FUNCTIONAL LAW OF THE ITERATED LOGARITHM TYPE FOR A SKEW BROWNIAN MOTION

UDC 519.21

I. H. KRYKUN

Abstract. The functional law of the iterated logarithm is proved for a skew Brownian motion.

1. Introduction

The functional law of the iterated logarithm for the Wiener process was proved in a well-known paper by Strassen [13]. A modification of this result for more general normalizing functions was proposed by Bulinski˘ı [1]. A functional law of the iterated logarithm for solutions of Itô stochastic differential equations with a jump process was obtained by Makhno [11].

The skew Brownian process studied in this paper was introduced by Itô and McKean [9] in terms of elliptic differential operators of the first order according to the Feller classification of one-dimensional diffusion processes. The skew Brownian motion has been studied by many authors since then. Among those authors are, to mention a few, Harrison and Shepp [8] and Le Gall [10], who considered this process as a solution of a stochastic equation with local time. In [10], as well as in [4] and [7], some interrelations were proposed between the solutions of stochastic equations with local time and solutions of Itô’s equations.

The functional law of the iterated logarithm for a skew Brownian motion is studied in this paper. In doing so, we follow the approach of the paper [4].

The paper is organized as follows. Notation and the main results are given in Section 2. An auxiliary Theorem 2 is proved in Section 3. Section 4 is devoted to the proof of some lemmas and Theorem 1.

2. Main results

Consider a skew Brownian motion as a solution of the following stochastic differential equation with local time:

\[ \xi(t) = x + \beta L_\xi(t,0) + w(t), \quad t \in [0, 1]. \]
If $|\beta| \leq 1$, then equation (1) has a strong solution [8]. This means that there exists a continuous semimartingale $(\xi(t), \mathcal{F}_t)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a flow of $\sigma$-algebras $\mathcal{F}_t$, $t \in [0, 1]$, where a standard one-dimensional Wiener process $(w(t), \mathcal{F}_t)$ leaves, such that the symmetric local time

$$L^2(t, 0) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_0^t I(-\delta, \delta)(\xi(s)) \, ds$$

exists almost surely and equation (1) is satisfied almost surely.

In relation (2) and throughout this paper, $I_A(x)$ denotes the indicator of a set $A$. Let $R$ be the real line and let $\mathcal{B}(R)$ be the Borel $\sigma$-algebra in $R$. The space of continuous functions $f$ on $[0, 1]$ assuming values in $R$ is denoted by $C[0, 1]$. Let $\mathcal{B}(C[0, 1])$ be the Borel $\sigma$-algebra of $C[0, 1]$ and let the norm in $C[0, 1]$ be given by $\|x\| = \sup_{t \in [0, 1]} |x(t)|$.

In what follows we use the standard notation $\dot{f}$ for the density of an absolutely continuous function $f$, namely

$$f(t) = f(a) + \int_a^t \dot{f}(s) \, ds.$$

Further, let

$$H^2[0, 1] = \left\{ f : f(t) \text{ is absolutely continuous and such that } \int_0^1 |\dot{f}(t)|^2 \, dt < \infty \right\}.$$

Recall the following property of absolutely continuous functions (throughout this paper, the symbol Leb$(A)$ denotes the Lebesgue measure of a set $A$):

$$\text{Leb}\left\{ t \in [0, 1] : f(t) = 0, \dot{f}(t) \neq 0 \right\} = 0.$$

We put

$$\text{sgn } x = \begin{cases} -1, & \text{for } x < 0, \\ 0, & \text{for } x = 0, \\ 1, & \text{for } x > 0. \end{cases}$$

Let $(X, \mathcal{B}(X))$ be a metric space equipped with a metric $\rho$, where $\mathcal{B}(X)$ is the Borel $\sigma$-algebra in the space $X$. Let $I(x) : X \to [0, \infty]$ be a lower semicontinuous functional such that $\{x : I(x) \leq a\}$ is a compact set for all $a > 0$.

