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THE STRUCTURE OF THE STOPPING REGION IN A LÉVY MODEL

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Abstract. The optimal stopping problem in a Lévy model is investigated. We show that the stopping region is nonempty for a wide class of models and payoff functions. In the general case, we establish sufficient conditions on the payoff function that provide nonemptiness of the stopping region. For a zero discounting rate we also give conditions for the stopping region to have a threshold structure.

1. Introduction

We consider a stochastic process $\{X_t, t \in [0,T]\}$ with independent increments and with an initial value $X_0 \in \mathbf{R} = (-\infty, \infty)$. This process is defined on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ equipped with the natural filtration $\mathcal{F}_t = \sigma\{X_s, s \le t\}$. The discounting rate is constant and equal to $q \ge 0$.

The optimal stopping problem for a payoff function g is to maximize the mean discounted payoff

$$\mathsf{E}(g(X_\tau)e^{-q\tau})$$

in the class M_T of all (F_t)-stopping times τ taking values in $[0,T]$. In other words, the problem is to find the cost function

$$V(T,x) = \sup_{\tau \in M_T} \mathsf{E}_x(g(X_\tau)e^{-q\tau}),$$

where E_x denotes the conditional expectation given $X_0 = x$.

A stopping time τ^* such that

$$V(T,x) = \mathsf{E}_x\left(g(X_{\tau^*})e^{-q\tau^*}\right)$$

is called the optimal stopping time.

According to the general American option pricing theory, the minimum optimal stopping time is given by $\tau^* = \inf\{t\colon (t, X_t) \in G\}$, where the set

$$G = \{(t,x) \in [0,T] \times \mathbf{R} \mid V(T-t,x) \le g(x)\} \tag{1}$$

is called the *optimal stopping region* for an option (see, for example, [1]). The complement of the set G,

$$C = \{(t,x) \in [0,T] \times \mathbf{R} \mid V(T-t,x) > g(x)\},$$

is called the *continuation region.* Define also the t-section of G as

$$G_t = \{x \in \mathbf{R} \mid V(T-t,x) \le g(x)\}.$$

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It is clear that

$$G = \bigcup_{t<T} \{t\} \times G_t.$$

For a numerical construction of an optimal stopping region, it is important to know apriori that it has certain properties, for example, it is nonempty or has a threshold structure, i.e., $G_t = [c(t), +\infty)$ or $G_t = (-\infty, c(t)]$.

The structure of stopping regions is studied in several papers. For example, Villeneuve [1] investigates nonemptiness of the optimal stopping region and its structure for an American option on several assets in a diffusion model. Lamberton and Mikou [2] study the behavior of the critical price and that of the stopping region boundary for an American put option in the exponential Lévy model if the underlying stock pays dividends. It is proved in the paper [3] that the stopping region has a threshold structure in the reselling problem for a European call option in discrete time. Conditions for the stopping region of an American option to have the threshold structure are found in [5, 6] for a discrete time model where the price process is modelled by a homogeneous Markov process.

We consider the optimal stopping problem in a Lévy model. We find conditions on the payoff function g implying that the stopping region is nonempty and conditions ensuring that the region has a threshold structure.

The paper is organized as follows. Section 2 contains some definitions and auxiliary results concerning Lévy processes. In Section 3, a theorem asserting nonemptiness of the stopping region for a Lévy process is proved. In Section 4, we find conditions on the payoff function under which the stopping region has the threshold structure.

