{ \left( \int_1^{M(t, p)} \exp \left\{ \frac{v^2 \lambda^2 \sigma^2(K(x, p))}{2} + v \lambda \delta(K(x, p)) \right\} \, dx \right)^{\frac{1}{\beta}}, \right. \]

\[ \left. (M(t, p) - 1)^{\frac{1}{\beta}} \inf_{w > 1} \exp \left\{ \frac{w \lambda^2 \gamma^2 w^2}{2(w - 1)} \right\} \right\} \]

\[ \leq \min \left\{ \left( \int_1^{M(t, p)} \exp \left\{ \frac{\lambda^2 \sigma^2(K(x, p))}{2(1-p)^2} + \frac{\lambda \delta(K(x, p))}{1-p} \right\} \, dx \right)^{1-p}, \right. \]

\[ \left. (M(t, p) - 1)^{2\lambda^2 \gamma^2} \right\}, \]
where
\[ M(t, p) = \frac{t - a}{\sigma(-1)(\beta p)} + 1 \]
and where \( K(x, p) \) is defined in Example 4.1. Moreover,
\[ \int_{0}^{3p^2} \zeta_{\varphi} \left( H_{B_{t}} \left( \sigma(-1)(u) \right) \right) du = \int_{0}^{3p^2} \left( 2 \ln \left( \frac{t - a}{\sigma(-1)(u)} \right) \right)^{\frac{1}{2}} du. \]

Therefore we proved the following result.

**Theorem 4.1.** Let \( X(t) \) be a stationary sub-Gaussian stochastic process defined on a finite interval and let condition (7) hold. Further let \( f = \{f(t), t \in B\} \) be a continuous function such that \(|f(u) - f(w)| \leq \delta(u, w)\), where \( \delta = \{\delta(s), s > 0\} \) is an increasing nonnegative function. If
\[ \int_{0}^{p} \left( \ln \left( \frac{t - a}{\sigma(-1)(u)} \right) \right)^{\frac{1}{2}} du < \infty \]
for all \( p \in (0, 1) \) and \( x > 0 \), then
\[ P \{Q(t) > x\} \leq \inf_{\lambda > 0} W_{3}(\lambda, t, p) \]
\[ = \exp \left\{ \frac{\lambda^{2} \beta^{2} p}{2(1 - p)^{2}} + \frac{\lambda \sqrt{2}}{p(1 - p)} \int_{0}^{p^{2}} \left( \ln \left( \frac{t - a}{\sigma(-1)(u)} \right) \right)^{\frac{1}{2}} du \right\} \times \exp \left\{ \lambda \sum_{k=1}^{\infty} \delta \left( \sigma\left( \frac{1}{\beta^{k-1}} \right) \right) - \lambda x \right\}, \]
where \( W_{3}(\lambda, t, p) = \min \{X_{\lambda, t, p}; Y_{\lambda, t, p}\} \) and where
\[ X_{\lambda, t, p} = \left( \int_{1}^{\frac{t - a}{\sigma(-1)(\beta p)}} \exp \left\{ \frac{\lambda^{2} \sigma^{2}(2x\sigma(-1)(\beta p))}{2(1 - p)^{2}} + \frac{\lambda \delta(2x\sigma(-1)(\beta p))}{1 - p} \right\} dx \right)^{1 - p}, \]
\[ Y_{\lambda, t, p} = \left( \int_{1}^{\frac{t - a}{\sigma(-1)(\beta p)}} \exp \left\{ \frac{\lambda^{2} \sigma^{2}(2x\sigma(-1)(\beta p))}{2(1 - p)^{2}} + \frac{\lambda \delta(2x\sigma(-1)(\beta p))}{1 - p} \right\} dx \right)^{1 - p}. \]