We say that a family of probability measures $\{\mu_\varepsilon\}$, $\varepsilon > 0$, defined on $X$ satisfies the large deviation principle with a normalizing coefficient $k(\varepsilon)$ such that $\lim_{\varepsilon \to 0} k(\varepsilon) = +\infty$ and with an action functional $I(x)$ if

a) for every open set $G \in \mathcal{B}(X)$,

$$\liminf_{\varepsilon \to 0} \frac{1}{k(\varepsilon)} \ln \mu_\varepsilon(G) \geq - \inf\{I(x), x \in G\};$$

b) for every closed set $F \in \mathcal{B}(X)$,

$$\limsup_{\varepsilon \to 0} \frac{1}{k(\varepsilon)} \ln \mu_\varepsilon(F) \leq - \inf\{I(x), x \in F\}.$$

Next we formulate the contraction principle (see [2] Theorem 5.3.1). Let measures $\{\mu_\varepsilon\}$ on $X$ be generated by some random elements $\{X_\varepsilon\}$. Assume that the family $\{\mu_\varepsilon\}$ satisfies the large deviation principle with an action functional $I(x)$. Further, let $F(x)$ be a continuous mapping acting from $X$ to $X'$. Then the family of measures $\{\mu_\varepsilon'\}$ on $X'$ generated by the random elements $\{F(X_\varepsilon)\}$ satisfy the large deviation principle with the action functional

$$I'(x) = \inf_{y : F(y) = x} \{I(y)\}.$$
Now we introduce the class $\Phi$ of increasing functions $\phi(T)$ such that
\[ \lim_{T \to \infty} \phi(T) = \infty, \quad \lim_{T \to \infty} \frac{\phi(T)}{\sqrt{T}} = 0. \]
Throughout the paper we use the notation $\psi(T) = \phi(T)\sqrt{T}$.
Consider the functional
\[ J^*(\phi, h, c) = \sum_{k=1}^{\infty} \exp \left\{ -h \phi^2 \left( c^k \right) \right\}, \quad c > 1. \]
Note that if $J^*(\phi, h, c_0) < \infty$ for some number $c_0 > 1$, then $J^*(\phi, h, c) < \infty$ for all $c > 1$.
Given $\phi \in \Phi$, let
\[ (4) \quad G^2(\phi) = \inf \{ h > 0 : J^*(\phi, h, c) < \infty \}. \]
We agree that $G^2(\phi) = \infty$ if there is no $h < \infty$ such that $J^*(\phi, h, c) < \infty$. In what follows, the numbers $G$, $G^2$, or $G^2(\cdot)$ are always defined according to relation (4).
Put
\[ (5) \quad Y(f) = \begin{cases} \frac{1}{2} \int_0^1 |\dot{f}(t)|^2 \, dt, & \text{if } f \in H^2[0, 1], \quad f(0) = 0, \\ +\infty, & \text{otherwise} \end{cases} \]
and
\[ \mathcal{F}_D = \left\{ h \in C[0, 1]: h(0) = 0; Y(h) \leq D^2 \right\}. \]
If $D^2 = \infty$, then $\mathcal{F}_\infty = \{ h \in C[0, 1]: h(0) = 0 \}$.
For an arbitrary $T > 0$, consider the following stochastic process:
\[ (6) \quad \xi_T(t) = \frac{\xi(Tt) - x}{\sqrt{T} \phi(T)} = \frac{\beta L^\xi(tT, 0) + w(tT)}{\sqrt{T} \phi(T)}. \]
The following is the main result of the paper.

**Theorem 1.** Let $|\beta| < 1$, $\phi \in \Phi$, and let $G$ be defined by (4). Then the set of cluster points of the family $\{\xi_T(t)\}$ for the almost sure convergence as $T \to \infty$ coincides in $C[0, 1]$ with $\mathcal{F}_G$.