2. Notation and auxiliary results

2.1. **Setting of the problem and assumptions.** Let $\{X_t, t \in [0, T]\}$ be a homogeneous process with independent increments defined on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a natural filtration $\mathcal{F}_t = \sigma\{X_s, s \le t\}$. The Lévy–Itô decomposition for the process X_t is given by

$$X_t = X_0 + at + \sigma W_t + \int_0^t \int_{\{|x|\le 1\}} x\,(\mu - \lambda \otimes \nu)(ds, dx) + \int_0^t \int_{\{|x|>1\}} x\,\mu(ds, dx), \tag{2}$$

where λ is Lebesgue measure, μ is Poisson measure on $\mathbf{R}_+ \times \mathbf{R}$,

$$\mu = \sum_{t\ge 0} \delta_{(t,\Delta X_t)} \mathbf{I}\{\Delta X_t \ne 0\},$$

δ_z is Dirac measure, ν is the Lévy measure of the process X_t, and where W_t is a standard Brownian motion. Let the Lévy measure of the process X_t be such that

(I) $\int_{\{|x|\ge 1\}} e^{p|x|}\,\nu(dx) < \infty$ for all $p \ge 0$.

2.2. **The optimal stopping problem if the interest rate is zero.** Assume that the discounting rate is zero, that is, $q = 0$.

For bounded functions $f \in C^2(\mathbf{R})$, we introduce the operator

$$\begin{aligned} Af(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{|y|\le 1} (f(x+y) - f(x) - yf'(x))\,\nu(dy) \\ + \int_{|y|>1} (f(x+y) - f(x))\,\nu(dy), \end{aligned} \tag{3}$$

called the generator of the process X. The action of the operator A for functions of the space $C(\mathbf{R})$ is understood in the sense of generalized functions.

Let $O \subset \mathbf{R}$. Put

$$\tau_O = \inf\{t \geq 0 \mid X_t \notin O\}.$$

Proposition 2.1. *Let a function $f \in C^2(\mathbf{R})$ be bounded and let the set $O \subset \mathbf{R}$ be open. Then the following two conditions are equivalent:*

1) *If $X_0 \in O$, then the process $\{f(X_{t\wedge\tau_O}), t \geq 0\}$ is a supermartingale.*
2) *$Af(x) \leq 0$ for all $x \in O$.*

Proposition 2.2. *Let $f \in C(\mathbf{R})$ and let the set $O \subset \mathbf{R}$ be open. Then the following two conditions are equivalent:*

1) *If $X_0 \in O$, then the process $\{f(X_{t\wedge\tau_O}), t \geq 0\}$ is a supermartingale.*
2) *The distribution Af is a nonnegative measure on O.*

In what follows we assume that the payoff function g has a subpolynomial growth, that is,

(II) $g \in C(\mathbf{R})$ and $|g(x)| \leq c(1 + |x|^\alpha)$ for some constants $c > 0$ and $\alpha > 0$.

By M_t, we denote the set of stopping times assuming values in $[0, t]$.

Proposition 2.3. *The function*

$$V(t, x) = \sup_{\tau\in M_t} \mathsf{E}_x(g(X_\tau)), \qquad (t, x) \in [0, \infty) \times \mathbf{R},$$

is continuous in $[0, \infty)\times\mathbf{R}$, and the process $\{V(T-t, X_t), 0 \leq t \leq T\}$ is the Snell envelope with horizon T for the process $\{g(X_t), 0 \leq t \leq T\}$, that is,

$$V(T - t, X_t) = \operatorname*{ess\,sup}_{\tau\in M_t} \mathsf{E}(g(X_\tau) \mid \mathcal{F}_t), \qquad t \in [0, T].$$

Proposition 2.4. *The function $U(t, x) = V(T - t, x)$ is a unique bounded continuous function defined in $[0, T) \times \mathbf{R}$ that satisfies the following conditions:*

1) *$U(T, \cdot) = g(\cdot)$;*
2) *$U \geq g$;*
3) *$\partial_t U + AU \leq 0$ in $(0, T) \times \mathbf{R}$;*
4) *$\partial_t U + AU = 0$ in the open set $\{(t, x) \in (0, T) \times \mathbf{R} \mid U(t, x) > g(x)\}$.*

The latter result is proved in [2] under the extra assumption that g is bounded. The latter assumption is used in [2] to prove that the expectation $\mathsf{E}[g(X_\tau) - g_n(X_\tau)]$ vanishes asymptotically as $n \to \infty$, where $g_n(x) = g(x)\phi_n(x)$, $\phi_n(x) = \phi(x/n)$, $\phi\colon \mathbf{R} \to [0, 1]$ is an infinitely differentiable finitely supported function which equals 1 throughout in $[-1, 1]$, and where τ is a bounded stopping time (without loss of generality one can assume that $\tau \leq T$).

