

# IMPROVEMENT OF THE STABILITY OF SOLUTIONS OF AN INHOMOGENEOUS PERTURBED RENEWAL EQUATION ON THE SEMIAXIS

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**ABSTRACT.** We consider a generalized inhomogeneous continuous-time renewal equation on the semiaxis that reduces to the Volterra integral equation with a nonnegative bounded (or substochastic) kernel. It is assumed that the kernel can be approximated in a large time scale by a convolution kernel generated by a stochastic distribution. Under some asymptotic assumptions imposed on the perturbation we find an improved condition for the boundedness; the latter condition is used to prove the existence of the limit of a solution of the perturbed equation and to establish estimates for the deviation from a solution of a nonperturbed equation.

Some examples are discussed.

## 1. INTRODUCTION

Risk processes in an inhomogeneous environment are studied in [2]–[11].

We consider a time inhomogeneous generalization of the classical continuous-time renewal equation on the semiaxis that reduces to the Volterra integral equation with a nonnegative bounded (or substochastic) kernel. We assume that this kernel is approximated in variation by a convolution kernel generated by a stochastic distribution on the positive semiaxis.

The setting of the problem is motivated by some questions concerning the asymptotic behavior of the ruin function for the risk processes with varying premium rates discussed in [7], for instance.

The proofs below use some ideas of the the author's earlier papers [12, 13] and those of Schmidli [5]. The main results of the current paper generalize and improve to some extent the corresponding results presented in [10].

## 2. PERTURBED RENEWAL EQUATION

Consider the semiaxis  $\mathbb{R}_+ = [0, \infty)$  equipped with the  $\sigma$ -algebra  $\mathfrak{B}_+ = \mathfrak{B}(\mathbb{R}_+)$ . We introduce the following classes of functions:

$$B_0 \equiv \left\{ x: \mathbb{R}_+ \rightarrow \mathbb{R}, x \text{ is a Borel function such that } \sup_{s \leq t} |x(s)| < \infty \text{ for all } t \geq 0 \right\},$$

$$B_0^+ \equiv B_0 \cap \{x: \mathbb{R}_+ \rightarrow \mathbb{R}_+\},$$

$$L_1^0 \equiv \left\{ y \in B_0: \lim_{t \rightarrow \infty} y(t) = 0, \int_{[0, \infty)} |y(s)| ds < \infty \right\}.$$

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Then  $L_1^0$  is a Banach space with the norm

$$(1) \quad \|y\|_{01} = \|y\|_0 + \|y\|_1, \quad \|y\|_0 = \sup_{s \geq 0} |y(s)|, \quad \|y\|_1 = \int_{[0, \infty)} |y(s)| \, ds.$$

Let  $G$  be a stochastic measure in  $\mathfrak{B}_+$  such that

$$(2) \quad m \equiv \int_{[0, \infty)} s G(ds) < \infty, \quad m_2 \equiv \int_{[0, \infty)} s^2 G(ds).$$

The integral equation

$$(3) \quad x_0(t) = y(t) + \int_{[0, t)} x_0(t-s) G(ds), \quad t \geq 0,$$

with respect to an unknown function  $x_0$  of the class  $B_0$  is called the *renewal equation* on the semiaxis  $\mathbb{R}_+$  generated by the distribution  $G$ .

Note that equation (3) has a unique solution  $x_0 \in B_0$  for all functions  $y \in B_0$ . This solution is written as the *convolution*

$$(4) \quad x_0(t) = y * U(t) \equiv \int_{[0, t)} y(t-s) U(ds)$$

(see [13]–[16]). The  $\sigma$ -finite *renewal measure*  $U$  is defined by

$$(5) \quad U(B) = \sum_{n \geq 0} G^{*n}(B).$$

Recall that the measure  $G$  is of the *absolutely continuous type* if the convolution  $G^{*m}$  has an absolutely continuous component for some  $m \geq 1$  (see [14, 17]). The latter property means that the inequality  $G^{*m}(B) \geq \nu(B)$  holds for all Borel sets  $B$  and some nonnegative nonzero absolutely continuous measure  $\nu$ .

Stone [18] proves the following representation:

$$(6) \quad U = m^{-1}L + V$$

for an absolutely continuous  $G$ , where  $L$  is the Lebesgue measure on  $\mathfrak{B}_+$  and  $V$  a signed measure that is finite if  $m_2 < \infty$ .

Let  $(F(t, B), t \geq 0, B \in \mathfrak{B}_+)$  be a bounded kernel on  $(\mathbb{R}_+, \mathfrak{B}_+)$  and let  $x \in B_0$ . Consider the Volterra linear operator acting from  $B_0$  to  $B_0$ ,

$$F[x](t) \equiv \int_0^t F(t, ds)x(t-s), \quad t \in \mathbb{R}_+.$$

**Definition 1.** The generalized Volterra integral equation

$$(7) \quad x(t) = y(t) + F[x](t), \quad t \in \mathbb{R}_+,$$

with respect to an unknown function  $x \in B_0$  and a given function  $y \in B_0$  is called the inhomogeneous perturbation of the renewal equation (3).

We assume that the kernel  $F$  is close to the measure  $G$  in variation. This means that the perturbation

$$(8) \quad \Delta(t, B) \equiv F(t, B) - G(B)$$

is uniformly small with respect to  $B \in \mathfrak{B}_+ \cap [0, t]$  in the scheme of series as  $t \rightarrow \infty$ . In other words, the perturbation function

$$(9) \quad \delta(t) \equiv |\Delta|(t, [0, t]) = \sup_{|x(\cdot)| \leq 1} \left| \int_0^t \Delta(t, ds)x(s) \right| \rightarrow 0, \quad t \rightarrow \infty,$$

is small for the norms defined in (1).