3. Auxiliary results

The solution of equation (4) is closely related to the solution of the Itô stochastic differential equation. Put
\[ (7) \quad \kappa(x) = \begin{cases} (1 - \beta)x, & x \leq 0, \\ (1 + \beta)x, & x \geq 0 \end{cases} \]
and let
\[ \varphi(x) = \begin{cases} \frac{x}{1 - \beta}, & x \leq 0, \\ \frac{x}{1 + \beta}, & x \geq 0 \end{cases} \]
be the inverse function to $\kappa(x)$.
Consider the following Itô stochastic differential equation:
\[ (8) \quad \eta(t) = \varphi(x) + \int_0^t \frac{dw(s)}{1 + \beta \text{sgn} \eta(s)}, \quad t \in [0, 1]. \]
Note that the diffusion coefficient of this equation is a discontinuous function of bounded variation for which a unique strong solution of equation (8) exists according to a result from [12].
It is known that
\[
\eta(t) = \varphi(\xi(t)) \quad \text{or} \quad \xi(t) = \kappa(\eta(t))
\] (see [4]).

Now we consider the processes
\[
\eta_T(t) = \eta(Tt) - \varphi(x) = \frac{1}{\sqrt{T} \phi(T)} \int_0^{Tt} dw(s) \frac{1 + \beta \text{sgn}(s)}{1 + \beta \text{sgn}(s)}, \quad t \in [0, 1].
\]

Let
\[
L(f(s), \dot{f}(s)) = (1 + \beta \text{sgn}(s))^2 \dot{f}(s)^2
\]
and introduce the functional \(J(f)\) as follows:
\[
J(f) = \begin{cases} 
\frac{1}{2} \int_0^1 L(f(t), \dot{f}(t)) dt, & \text{if } f \in H^2[0, 1], \ f(0) = 0, \\
+\infty, & \text{otherwise}.
\end{cases}
\]

Further, let
\[
K_D = \left\{ f \in C[0, 1]: f(0) = 0; J(f) \leq \frac{D^2}{2} \right\}.
\]
If \(D^2 = \infty\), then \(K_{\infty} = \{ f \in C[0, 1]: f(0) = 0 \} \) and
\[
L(f(s), \dot{f}(s)) = \left( \frac{d\kappa(f(s))}{ds} \right)^2.
\]

Remark 1. It follows from relation (11) that \(Y(\kappa(f)) = J(f)\), whence \(\kappa(f) \in \mathcal{F}_D\) in view of \(f \in K_D\).

Lemma 1. Let \(|\beta| < 1\) and let the measures \(\{\nu_T\}\) be generated by the processes \(\{\eta_T(t)\}\). Then the family of measures \(\{\nu_T\}\) satisfies the large deviation principle in the space \((C[0, 1], \mathcal{B}(C[0, 1]))\) with the normalizing coefficient \(\phi^2(T)\) and action functional \(J(\phi)\).

Proof. Using relation (3), the proof follows from [6, Theorem B] with \(\varepsilon = 1/\phi(T)\), since the infimum in Theorem B is attained at either \(\rho = 0\) or \(\rho = 1\) (note that the infimum itself equals 0).

Lemma 1 is proved.

Consider the sequence of functions \(z_k(t) = \eta_{c^k}(t)\), that is,
\[
z_k(t) = \frac{1}{\psi(c^k)} \int_0^{c^k t} dw(s) \frac{1 + \beta \text{sgn}(s)}{1 + \beta \text{sgn}(s)}.
\]
Put
\[
u(t) = \int_0^t dw(s) \frac{1 + \beta \text{sgn}(s)}{1 + \beta \text{sgn}(s)}.
\]
Then
\[
z_k(t) = \frac{u(c^k t)}{\psi(c^k)}.
\]

Theorem 2. Let \(|\beta| < 1, \ \phi \in \Phi\), and let \(G\) be defined by equality (4). Then the set of cluster points of the family \(\{\eta_T(t)\}\) with respect to the almost sure convergence as \(T \to \infty\) coincides with \(K_G\) in \(C[0, 1]\).

Proof. The proof consists of the following three standard steps.
Step 1. First we prove that, for $G^2(\phi) < \infty$, for all $c > 1$, and for an arbitrary $\varepsilon > 0$, there exists a number $k_0$ such that

$$\rho(z_k, K_G) < \varepsilon$$

almost surely for all $k > k_0$. Note that $\{f : J(f) \leq a\}$ is a compact set in $C[0, 1]$ whatever a number $a < \infty$.