Taking into account the growth assumption, $|g(x)| \leq c(1 + |x|^\alpha)$, we get

$$\begin{aligned} |\,\mathsf{E}[g(X_t) - g_n(X_t)]| &= |\,\mathsf{E}[g(X_t)(1 - \phi_n(X_t)]| \\ &\leq \mathsf{E}[|g(X_t)|\mathbf{I}\{|X_t| \geq n\}] \\ &\leq c\,\mathsf{E}\left(\left(1 + \sup_{t\leq T} |X_t|^\alpha\right) \mathbf{I}\left\{\sup_{t\leq T} |X_t| \geq n\right\}\right). \end{aligned}$$

This means that one only needs to prove that $\mathsf{E}\sup_{t\leq T} |X_t|^\alpha < \infty$. All the moments of the process X_t are finite by assumption (I) (see [7]). Now applying an estimate of [4, Lemma 1], we obtain

$$\mathsf{E}\sup_{t\leq T} |X_t|^\alpha \leq C_\alpha \left(|X_0|^\alpha + \mathsf{E}\sup_{t\leq T} |X_t - X_0|^\alpha\right) \leq K_\alpha(|X_0|^\alpha + T \vee T^\alpha)$$

and we are done. Moreover, the latter inequality yields the bound

$$|V(t,x)| \le C(1+|x|^\alpha) \tag{4}$$

uniformly in $t \in [0,T]$.

2.3. **The optimal stopping problem for a nonzero discounting rate.** We prove similar results for the case of $q > 0$. For $g \in C(\mathbf{R})$, introduce the operator

$$\begin{aligned} A_q f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{|y|\le 1} (f(x+y) - f(x) - yf'(x))\,\nu(dy) \\ + \int_{|y|>1} (f(x+y) - f(x))\,\nu(dy) - qf(x). \end{aligned}$$

Proposition 2.5. *Let a function $f \in C^2(\mathbf{R})$ be bounded and let a set $O \subset \mathbf{R}$ be open. Then the following two conditions are equivalent:*

1) *if $X_0 \in O$, then the process $\{e^{-qt\wedge\tau_O} f(X_{t\wedge\tau_O}), t \ge 0\}$ is a supermartingale;*
2) *$A_q f(x) \le 0$ for all $x \in O$.*

Proof. 1) $\Rightarrow$ 2). Fix $x \in O$. Since the process $\{e^{-qt\wedge\tau_O} f(X_{t\wedge\tau_O}), t \ge 0\}$ is a supermartingale,

$$\frac{1}{t}\,\mathsf{E}_x[e^{-qt\wedge\tau_O} f(X_{t\wedge\tau_O}) - f(x)] \le 0.$$

Applying the Itô formula, we conclude that

$$\frac{1}{t}\,\mathsf{E}_x\left[\int_0^{t\wedge\tau_O} A_q f(X_s)\,ds\right] \le 0. \tag{5}$$

Since X_t is a right continuous process, we get $\tau_O > 0$ almost surely. Letting $t \to 0$ in (5), we prove that $A_q f(x) \le 0$.

2) $\Rightarrow$ 1). We use the Itô formula for $s < t$:

$$\begin{aligned} \mathsf{E}[e^{-qt\wedge\tau_O} f(X_{t\wedge\tau_O}) \mid \mathcal{F}_s] &= \mathsf{E}\left[e^{-qs\wedge\tau_O} f(X_{s\wedge\tau_0}) + \int_{s\wedge\tau_O}^{t\wedge\tau_O} e^{-qu} A_q f(X_u)\,du \,\middle|\, \mathcal{F}_s\right] \\ &= e^{-qs\wedge\tau_O} f(X_{s\wedge\tau_0}) + \int_{s\wedge\tau_O}^{t\wedge\tau_O} \mathsf{E}[e^{-qu} A_q f(X_u) \mid \mathcal{F}_s]\,du \\ &\le e^{-qs\wedge\tau_O} f(X_{s\wedge\tau_O}), \end{aligned}$$

since $A_q f(X_u) \le 0$ almost surely for $u \in (s\wedge\tau_O, t\wedge\tau_O)$. □

The proof of the following results follows the lines of the proof of the corresponding results for the case of a zero discounting rate.