To obtain a better estimate of the stability of equation (3) we introduce the following indices:

$$(10) \quad \varepsilon_L^\pm = m^{-1} \sup_{t \geq 0} \sup_{x(\cdot) \in [0,1]} \left( \int_0^t ds \int_0^s \Delta(s, du)x(s-u) \right)^\pm,$$

$$(11) \quad \varepsilon_L = m^{-1} \sup_{t \geq 0} \sup_{x(\cdot) \in [-1,1]} \left| \int_0^t ds \int_0^s \Delta(s, du)x(s-u) \right| \leq \varepsilon_L^+ + \varepsilon_L^-,$$

$$(12) \quad \delta_V(t) \equiv |V| * \delta(t), \quad \varepsilon_V = \sup_{t \geq 0} \delta_V(t).$$

Here and in what follows,  $|V|$  denotes the full variation of the signed measure  $V$ .

*Remark 1.* Definitions (11) and (12) together with (1) and (9) imply that  $\varepsilon_L \leq m^{-1} \|\delta\|_1$  and  $\varepsilon_V = O(\|\delta\|_0)$  if  $m_2 < \infty$ . The first of these inequalities is essential in the case of the perturbation  $\Delta$ . Moreover, in view of (6),

$$(13) \quad \varepsilon_V + \varepsilon_L \leq \|\delta\|_G \equiv \sup_{t \geq 0} U * \delta(t)$$

(see [10, equality (17)]).

**Theorem 1.** *Let a stochastic measure  $G$  be of the absolutely continuous type and  $m < \infty$ . We further assume that perturbation (8) satisfies the condition*

$$(14) \quad \varepsilon_V + \varepsilon_L < 1$$

and that  $y \in L_1^0$ . Then

(a) Equation (7) has a unique solution  $x \in B_0$ . Moreover,

$$(15) \quad \sup_{t \geq 0} |x(t)| \leq (1 - \varepsilon_V - \varepsilon_L)^{-1} \sup_{t \geq 0} |x_0(t)| < \infty,$$

$$(16) \quad \sup_{t \geq 0} |x(t) - x_0(t)| \leq (\varepsilon_V + \varepsilon_L) \sup_{t \geq 0} |x(t)|,$$

$$(17) \quad \overline{\lim}_{t \rightarrow \infty} |x(t) - x_0(t)| \leq \left( \overline{\lim}_{t \rightarrow \infty} \delta_V(t) + \varepsilon_L \right) \sup_{t \geq 0} |x(t)|,$$

where  $x_0 \in B_0$  is a solution of equation (3).

(b) If  $x$  and  $x_0$  are nonnegative, then estimates (15)–(17) hold with  $\varepsilon_L^+$  and  $(\cdot)^+$  instead of  $\varepsilon_L$  and  $|\cdot|$ , respectively (the same substitution should be made in inequality (14)).

(c) If the kernel  $F$  is nonnegative and  $\delta \in L_1^0$ , then the limit  $\lim_{t \rightarrow \infty} x(t)$  exists.

Assume that the following  $\sigma$ -finite measures  $\Lambda_{0\pm}$  are majorized by some nonnegative measures  $\Lambda_\pm$  for all  $B \in \mathfrak{B}_+$ , namely

$$\Lambda_{0\pm}(B) \equiv m^{-1} \sup_{t \geq 0} \sup_{A \in B \cap \mathfrak{B}_+} \left( \int_0^t \Delta(s, s - A \cap [0, s]) ds \right)^\pm \leq \Lambda_\pm(B),$$

$$(18) \quad \Lambda(B) = \max(\Lambda^+, \Lambda^-)(B).$$

Note the following relationship between (10) and (18):

$$(19) \quad \Lambda_{0\pm}(\mathbb{R}_+) = \varepsilon_L^\pm.$$

Now we introduce

$$(20) \quad \begin{aligned} \delta_V(t) &\equiv |V| * \delta(t), & \varepsilon_V &= \sup_{t \geq 0} \delta_V(t), \\ \varepsilon_\Lambda^\pm &\equiv \int_0^\infty \delta_V(t) \Lambda_\pm(dt), & \varepsilon_\Lambda &= \max(\varepsilon_\Lambda^+, \varepsilon_\Lambda^-). \end{aligned}$$

The index  $\varepsilon_\Lambda$  is finite if  $\|\delta\|_0 < \infty$ ,  $m_2 < \infty$ , and  $\Lambda(\mathbb{R}_+) = O(\|\delta\|_1) < \infty$ . However one can expect that  $\varepsilon_\Lambda$  is finite (or even small) if  $\Lambda(\mathbb{R}_+) = \infty$  and  $\delta$  is integrable.

**Theorem 2.** *Let a stochastic measure  $G$  be of the absolutely continuous type and  $m_2 < \infty$ . Assume that its perturbation (8) satisfies the following condition:*

$$(21) \quad \varepsilon_V + \varepsilon_\Lambda < 1$$

and that  $y \in L_1^0$ . Then

(a) equation (7) has a unique solution  $x \in B_0$ . Moreover,

$$(22) \quad \sup_{t \geq 0} \exp(-\Lambda(t)) |x(t)| \leq (1 - \varepsilon_V - \varepsilon_\Lambda)^{-1} \sup_{t \geq 0} |x_0(t)| < \infty,$$

$$(23) \quad \sup_{t \geq 0} \exp(-\Lambda(t)) |x(t) - x_0(t)| \leq \varepsilon_V \sup_{t \geq 0} |x(t)| + \int_0^\infty |x_0(s)| \exp(-\Lambda(s)) d\Lambda(s),$$

$$(24) \quad \sup_{t \geq 0} |x(t) - x_0(t)| \leq (1 - \varepsilon_V - \varepsilon_\Lambda)^{-1} \sup_{t \geq 0} |U * \Delta[x_0](t)|,$$

$$(25) \quad \overline{\lim}_{t \rightarrow \infty} |x(t) - x_0(t)| \leq \varepsilon_L \exp(\varepsilon_L) (1 - \varepsilon_V - \varepsilon_\Lambda)^{-1} \sup_{t \geq 0} |x_0(t)|,$$

where  $x_0 \in B_0$  is a solution of equation (3);