**Example 4.3.** Let a stochastic process \( X \) in Theorem 4.1 be such that \( \sigma(u) = u^{1/2} \) and let \( f(u) = \delta(u) = cu \), where \( c > 0 \) is a constant. Then
\[ W_{3}(\lambda, t, p) = \min \left\{ D(t, p); \left( \int_{1}^{\frac{t - a}{\beta^{2} p^{2}}} \exp \left\{ x \left( \frac{\lambda^{2} \beta p}{(1 - p)^{2}} + \frac{2 \lambda^{2} \beta^{2} p^{2}}{1 - p} \right) \right\} dx \right)^{1 - p} \right\} \]
\[ = \min \left\{ D(t, p); \left( \frac{(1 - p)^{2}}{\lambda \beta p(\lambda + 2 \beta p(1 - p))} \right)^{1 - p} \exp \left\{ \lambda \beta p \left( \frac{\lambda}{1 - p} + 2 \beta p \right) \right\} \times \left( \exp \left\{ \frac{t - a}{\beta p} \left( \frac{\lambda^{2}}{1 - p} + 2 \lambda \right) \right\} - 1 \right) \right\}, \]
where
\[ D(t, p) = \left( \frac{t - a}{\beta^{2} p^{2}} \right)^{2 \lambda^{2} \gamma^{2}}. \]

Changing the variables
\[ s = \left( \ln \frac{t - a}{u^{2}} \right)^{\frac{1}{2}}, \]
we obtain
\[
\int_0^{\beta p^2} \left( \ln \left( \frac{t-a}{\sigma(-1)(u)} \right) \right) \frac{1}{2} du
= \int_0^{\beta p^2} \left( \ln \left( \frac{t-a}{u^2} \right) \right) \frac{1}{2} du
= (t-a) \frac{1}{2} \int_0^\infty \left( \ln \left( \frac{t-a}{u^2} \right) \right) \frac{1}{2} s^2 \exp \left\{ -\frac{s^2}{2} \right\} ds
= (t-a) \frac{1}{2} \left( -s \exp \left\{ -\frac{s^2}{2} \right\} \right) \int_0^\infty \left( \ln \frac{t-a}{u^2} \right) \frac{1}{2} \exp \left\{ -\frac{s^2}{2} \right\} ds
= (t-a) \frac{1}{2} \left( \ln \frac{t-a}{\beta p^2} \right) \frac{1}{2} \exp \left\{ -\frac{1}{2} \ln \frac{t-a}{\beta p^2} \right\} + \frac{\sqrt{\pi}}{2} - \int_0^\infty \left( \ln \frac{t-a}{\beta p^2} \right) \frac{1}{2} \exp \left\{ -\frac{s^2}{2} \right\} ds
\leq (t-a) \frac{1}{2} \frac{\sqrt{\pi}}{2}.
\]

Finally
\[
\sum_{k=1}^\infty \delta \left( \sigma(-1)(\beta p^k) \right) = \sum_{k=1}^\infty c_\beta^2 p^{2(k-1)} = \frac{c_\beta^2}{1-\beta^2}.
\]

Therefore bound (34) becomes of the following form in the case of a stationary sub-Gaussian process:
\[
P \{ Q(t) > x \} \leq \inf_{\lambda > 0, \rho \in (0,1)} \min \{ X_{\lambda,p,t}, Y_{\lambda,p,t} \},
\]
\[
X_{\lambda,p,t} = \left( \frac{t-a}{\beta p^2} \right)^{2\lambda^2 p^2};
\]
\[
Y_{\lambda,p,t} = \left( \frac{(1-p)^2}{\lambda \beta p (\lambda + 2 \beta p (1-p))} \right)^{1-p} \exp \left\{ \lambda \beta p \left( \frac{\lambda}{1-p} + 2 \beta p \right) \right\} \times \exp \left\{ \frac{t-a}{\beta p} \left( \frac{\lambda^2}{1-p} + 2 \lambda \right) \right\} - 1 \times \exp \left\{ \frac{\lambda^2 \beta^2 p^2}{2(1-p)^2} + \frac{\lambda \sqrt{\pi}(t-a)^{1/2}}{\sqrt{2p(1-p)}} + \frac{\lambda c_\beta^2}{1-\beta^2} - \lambda x \right\}.
\]

**Concluding remarks**

We studied the properties of stochastic processes $X$ belonging to the class $V(\varphi, \psi)$ and obtained some bounds for the distribution of the storage process
\[
Q(t) = \sup_{s \leq t} (X(t) - X(s) - (f(t) - f(s)))
\]
generated by $X$, where a continuous increasing function $f(t)$ is the service output rate. The results of the paper can be applied to a wide class of stochastic processes, in particular to Gaussian processes. As an example, the case of stationary sub-Gaussian processes is considered in the paper.
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