Proposition 2.6. *Let $f \in C(\mathbf{R})$ and let a set $O \subset \mathbf{R}$ be open. Then the following two conditions are equivalent:*

1) *if $X_0 \in O$, then the process $\{e^{-qt\wedge\tau_O} f(X_{t\wedge\tau_U}), t \ge 0\}$ is a supermartingale;*
2) *the distribution $A_q(f)$ is a nonnegative measure on O.*

Proposition 2.7. *The function*

$$V_q(t,x) = \sup_{\tau\in M_t} \mathsf{E}e^{-q\tau}(g(X_\tau)), \qquad (t,x) \in [0,\infty)\times\mathbf{R},$$

is continuous in $[0,\infty)\times\mathbf{R}$, and the process

$$\{V_q(T-t, X_t), 0 \le t \le T\}$$

is the Snell envelope with horizon T for the process $e^{-qt}(g(X_t))_{0\le t\le T}$, that is,

$$V_q(T-t, X_t) = \operatorname*{ess\,sup}_{\tau\in M_t} \mathsf{E}(e^{-q\tau} g(X_\tau) \mid \mathcal{F}_t), \qquad 0 \le t \le T.$$

Proposition 2.8. *The function $U_q(t,x) = V_q(T-t,x)$ is a unique continuous bounded function defined in $[0,T)\times\mathbf{R}$ that satisfies the following conditions:*
1) $U_q(T,\cdot) = e^{-qT}g$;
2) $U_q(t,x) \ge e^{-qt}g(x)$;
3) $\partial_t U_q + A_q U_q \le 0$ *in* $(0,T)\times\mathbf{R}$;
4) $\partial_t U_q + A_q U_q = 0$ *in the open set* $\{(t,x)\in(0,T)\times\mathbf{R} \mid U_q(t,x) > e^{-qt}g(x)\}$.

3. Nonemptiness of the stopping region for Lévy processes

In this section, we study conditions under which the stopping region defined by (1) is nonempty.

We assume that the process X and the payoff function g satisfy assumptions (I) and (II).

For an arbitrary set O, denote by O° and $\overline{O}$ its interior and closure, respectively. The action of the operator A is understood in the sense of generalized functions.

Theorem 3.1. *The operator A is such that*
1) $Ag \le 0$ *in the set* $\bigcup_{t<T} G_t^\circ$.
2) $Ag \ge 0$ *in the set* $\mathbf{R}\setminus\big(\bigcup_{t<T}\overline{G_t}\big)$.

Proof. 1. Let $t\in[0,T)$ be such that G_t° is nonempty (if such a number t does not exist, then the result is obvious). Clearly, the sets G_t increase with respect to t. Thus $(t,T)\times G_t^\circ \subset G$, whence $U = g$ in $(t,T)\times G_t^\circ$ and $Ag = \frac{\partial U}{\partial t} + AU$. Therefore, $Ag \le 0$ in G_t° by Proposition 2.4.