(b) if  $x$  and  $x_0$  are nonnegative, then bounds (22), (23), and (25) hold with  $\Lambda_+(t)$ ,  $\varepsilon_\Lambda^+$ ,  $\varepsilon_L^+$ , and  $(\cdot)^+$  instead of  $\Lambda(t)$ ,  $\varepsilon_\Lambda$ ,  $\varepsilon_L$ , and  $|\cdot|$ , respectively. The same changes should be made in (21);

(c) if the kernel  $F$  is nonnegative and  $\delta \in L_1^0$ , then the limit  $\lim_{t \rightarrow \infty} x(t)$  exists.

*Remark 2.* In contrast to (15), (22) is asymptotically bounded in the case of  $m_2 < \infty$  if  $\|\delta\|_0 = o(1)$  and  $\|\delta\|_1 = O(1)$  (instead of  $\|\delta\|_1 = o(1)$ ) in the scheme of series. This follows from the estimates

$$(26) \quad \varepsilon_V \leq \text{var } V \|\delta\|_0, \quad \varepsilon_\Lambda \leq \text{var } V \|\delta\|_0 (1 + \|\delta\|_1).$$

### 3. ABSOLUTELY CONTINUOUS PERTURBATION

Consider the case where perturbation (8) is absolutely continuous with respect to the initial distribution  $G$ ,

$$(27) \quad \Delta(t, B) = \int_B \gamma(t, s) G(ds), \quad B \in \mathfrak{B}_+,$$

with the locally bounded density  $\gamma(t, s)$ .

Now we introduce locally bounded Borel functions and the following indices:

$$(28) \quad \lambda_t(s) \equiv m^{-1} \int_0^{t-s} \gamma(s+u, u) G(du), \quad 0 \leq s \leq t, \quad \delta(t) = \int_0^t |\gamma(t, s)| G(ds),$$

$$\lambda_\pm(s) \geq \sup_{t \geq s} (\lambda_t(s))^\pm, \quad \lambda(s) = \max(\lambda_+(s), \lambda_-(s)), \quad s \geq 0,$$

$$(29) \quad \Lambda_\pm(t) \equiv \int_0^t \lambda_\pm(s) ds, \quad \Lambda(t) \equiv \int_0^t \lambda(s) ds,$$

$$(30) \quad \delta_V(t) \equiv |V| * \delta(t), \quad \varepsilon_V \equiv \sup_{t \geq 0} \delta_V(t),$$

$$(31) \quad \varepsilon_\Lambda^\pm \equiv \int_0^\infty \delta_V(t) \lambda_\pm(t) dt, \quad \varepsilon_\Lambda = \max(\varepsilon_\Lambda^+, \varepsilon_\Lambda^-).$$

**Theorem 3.** *Let a stochastic measure  $G$  be of the absolutely continuous type and*

$$m_2 < \infty.$$

Assume that representation (27) holds, perturbation function (28) is such that

$$\varepsilon_V + \varepsilon_\Lambda < 1,$$

and  $y \in L_1^0$ . Then:

- (a) Equation (7) has a unique solution  $x \in B_0$ . Moreover, inequalities (22), (23), (24), and (25) hold with indices (28), (29), (31) and (10), where  $x_0 \in B_0$  is a solution of equation (3).
- (b) If  $x$  and  $x_0$  are nonnegative, then estimates (22), (23), and (25) hold with  $\Lambda_+(t)$ ,  $\varepsilon_\Lambda$ ,  $\varepsilon_L$ , and  $(\cdot)^+$  instead of  $\Lambda(t)$ ,  $\varepsilon_\Lambda^+$ ,  $\varepsilon_L^+$ , and  $|\cdot|$ , respectively. The same changes should be made in (21).
- (c) If the kernel  $F$  is nonnegative (that is, if  $\gamma(t, s) \geq -1$ ) and  $\delta \in L_1^0$ , then the limit  $\lim_{t \rightarrow \infty} x(t)$  exists.

**Example 1.** Consider the nonperturbed uniform distribution

$$G(ds) = a^{-1} 1_{s \in [0, a]} ds$$

and related perturbation (27) with the density

$$(32) \quad \gamma(t, s) = \varepsilon(1+t)^{-1/2} (-1)^{[s(t-s+1)]} 1_{0 \leq s \leq \min(t, a)} ds,$$

where  $[x]$  is the integer part of a real number  $x$ . Note that  $|\gamma| \leq 1$  for  $\varepsilon \in (0, 1)$ ; that is, the corresponding perturbed kernel  $F$  is nonnegative.

Definition (32) implies that the perturbation function

$$\delta(t) = \varepsilon \min(1, t/a) (1+t)^{-1/2}$$

does not belong to  $L_1(\mathbb{R}_+)$ , whence  $\|\delta\|_1 = \infty$  and  $\|\delta\|_G = \infty$ . Thus the estimate for the stability obtained in [10] does not apply for such a perturbation. On the other hand,  $\|\delta\|_0 = \varepsilon \rightarrow 0$  in the scheme of series.