2. Let $H_t = \bigcup_{t<T}\overline{G}$ and let $\Lambda = \mathbf{R}\setminus H_T$. Note that $U > g$ and $U(\cdot,x)$ does not increase in the open set $(0,T)\times\Lambda$. Hence $\frac{\partial U}{\partial t} \le 0$, since $AU(t,\cdot)\ge 0$ in $(0,T)\times\Lambda$. This means that $\langle U(t,\cdot), A^*\theta\rangle = \langle AU(t,\cdot),\theta\rangle \ge 0$ for all finitely supported positive functions $\theta\in C^\infty(\mathbf{R})$ vanishing outside Λ. Since U is continuous, $U(t,x)\to g(x)$ as $t\to T-$. Using (I), we easily get that $|A^*\theta(x)| \le K_p e^{-p|x|}$ for all $p>0$ as θ has a finite support. On the other hand, $|U(t,x)| \le C(1+|x|^\alpha)$ in view of (4). Therefore, $\langle U(t,\cdot), A^*\theta\rangle \to \langle g, A^*\theta\rangle = \langle Ag,\theta\rangle$ as $t\to T$ by the dominated convergence theorem. Thus $\langle Ag,\theta\rangle \ge 0$ for all finitely supported positive functions $\theta\in C^\infty(\mathbf{R})$ vanishing outside Λ. This means that $Ag\ge 0$ in Λ. $\square$

In what follows we need the following auxiliary result.

Lemma 3.1. *If $Ag \ge 0$ (or, if $Ag = 0$) in $\mathbf{R}$, then the process $g(X_t)$ is a submartingale (or, a martingale).*

Proof. The proof follows directly from the Itô decomposition for the process $g(X_t)$. $\square$

First we state a result for the case of $q=0$.

Theorem 3.2. *Let $q=0$. The stopping region is empty if and only if Ag is a nonzero nonnegative measure on $\mathbf{R}$.*

Proof. a) Necessity. If the stopping region is empty, we have $Ag\ge 0$ according to Theorem 1. Assume that $Ag = 0$ in $\mathbf{R}$. Then, for all $x\in\mathbf{R}$, the process $g(X_t)$ is a martingale in view of Lemma 3.1. Further, the definition of the Snell envelope and the theorem on the optimal stopping time imply that $U(t,x) = g(x)$ for all $(t,x)\in[0,\infty)\times\mathbf{R}$. The latter result contradicts the assumption that G is empty.

b) Sufficiency. Let Ag be a nonzero nonnegative measure in $\mathbf{R}$, let there exist t such that $G_t\ne\varnothing$, and let $x\in G_t$. We have $U(t,x) = g(x) \ge \mathsf{E}\, g(X_\tau)$ for all $\tau\in M_{T-t}$. On the other hand, Lemma 3.1 and the the optimal stopping time theorem imply that $g(x)\le \mathsf{E}\, g(X_\tau)$ for all $\tau\in M_{T-t}$ as the process $g(X_\tau)$ is a martingale. Since the Snell envelope is a minimal supermartingale, we get $U(u, X_u) = g(X_u)$ almost surely. Using the continuity of $U(u, \cdot)$ and g, we prove that $U(u, y) = g(y)$ in $(0, T-t) \times \mathbf{R}$. Then $Ag \le 0$ in $\mathbf{R}$ by Theorem 3.1, whence $Ag = 0$ in $\mathbf{R}$, which is a contradiction. The theorem is proved. $\square$

The following results for nonzero interest rate can be proved similarly.

Theorem 3.3. *The operator A_q is such that*
1) $A_q g \le 0$ *in* $\bigcup_{t<T} G_t^\circ$*;*
2) $A_q g \ge 0$ *in the set* $\mathbf{R} \setminus \left(\bigcup_{t<T} \overline{G_t}\right)$.

Theorem 3.4. *The stopping region is empty if and only if $A_q g$ is a nonzero positive measure on* $\mathbf{R}$.

It is possible to prove that the stopping region is nonempty for a wide class of processes and payoff functions. Assume that a payoff function g satisfies assumption (II) and

(i) $g \in C^1(\mathbf{R})$;
(ii) there exists $\beta \in \mathbf{R}$ such that $g(x) > x^\beta$ for sufficiently large x (the case of $\beta < 0$ is also included);
(iii) $\lim_{x\to\infty} \sup_{|a|\le \ln x} g(x+a)/g(x) = 1$.

The latter two conditions do not restrict essentially the class of functions g. For example, all the functions $g(x) = |x|^\alpha L(x)$, where $L(x)$ is a slowly varying function at $+\infty$, satisfy these conditions.

We further assume that the process X is such that $\sigma = 0$ and $\nu(\mathbf{R}) < \infty$ in (2).

Theorem 3.5. *The stopping region is nonempty for a Lévy process if its Lévy measure is symmetric and finite, the process does not contain a Brownian component, and if the payoff function satisfies conditions (II) and (i)–(iii).*

Proof. Let $\sigma = 0$ in expansion (2) and let the Lévy measure ν be finite and symmetric. Then

$$\begin{aligned} Ag(x) &= ag'(x) + \int_{-\infty}^{\infty} (g(x+y) - g(x))\,\nu(dy) - qg(x) \\ &= ag'(x) - \frac{q}{2} g(x) + \mathsf{E}\left[g(x+\xi) - \left(1 + \frac{q}{2\nu(\mathbf{R})}\right) g(x)\right], \end{aligned}$$

where ξ is a random variable whose distribution is the same as that of a jump of the process X_t.

Put $d = q/(2\nu(\mathbf{R})) > 0$. We have

$$\mathsf{E}\, g(x+\xi) = \mathsf{E}\, g(x+\xi)\mathbf{I}\{|\xi| \le \ln x\} + \mathsf{E}\, g(x+\xi)\mathbf{I}\{|\xi| > \ln x\}.$$

Conditions (ii) and (iii) imply that

$$\mathsf{E}\, g(x+\xi)\mathbf{I}\{|\xi| \le \ln x\} \le (1 + d/3) g(x) \tag{6}$$

for sufficiently large x.

Next, condition (I) implies that $\mathsf{P}(|\xi| > A) \le c_p e^{-pA}$ for all $p > 2(\alpha+\beta)$, whence

$$\begin{aligned} \mathsf{E}\,|g(x+\xi)|\mathbf{I}\{|\xi| > \ln x\} &\le \left(\mathsf{E}(|g(x+\xi)|)^2\right)^{1/2} \mathsf{P}(|\xi| > \ln x)^{1/2} \\ &\le c_{p,\alpha} \left(\mathsf{E}\left(1 + |x|^{2\alpha} + |\xi|^{2\alpha}\right)\right)^{1/2} e^{-(p \ln x)/2} \\ &\le c_{p,\alpha} \left(1 + |x|^\alpha + \mathsf{E}\left(|\xi|^{2\alpha}\right)\right)^{1/2} x^{-p/2} \to 0. \end{aligned}$$

Since $p > 2(\alpha+\beta)$, we derive from (ii) that

$$\mathsf{E}\,|g(x+\xi)|\mathbf{I}\{|\xi| > \ln x\} \le dg(x)/3$$

for sufficiently large x. Combining this bound with (6), we obtain

$$\mathsf{E}\Big[g(x+\xi)-(1+d)g(x)\Big]<0.$$

If the stopping region is empty, then

$$ag'(x)-\frac{q}{2}g(x)>0$$

for all $x\ge x'$ and some $x'>0$. Without loss of generality one can assume that $g(x')>0$. It is clear that $a\ne 0$.

Let $a>0$. Then $g'(x)\ge kg(x)$ for $x\ge x'$, where $k=a/(2q)$. The comparison theorem for ordinary differential equations implies that $g(x)\ge g(x')e^{k(x-x')}$, $x\ge x'$, which contradicts condition (II).

If $a<0$, then $g'(x)\le kg(x)$ for $x>x'$, whence $g(x)\le g(x')e^{k(x-x')}$, which contradicts condition (i), since $k<0$ in this case. □

4. The threshold structure of the stopping region for Lévy processes if the interest rate is zero

Let the process X_t and the payoff function $g(x)$ satisfy conditions (I)–(II). Assume that $q=0$. We additionally assume that $g\in C^2(\mathbf{R})$ and $|g'(x)|\le c(1+|x|^\alpha)$.