On the other hand, the functions

$$\lambda_t(s) = 2a^{-2} \int_0^{\min(t-s, a)} \varepsilon(1+u+s)^{-1/2} (-1)^{[u(s+1)]} du$$

coincide with the partial sums of sign alternating series with monotone terms, since the absolute value of the function under the sign of the integral is monotone and since the points where the sign changes are equidistributed by definition (28). Since the sums mentioned above do not exceed their first terms,

$$|\lambda_t(s)| \leq \lambda(s) \equiv 2\varepsilon a^{-2} (1+s)^{-3/2} \in L_1(\mathbb{R}_+), \quad \forall t \geq s.$$

Therefore the estimate of the stability obtained in [10] does not apply for this example, while Theorem 1 does apply.

Further, under the assumptions of Theorems 1 and 2,

$$\varepsilon_L = \|\lambda\|_1 = O(\varepsilon), \quad \varepsilon_V \leq \text{var} V \|\delta\|_0 = O(\varepsilon), \quad \varepsilon_\Lambda \leq \|\lambda\|_1 \varepsilon_V = O(\varepsilon^2)$$

as  $\varepsilon \rightarrow 0$ . The latter properties exhibit the advantage of the estimate of stability given in Theorem 2.

#### 4. AN APPLICATION TO THE RISK PROCESS

Consider an application of Theorem 3 for a model of the risk process. This model is nonstandard to some extent, since it is based on a renewal process as a sequence of independent claims, while the ruin is neither related to the process of premiums nor to the balance between claims and premiums. Rather the ruin is related to the ruin probabilities at sequential moments of claims. These moments depend on the current capital of the insurance company and on amounts to be paid according to the claims.

Consider an insurance company with an initial capital  $t > 0$  and assume that amounts  $(\xi_k, k \geq 1)$  to be paid according to the claims are independent random variables that have a common distribution  $G$  of the absolutely continuous type whose moments are given by (2). Put  $S_n = \xi_1 + \dots + \xi_n, S_0 = 0$ .

After a  $k$ -th claim, a ruin occurs if  $S_k \geq t$ , while if  $S_k < t$ , then an unexpected ruin occurs with probability  $\pi(t - S_k, \xi_k)$  that depends on the current capital available at this moment and on the amount to be paid according to the  $k$ -th claim. In the second case, the company continues functioning with probability  $1 - \pi(t - S_k, \xi_k)$ .

Note that the event of an unexpected ruin is independent of the sequence of amounts to be paid according to the claims given the current capital of the company and the amount to be paid according to the corresponding claim.

We denote by  $\nu(t)$  the number of claims up to the unexpected ruin. Then

$$P(\nu(t) = n \mid S_1, \dots, S_n) = \left[ \prod_{k=1}^{n-1} (1 - \pi(t - S_k, \xi_k)) \right] \pi(t - S_n, \xi_n), \quad n \geq 1,$$

if  $S_n < t$ .

Our aim is to establish the distribution function  $x_b(t)$  of the rest of the capital at a moment of an unexpected ruin and the probability  $z(t)$  of the regular ruin, namely

$$x_b(t) = P(t - b \leq S_{\nu(t)} < t) = P(0 < t - S_{\nu(t)} \leq b),$$

$$z(t) = P\left( \bigcup_{n < \nu(t)} \{S_n \geq t\} \right).$$

These two probabilities are solutions of the inhomogeneous renewal equations

$$(33) \quad x_b(t) = \int_{(t-b)^+}^t \pi(t - s, s) G(ds) + \int_0^t (1 - \pi(t - s, s)) x_b(t - s) G(ds),$$

$$(34) \quad z(t) = 1 - G(t) + \int_0^t (1 - \pi(t - s, s)) z(t - s) G(ds).$$

The renewal kernel in (7) related to equations (33) and (34) is equal to

$$(35) \quad F(t, ds) = (1 - \pi(t - s, s)) G(ds) 1_{s < t}.$$

Assume that

$$(36) \quad \theta(s) \equiv \lim_{t \rightarrow \infty} \pi(t - s, s) \in (0, 1) \quad \text{exists and is such that} \quad \sup_{s \geq 0} \theta(s) < 1.$$

Thus the kernel (35) satisfies condition (8) with the limit measure

$$\int_B (1 - \theta(s)) G(ds)$$

(as  $t \rightarrow \infty$ ).

Assume the following Cramér condition:

$$(37) \quad \exists \delta > 0: \quad \widehat{G}(\delta) \equiv \mathbf{E} \exp(\delta \xi_1) < \infty.$$

We further assume that the Cramér–Lundberg index  $\alpha > 0$  exists. This index is defined as a unique solution of the equation

$$(38) \quad \int_0^\infty \exp(\alpha s) (1 - \theta(s)) G(ds) = 1.$$

Consider the probability measure

$$(39) \quad G_\alpha(B) \equiv \int_B \exp(\alpha s) (1 - \theta(s)) G(ds)$$

and assume that the moments

$$m_\alpha \equiv \int_0^\infty s G_\alpha(ds), \quad m_{2\alpha} \equiv \int_0^\infty s^2 G_\alpha(ds)$$

are finite.