The problem is to find conditions on g under which the region G_t has the threshold structure. In other words, we want to find conditions under which

$$G_t=[c(t),+\infty)\quad\text{or}\quad G_t=(-\infty,c(t)].$$

The first of these equalities is equivalent to the condition that, for all $x\in G_t$ and $y>x$, we have $y\in G_t$ or

$$\forall\tau\ge t\colon\quad \mathsf{E}[g(X_\tau)\mid X_0=x]\le g(x)\Longrightarrow \mathsf{E}[g(X_\tau)\mid X_0=y]\le g(y). \tag{7}$$

Theorem 4.1. *Assume that one of the following conditions holds.*

(i) *The jumps of the process X_t are bounded from below.*
(ii) *The measure ν is symmetric.*

Then

1) *if the payoff function g is such that $g''(x)\le 0$ for all $x\in\mathbf{R}$ and if $g''(x)$ is nonincreasing in x, then*
$$G_t=[c(t),\infty);$$
2) *if the payoff function g is such that $g''(x)\ge 0$ for all $x\in\mathbf{R}$ and if $g''(x)$ is nondecreasing in x, then*
$$G_t=(-\infty,c(t)].$$

Proof. 1. Without loss of generality one can assume that $t=0$. Consider the function

$$\gamma(x)=\mathsf{E}[g(X_\tau)-g(X_0)\mid X_0=x].$$

Since the function $\gamma(x)$ is nonincreasing, condition (7) holds. Applying the Itô formula,

$$\begin{aligned}\gamma(x)&=\mathsf{E}[g(X_\tau)-g(X_0)\mid X_0=x]\\&=\mathsf{E}\bigg[\int_0^\tau Ag(X_y)\,dy+\sigma\int_0^\tau g'(X_y)\,dW_y\\&\qquad+\int_s^t\int_{\mathbf{R}}(g(X_{y_-}+z)-g(X_{y_-}))\tilde\nu(dy,dz)\;\bigg|\;X_0=x\bigg]\\&=\mathsf{E}\bigg[\int_0^\tau Ag(X_y)\,dy\;\bigg|\;X_0=x\bigg].\end{aligned}\tag{8}$$

Since

$$
\begin{aligned}
Ag(x) &= ag'(x) + \frac{\sigma^2}{2}g''(x) + \int_{|y|\le 1} (g(x+y) - g(x) - yg'(x))\,\nu(dy) \\
&\quad + \int_{|y|>1} (g(x+y) - g(x))\,\nu(dy),
\end{aligned}
\tag{9}
$$

it is sufficient to show that the last two terms in (9) are nonincreasing under the assumptions of the theorem.

(i) Suppose the jumps of the process X_t are bounded from below. Without loss of generality one can assume that $\Delta X \ge -1$. Otherwise, one makes the change $x = x/c$ and obtains

$$g(\cdot) = g(c\,\cdot).$$

The Taylor formula with remainder written in the integral form implies that

$$
\begin{aligned}
g(x+y) - g(x) - yg'(x) &= \int_x^{x+y} g''(t)(x+y-t)\,dt = \int_0^y g''(t+x)(y-t)\,dt, \\
g(x+y) - g(x) &= \int_x^{x+y} g'(t)\,dt = \int_0^y g'(t+x)\,dt.
\end{aligned}
$$

Thus the last two terms in (8) are nonincreasing in x, since $g''(x) \le 0$ and $g''(x)$ is nonincreasing by the assumption of the theorem.

(ii) Now let the measure ν be symmetric. If $\Delta X < -1$, we rewrite the last term in (9) as

$$
\begin{aligned}
\int_{|y|>1} (g(x+y) - g(x))\,\nu(dy) &= \int_{-\infty}^{-1}\int_0^y g'(t+x)\,dt\,\nu(dy) \\
&= -\int_{\infty}^{1}\int_0^{-z} g'(t+x)\,dt\,\nu(dz) \\
&= -\int_1^{\infty}\int_{-z}^{0} g'(t+x)\,dt\,\nu(dz) \\
&= -\int_1^{\infty}\int_0^{z} g'(t+x)\,dt\,\nu(dz).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\int_{|y|\le 1} (g(x+y) - g(x) - yg'(x))\,\nu(dy) + \int_{|y|>1} (g(x+y) - g(x))\,\nu(dy) \\
&\quad = \int_1^{\infty}\int_0^{z} (g'(t+x) - g'(t+x-z))\,dt\,\nu(dz) \\
&\quad = \int_{-z}^{0} (g''(t+x) - g'(s+t+x))\,ds.
\end{aligned}
$$

Thus the last two terms in (9) are nonincreasing in x, since $g''(x) \le 0$ and $g''(x)$ is nonincreasing by the assumption of the theorem.

2. The proof is analogous to the preceding one. The difference is that one needs to use condition (7) for all $y < x$ and $x \in G_t$. Note that this condition holds, since the terms in (9) are nondecreasing. $\square$

Example 4.1. Let assumption (II) hold and let the price process

$$X_t = X_0 + at + \int_0^t\int_{\{|x|\le 1\}} x\,(\mu - \lambda\otimes\nu)(ds,dx) + \int_0^t\int_{\{|x|>1\}} x\,\mu(ds,dx)$$

be a Lévy process without Brownian component. We also assume that the Lévy measure ν is finite and symmetric. Let the payoff function be as follows:

$$g(x) = \begin{cases} k + (x+1)^2, & x \geq 0, \\ k - 3 + 8(2-x)^{-1}, & x < 0, \end{cases}$$

where $k \in \mathbf{R}$. Then the stopping region has the threshold structure.

Indeed, conditions (I) hold for the function $g(x)$. Theorem 3.4 implies that the stopping region is nonempty. Moreover, $g''(x) \geq 0$ and $g''(x)$ is nondecreasing for all $x \in \mathbf{R}$, thus the stopping region is of the form $G_t = (-\infty, c(t)]$ by Theorem 4.1.

5. Concluding remarks

We consider the problem of optimal stopping for an American type contingent claim which runs for perpetuity in the Lévy model. We study the conditions under which the payoff region is nonempty for the case of zero interest rate, or if the interest rate is constant, or if the payoff region is of the threshold structure and if the interest rate is zero.

Bibliography

1. S. Villeneuve, *Exercise regions of American options on several assets*, Finance Stoch. **3** (1999), no. 3, 295–322.
2. D. Lamberton and M. Mikou, *The critical price for the American put in an exponential Lévy model*, Finance Stoch. **12** (2008), no. 4, 561–581. MR2447412 (2009j:91100)
3. A. Kukush, Yu. Mishura, and G. Shevchenko, *On reselling of European option*, Theory Stoch. Process. **12(28)** (2006), no. 1–2, 75–87. MR2316567 (2008e:62171)
4. A. Moroz and G. Shevchenko, *Asymptotic behavior of the American type option prices in the Lévy model if the time interval is extending unboundedly*, Visn. Kyiv Univ. Mat. Mech. **24** (2010), 39–43. (Ukrainian)
5. H. Jönsson, A. G. Kukush, and D. S. Silvestrov, *Threshold structure of optimal stopping strategies for American type option. I*, Teor. Imovir. Matem. Statist. **71** (2004), 82–92; English transl. in Theor. Probability and Math. Statist. **71** (2005), 93–103. MR2144323 (2006h:91075)
6. H. Jönsson, A. G. Kukush, and D. S. Silvestrov, *Threshold structure of optimal stopping strategies for American type option. II*, Teor. Imovir. Matem. Statist. **72** (2005), 42–53; English transl. in Theor. Probability and Math. Statist. **72** (2006), 47–58. MR2168135 (2006i:62070)
7. A. Papapantoleon, *An Introduction to Lévy Processes with Applications in Finance*, Lecture notes, 2008; arXiv/0804.0482.
8. P. E. Protter, *Stochastic Integration and Differential Equations*, Springer-Verlag, Berlin–Heidelberg, 2004. MR2020294 (2005k:60008)

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