Equality (38) holds and the above two moments are finite if, for example, there exists a number  $\delta > 0$  such that

$$(1 - \sup \theta(s))\widehat{G}(\delta) \in (1, \infty).$$

Consider the perturbation function

$$(40) \quad \delta_\alpha(t) \equiv \int_0^t |\theta(s) - \pi(t-s, s)| \exp(\alpha s) G(ds)$$

and the Borel functions

$$(41) \quad \begin{aligned} \lambda_{\alpha\pm}(s) &\equiv m_\alpha^{-1} \int_0^\infty (\theta(u) - \pi(s, u))^\pm \exp(\alpha u) G(du), \\ \lambda_\alpha(s) &= \max(\lambda_{\alpha+}(s), \lambda_{\alpha-}(s)). \end{aligned}$$

**Theorem 4.** *Let a stochastic measure  $G$  be of the absolutely continuous type and conditions (36) and (37) hold. Further let the constant  $\alpha > 0$  be defined by (38) and let  $x_b$  and  $z$  be solutions of equations (33) and (34), respectively. Then:*

(a) *If  $\lambda_\alpha \in L_1(\mathbb{R}_+)$ , then there are some constants  $C_i = C_i(\pi, G)$ ,  $i = 0, 1$ , such that*

$$(42) \quad \overline{\lim}_{t \rightarrow \infty} |\exp(\alpha t)x_b(t) - l_{b\alpha}| \leq C_1 \|\lambda_\alpha\|_1,$$

$$(43) \quad \overline{\lim}_{t \rightarrow \infty} |\exp(\alpha t)z(t) - l_\alpha| \leq C_1 \|\lambda_\alpha\|_1$$

*if  $C_0 \|\delta_\alpha\|_0 < 1$ .*

(b) *If  $\lambda_{\alpha+} \in L_1(\mathbb{R}_+)$ , then there are some constants  $C_i = C_i(\pi, G)$ ,  $i = 0, 1$ , such that*

$$(44) \quad \overline{\lim}_{t \rightarrow \infty} (\exp(\alpha t)x_b(t) - l_{b\alpha}) \leq C_1 \|\lambda_{\alpha+}\|_1,$$

$$(45) \quad \overline{\lim}_{t \rightarrow \infty} (\exp(\alpha t)z(t) - l_\alpha) \leq C_1 \|\lambda_{\alpha+}\|_1$$

*if  $C_0 \|\delta_\alpha\|_0 < 1$ .*

(c) *If  $\lambda_\alpha \in L_1(\mathbb{R}_+)$ , then the limits*

$$\lim_{t \rightarrow \infty} \exp(\alpha t)x_b(t), \quad \lim_{t \rightarrow \infty} \exp(\alpha t)z(t)$$

*exist and are finite.*

Here the limit constants equal

$$(46) \quad l_{b\alpha} \equiv m_\alpha^{-1} \int_0^\infty \exp(\alpha t) \int_{(t-b)^+}^t \pi(t-s, s) G(ds) dt,$$

$$(47) \quad l_\alpha \equiv m_\alpha^{-1} \int_0^\infty \exp(\alpha t)(1 - G(t)) dt.$$

**Remark 3.** If  $\theta(s) \leq \pi(t, s)$  for  $t \geq s \geq 0$ , then the right hand sides of (44) and (45) are equal to zero.

**Example 2.** Let  $\pi(t, s) \equiv \pi(s)$ . Then  $\theta(s) = \pi(s)$ ,  $\delta_\alpha = \lambda_\alpha = 0$ , and thus the limit  $\lim_{t \rightarrow \infty} \exp(\alpha t)x_b(t)$  exists and is equal to  $l_{b\alpha}$ .

**Example 3.** Let  $\pi(t, s) \equiv \pi(t)$ . Then  $\theta(s) \equiv \theta = \lim_{t \rightarrow \infty} \pi(t)$  and  $\alpha > 0$  equals the classical Lundberg index. The condition for the existence of the limit

$$\lim_{t \rightarrow \infty} \exp(\alpha t)x_b(t)$$

reduces to  $\pi(t) - \theta \in L_1^0$  (a similar condition is used in [10]). However, the right hand side of inequality (44) (in contrast to [10]) is of order  $O(\|(\theta - \pi)^+\|_1)$ .

### 5. PROOFS

Recall that condition  $m_2 < \infty$  implies that the measure  $V = U - m^{-1}L$  is finite (see [17]–[19]).

Using (6), inequality (13) follows from the following bounds for the indices defined by (10) and (11):

$$(48) \quad \begin{aligned} \sup_{s \leq t} m^{-1}L * \Delta[x](t) &\leq \varepsilon_L^+ \sup_{s \leq t} x(s) \quad \text{for } x \geq 0, \\ \sup_{s \leq t} |m^{-1}L * \Delta[x](t)| &\leq \varepsilon_L \sup_{s \leq t} |x(s)|, \end{aligned}$$

and from

$$(49) \quad |\Delta[x](t)| \leq \delta(t) \sup_{s \leq t} |x(s)|,$$

$$(50) \quad |V * \Delta[x](t)| \leq \sup_{s \leq t} |V| * \delta(s) \sup_{s \leq t} |x(s)| \leq \varepsilon_V \sup_{s \leq t} |x(s)|$$

by definition (9).

Inequalities (48) follow from  $L * \Delta[x](t) = \int_0^t \Delta[x](s) ds$ , since the left hand side is linear with respect to  $x$ .

*Proof of Theorem 1.* Let  $x \in B_0$  be a solution of equation (7). Similarly to the method in [10], we use definition (8) and rewrite equation (7) in the form of the perturbed renewal equation

$$(51) \quad x(t) = y(t) + \Delta[x](t) + x * G(t).$$

The first two functions on the right hand side of (51) belong to the class  $B_0$ . Now we derive the equations

$$(52) \quad x(t) = U * (y(t) + \Delta[x](t)),$$

$$(52) \quad x(t) = x_0(t) + V * \Delta[x](t) + m^{-1}L * \Delta[x](t)$$

from (4) and (6). Taking into account (48), (50), (10), (12), and substituting  $t = s$  in (52) we get

$$(53) \quad \sup_{s \leq t} |x(s)| \leq \sup_{s \leq t} |x_0(s)| + (\varepsilon_V + \varepsilon_L) \sup_{s \leq t} |x(s)|,$$

whence

$$\sup_{s \leq t} |x(s)| \leq (1 - \varepsilon_V - \varepsilon_L)^{-1} \sup_{s \leq t} |x_0(s)|,$$

which proves (15).

To prove the uniqueness of a solution  $x \in B_0$  of equation (7), note that the difference  $z$  of two arbitrary solutions satisfies (7) with the function  $y = 0$ ; hence  $x_0 = 0$ , since equation (3) has a unique solution. Then we derive from (15) that  $z = 0$ .



If  $x$  and  $x_0$  are nonnegative, we apply analogs of estimates (53) without the sign of absolute values as well as estimates (48) and (50) for (52) and obtain the following inequality:

$$\sup_{s \leq t} x(s) \leq \sup_{s \leq t} x_0(s) + (\varepsilon_V + \varepsilon_L^+) \sup_{s \leq t} x(s),$$

whence we get (15) in the case under consideration.

Further, equality (52) implies that

$$(54) \quad x - x_0 = V * \Delta[x] + m^{-1}L * \Delta[x],$$

whence we establish (16) similarly to (53).

Applying the semi-additive transform  $()^+$  to equation (54) we get

$$(x - x_0)^+ \leq |V * \Delta[x]| + (m^{-1}L * \Delta[x])^+.$$

Now we derive inequality (16) for nonnegative  $x$  by using an appropriate case of inequality (48).

Finally, passing to the limit in (54) as  $t \rightarrow \infty$  and using (49) and (48), we prove the inequality

$$(55) \quad \overline{\lim}_{t \rightarrow \infty} |x(t) - x_0(t)| \leq \overline{\lim}_{t \rightarrow \infty} (\delta_V(t) + \varepsilon_L) \sup_{s \leq t} |x(s)|,$$

whence (17) follows. The proof for nonnegative  $x$  is similar.

The boundedness of  $x$  follows from the first statement of the theorem, since by definition of the function  $x_0 \in B_0$  it has a finite limit in view of the renewal theorem and according to the inclusion  $y \in L_1^0$ . Thus the limit of  $x$  exists by Theorem 1 in [10].  $\square$

**Lemma 1** (Gronwall). *Let  $z, w \in B_0$  and let  $\Lambda$  be a nondecreasing function on  $\mathbb{R}_+$ . If*

$$(56) \quad z(t) \leq w(t) + \int_0^t z(s) d\Lambda(s), \quad t \geq 0,$$

then, for all  $t \geq 0$ ,

$$(57) \quad \exp(-\Lambda(t))z(t) \leq \exp(-\Lambda(t))w(t) + \int_0^t \exp(-\Lambda(s))w(s) d\Lambda(s) \leq \sup_{s \leq t} w(s).$$

*Proof of Lemma 1.* Put  $\Psi(t) \equiv \exp(-\Lambda(t)) \int_0^t z(s) d\Lambda(s)$ . Then

$$d\Psi(t) = \left( z(t) - \int_0^t z(s) d\Lambda(s) \right) \exp(-\Lambda(t)) d\Lambda(t) \equiv W(t) \exp(-\Lambda(t)) d\Lambda(t),$$

where  $W(t) \leq w(t)$ ,  $t \geq 0$ . Integrating both sides of the latter equality and multiplying by  $\exp(\Lambda(t))$  we get

$$\int_0^t z(s) d\Lambda(s) = \exp(\Lambda(t))\Psi(t) = \exp(\Lambda(t)) \int_0^t W(s) \exp(-\Lambda(s)) d\Lambda(s).$$

Substituting this equality into (56) and applying the inequality  $W \leq w$  we prove the first inequality in (57). The second inequality follows from the first one by changing  $w$  on the right hand side by the corresponding upper bound and by evaluating the integral.  $\square$

*Proof of Theorem 2.* The proof of equality (19) reduces to an application of the inequality  $0 \leq x \leq 1_A$  for  $x \in [0, 1]$  in

$$(58) \quad L * \Delta[1_A](t) = \int_0^t ds \int_0^s \Delta(s, du) 1_A(s-u) = \int_0^t \Delta(s, s - A \cap [0, s]) ds$$

with  $A = \{x > 0\}$ .

Using the convexity, equality (58) implies that

$$(m^{-1}L * \Delta[x](t))^{\pm} \leq \int_0^t x(s) \Lambda_{\pm}(ds)$$

for all  $x \in B_0$ , and

$$(59) \quad |m^{-1}L * \Delta[x](t)| \leq \int_0^t |x(s)| \Lambda(ds)$$

for all  $t \geq 0$  and all  $x \in B_0$ .

It follows from equality (52) and inequality (59) that

$$(60) \quad |x(t)| \leq |x_0(t)| + |V * \Delta[x](t)| + \int_0^t |x(s)| d\Lambda(s).$$

Now lower and upper bounds (57) of Lemma 1 imply that

$$\begin{aligned} \exp(-\Lambda(t))|x(t)| &\leq \sup_{s \leq t} |x_0(s)| + \exp(-\Lambda(t)) |V * \Delta[x](t)| \\ &\quad + \int_0^t \exp(-\Lambda(s)) |V * \Delta[x](s)| d\Lambda(s). \end{aligned}$$

Since  $\Lambda(t)$  is monotone,

$$\exp(-\Lambda(t)) |V * \Delta[x](t)| \leq |V| * \delta(t) \sup_{s \leq t} \exp(-\Lambda(s)) |x(s)|$$

by (50). The preceding inequality and (20) imply that

$$(61) \quad \exp(-\Lambda(t))|x(t)| \leq \sup_{s \leq t} |x_0(s)| + (\varepsilon_V + \varepsilon_{\Lambda}) \sup_{s \leq t} \exp(-\Lambda(s)) |x(s)|.$$

Considering  $\limsup$  over  $t \leq T$  in (61) and passing to the limit as  $T \rightarrow \infty$  we derive inequality (22). The uniqueness of  $x \in B_0$  follows from (22) in the same way as in the proof of Theorem 1.

A similar method derives inequality (22) from the bound

$$x^+(t) \leq x_0^+(t) + \varepsilon_V \sup_{s \leq t} \exp(-\Lambda(s))x^+(s) + \int_0^t x^+(s) d\Lambda_+(s)$$

for nonnegative  $x$  and  $x_0$ . The latter bound is obtained by changing the sign of the absolute value  $|\cdot|$  in (60) by the sign of the positive part  $(\cdot)^+$  and by definition (18).

To prove (23), (24), and (25) put

$$(62) \quad h(t) = x(t) - x_0(t), \quad h_0(t) = U * \Delta[x_0](t).$$

Since (52) is linear,

$$(63) \quad h(t) = V * \Delta[x](t) + m^{-1}L * \Delta[x_0](t) + m^{-1}L * \Delta[h](t),$$

whence

$$(64) \quad |h(t)| \leq \delta_V(t) \sup_{s \leq t} |x(s)| + \int_0^t |x_0(s)| d\Lambda(s) + \int_0^t |h(s)| d\Lambda(s)$$

by (49) and (59). Now we apply the upper bound in (57) and prove (23).

The proof of (23) for nonnegative  $x$  and  $x_0$  is analogous and uses (18).

Further, the linearity of (52) implies that

$$(65) \quad h(t) = h_0(t) + V * \Delta[h](t) + m^{-1}L * \Delta[h](t).$$

Then (49) and (59) establish the inequality

$$(66) \quad |h(t)| \leq |h_0(t)| + |V * \Delta[h](t)| + \int_0^t |h(s)| d\Lambda(s).$$

This result implies (24) in the same way as that used to establish (61) from (60) (one needs to change  $x$  and  $x_0$  by  $h$  and  $h_0$ , respectively).

To prove (25), we use inequality (55), where  $\overline{\lim}_{t \rightarrow \infty} \delta_V(t) = 0$  in view of condition (9), since the measure  $V$  is finite (this property follows from  $m_2 < \infty$ ). Taking into account bound (22),

$$(67) \quad \overline{\lim}_{t \rightarrow \infty} |x(t) - x_0(t)| \leq \varepsilon_L \sup_{s \leq t} |x(s)| \leq \varepsilon_L \exp(\Lambda(\infty))(1 - \varepsilon_V - \varepsilon_\Lambda)^{-1} \sup_{s \leq t} |x_0(s)|,$$

whence the result follows by (18) and (19).

In the case of nonnegative  $x$  and  $x_0$ , one should change  $\varepsilon_L$ ,  $\Lambda(\infty)$ , and  $\varepsilon_\Lambda$  by  $\varepsilon_L^+$ ,  $\Lambda_+(\infty)$ , and  $\varepsilon_\Lambda^+$  respectively, and use definition (10) and the appropriate case of inequality (22).

The boundedness of  $x$  follows from the first statement of the theorem, since the function  $x_0 \in B_0$  has a finite limit according to the renewal theorem and in view of the inclusion  $y \in L_1^0$ . Thus the limit of  $x$  exists by Theorem 1 of [10].  $\square$

*Proof of Theorem 3.* Consider the function defined by (18) and use (27):

$$(68) \quad \begin{aligned} L * \Delta[x](t) &= \int_0^t \int_0^{t-s} \Delta(t-s, du)x(t-s-u) ds = \int_0^t \int_0^s \Delta(s, du)x(s-u) ds \\ &= \int_0^t \int_0^s \gamma(s, u)G(du)x(s-u) ds = \int_0^t G(du) \int_u^t \gamma(s, u)x(s-u) ds \\ &= \int_0^t G(du) \int_0^{t-u} \gamma(s+u, u)x(s) ds \\ &= \int_0^t \int_0^{t-s} \gamma(s+u, u)G(du)x(s) ds = m \int_0^t \lambda_t(s)x(s) ds. \end{aligned}$$

Substituting  $x = 1_A$  and using (58) and (29) we obtain an estimate for the measures (18) in Theorem 2:

$$(69) \quad \begin{aligned} \Lambda_{0\pm}(B) &= \sup_{t \geq 0} \sup_{A \in B \cap \mathfrak{B}_+} \left( \int_0^t \lambda_t(s)1_A(s)ds \right)^\pm \leq \int_B \lambda_\pm(s)ds = \Lambda_\pm(B), \\ \Lambda(B) &= \max(\Lambda_+, \Lambda_-)(B) = \int_B \lambda(s)ds. \end{aligned}$$

If  $x$  is nonnegative, we pass to the limsup over  $t$  on the right hand side of (68), estimate  $\lambda_t(s)$  by its limsup and prove the second statement of the theorem.

All other parameters and conditions of Theorems 3 and 2 coincide. Thus Theorem 3 follows from Theorem 2.  $\square$

*Proof of Theorem 4.* Introduce the functions

$$(70) \quad \begin{aligned} x_{b\alpha}(t) &\equiv \exp(\alpha t)x_b(t), & y_{b\alpha}(t) &\equiv \exp(\alpha t) \int_{(t-b)^+}^t \pi(t-s, s)G(ds), \\ z_\alpha(t) &\equiv \exp(\alpha t)z(t), & y_\alpha(t) &\equiv \exp(\alpha t)(1 - G(t)). \end{aligned}$$

Multiplying by  $\exp(\alpha t)$ , recalling the definition of  $G_\alpha$  in (39), and using Remark 1 of [10], equalities (33) and (34), and transformation (51) we derive the following equations:

$$(71) \quad \begin{aligned} x_{b\alpha}(t) &= y_{b\alpha}(t) + \Delta_\alpha[x_{b\alpha}](t) + G_\alpha * x_{b\alpha}(t), \\ z_\alpha(t) &= y_\alpha(t) + \Delta_\alpha[z_\alpha](t) + G_\alpha * z_\alpha(t), \end{aligned}$$

where the kernel  $\Delta_\alpha$  is of the form of the related perturbation (27) with the density  $\gamma_\alpha(t, s)$  with respect to the probability measure  $G_\alpha$  given by (39):

$$(72) \quad \Delta_\alpha(t, ds) = \exp(\alpha s)(\theta(s) - \pi(t - s, s))1_{s \leq t} G(ds) = \gamma_\alpha(t, s)1_{s \leq t} G_\alpha(ds),$$

$$(73) \quad \gamma_\alpha(t, s) \equiv (\theta(s) - \pi(t - s, s))/(1 - \theta(s)).$$

Recalling definition (41), the indices defined by (28)–(31) are of the form (40). Moreover,

$$(74) \quad \begin{aligned} \lambda_t(s) &= m_\alpha^{-1} \int_0^{t-s} (\theta(u) - \pi(s, u)) \exp(\alpha u) G(du), \\ \lambda(s) &= \max(\lambda_{\alpha+}(s), \lambda_{\alpha-}(s)) = \lambda_\alpha(s), \\ \delta_{V_\alpha}(t) &= V_\alpha * \delta_\alpha(t), \quad \varepsilon_{V_\alpha} = \sup_{t \geq 0} \delta_{V_\alpha}(t), \\ \varepsilon_{\Lambda_\alpha}^\pm &= \int_0^\infty \delta_{V_\alpha}(s) \lambda_{\alpha\pm}(s) ds, \quad \varepsilon_{\Lambda_\alpha} = \int_0^\infty \delta_{V_\alpha}(s) \lambda_\alpha(s) ds, \end{aligned}$$

where  $V_\alpha$  is the bounded charge in Stone’s decomposition (6) for the renewal measure

$$U_\alpha \equiv \sum_{n \geq 0} G_\alpha^{*n},$$

and where  $G_\alpha$  is the distribution of jumps (see (39)).

Condition (36) implies that

$$\int_0^\infty \exp(\alpha s) G(ds)$$

is finite. Applying the Lebesgue dominated convergence theorem to (40) we get  $\delta_\alpha(t) \rightarrow 0$  as  $t \rightarrow \infty$ . Further, Fubini’s theorem implies that

$$(75) \quad \|\delta_\alpha\|_1 = \int_0^\infty \delta_\alpha(s) ds \leq 2 \int_0^\infty \lambda_\alpha(s) ds = O(\|\lambda_\alpha\|_1) < \infty.$$

Thus  $\delta_\alpha \in L_1^0$ . Now the boundedness and (70) imply the inclusion  $y_{b\alpha} \in L_1^0$  for all  $b$ .

Since  $(G_\alpha)^{*m} \geq (1 - \sup_{s \geq 0} \theta(s))^m G^{*m}$ , the measure  $G_\alpha$  is of the absolutely continuous type.

We obtain from (74) and the boundedness of the measure  $V_\alpha$  that

$$(76) \quad \varepsilon_{V_\alpha} = O(\|\delta_\alpha\|_0), \quad \varepsilon_{\Lambda_\alpha} \leq \varepsilon_{V_\alpha} \int_0^\infty \lambda_\alpha(s) ds = O(\|\delta_\alpha\|_0), \quad \|\delta_\alpha\|_0 \rightarrow 0.$$

Thus the key condition of Theorem 3,

$$\varepsilon_V + \varepsilon_\Lambda \equiv \varepsilon_{V_\alpha} + \varepsilon_{\Lambda_\alpha} \leq \frac{1}{2} < 1,$$

holds for sufficiently small  $\|\delta_\alpha\|_0$ . At the same time, the function  $\Lambda(t) = \int_0^t \lambda_\alpha(s) ds$  is bounded in view of the inclusion  $\lambda_\alpha \in L_1(\mathbb{R}_+)$ .

To prove inequalities (42) and (43) we apply bound (25) under the conditions of part (a) of Theorem 3. By (69) and (19),

$$\varepsilon_L \leq \Lambda_+(\mathbb{R}_+) + \Lambda_-(\mathbb{R}_+) \leq 2 \int_0^\infty \lambda_\alpha(s) ds = O(\|\lambda_\alpha\|_1), \quad \|\lambda_\alpha\|_1 \rightarrow 0.$$

As shown above, the denominator in (25) is not less than 1/2. The last factor in (25) is finite, since  $(\alpha t)x_{0b\alpha}(t)$  is bounded. In turn, the latter property follows from Cramér–Lundberg’s theorem for the function  $x_{0b\alpha}$  in the nonperturbed renewal equation (71).

The proof of inequalities (44) and (45) is analogous to the part of the proof of Theorem 3 (b) where inequality (25) is applied. The nonnegativity of the functions  $x_{b\alpha}$ ,  $x_{0b\alpha}$ ,  $z_\alpha$ , and  $z_{0\alpha}$  is obvious.

The limit values (46) and (47) are obtained from

$$\lim_{t \rightarrow \infty} x_{0b\alpha}(t) = m_\alpha^{-1} \int_0^\infty y_{b\alpha}(t) dt = l_{b\alpha}, \quad \lim_{t \rightarrow \infty} z_{0\alpha}(t) = m_\alpha^{-1} \int_0^\infty y_\alpha(t) dt = l_\alpha.$$

Applying inequality (22) to equation (71) we prove the boundedness

$$\sup_{t \geq 0} x_{b\alpha}(t) = O(\|y_{b\alpha}\|_{01}) < \infty.$$

Thus the limit of  $x_{b\alpha}$  exists by Theorem 1 in [10].  $\square$

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