Put $N_\varepsilon = \{f : \rho(f, K_G) \geq \varepsilon\}$. Then there exists $\delta > 0$ such that

$$\inf_{f \in N_\varepsilon} J(f) \geq \frac{G^2(\phi)}{2} + \delta.$$

By Lemma [1], the family $\{\eta_T(t)\}$ satisfies the large deviation principle. Using property b) of the large deviation principle we get

$$\Pr\{z_k \in N_\varepsilon\} \leq \exp \left\{ -\phi^2(c^k) \left( \frac{G^2(\phi)}{2} + \delta \right) \right\}$$

for sufficiently large $k$. Then the definition of $G^2(\phi)$ and Borel–Cantelli lemma complete the proof of Step 1.

Step 2. We prove that every limit point of the family $\{\eta_T(t)\}$ almost surely belongs to $K_G$ if $G^2(\phi) < \infty$. This result is proved in Step 1 for $\{T\} = \{c^k\}$. Now let $T \in [c^k, c^{k+1}]$. Since the function $\psi(T)$ is non-decreasing with respect to $T$, we write

$$\frac{1}{\psi(T)} = \frac{\alpha(T, k)}{\psi(c^k)} + \frac{\beta(T, k)}{\psi(c^{k+1})},$$

where $\alpha(T, k) \geq 0$, $\beta(T, k) \geq 0$, and $\alpha(T, k) + \beta(T, k) = 1$. Put

$$\hat{\eta}_{T, k}(t) = \alpha(T, k)z_k(t) + \beta(T, k)z_{k+1}(t).$$

The desired result follows from the following bound: for every $\varepsilon > 0$, there exist two numbers $c_\varepsilon > 1$ and $k_0$ such that

$$\sup_{t \in [0, 1], T \in [c^k, c^{k+1}]} |\eta_T(t) - \hat{\eta}_{T, k}(t)| < \varepsilon$$

almost surely for all $k > k_0$ and $c \in (1, c_\varepsilon)$.

It follows from the definition of the family $\{\eta_T(t)\}$ and equality (13) that

$$\eta_T(t) = z_k \left( t \frac{T}{c^k} \right) \frac{\psi(c^k)}{\psi(T)} = \alpha(T, k)z_k \left( t \frac{T}{c^k} \right) + \beta(T, k)z_{k+1} \left( t \frac{T}{c^{k+1}} \right).$$

Note that $z_k, z_{k+1} \in \{f : \rho(f, K_G) < \delta\}$ for sufficiently large $k$ and for all $\delta$.

Then

$$|\eta_T(t) - \hat{\eta}_{T, k}(t)| \leq \alpha(T, k) \left| z_k(t) - z_k \left( t \frac{T}{c^k} \right) \right| + \beta(T, k) \left| z_{k+1}(t) - z_{k+1} \left( t \frac{T}{c^{k+1}} \right) \right|$$

and

$$\sup_{t \in [0, 1], T \in [c^k, c^{k+1}]} |\eta_T(t) - \hat{\eta}_{T, k}(t)| \leq \sup_{t \in [0, 1], s \in [t, ct]} |z_k(t) - z_k(s)| + \sup_{t \in [0, 1], s \in [t/c, t]} |z_{k+1}(t) - z_{k+1}(s)|.$$
This implies that
\[
P \left\{ \sup_{t \in [0,1], T \in [c^{k}, c^{k+1}]} |\eta_{T}(t) - \hat{\eta}_{T,k}(t)| \geq \varepsilon \right\} 
\]
\[
\leq P \left\{ \sup_{t \in [0,1], s \in [t, c^{k+1}]} |z_{k}(t) - z_{k}(s)| \geq \frac{\varepsilon}{2} \right\} 
+ P \left\{ \sup_{t \in [0,1], s \in [t, c^{k+1}]} |z_{k+1}(t) - z_{k+1}(s)| \geq \frac{\varepsilon}{2} \right\}.
\] (15)

To estimate the probabilities on the right hand side of (15) we apply Lemma 2 of [1]: there exists a constant $C$ such that
\[
P \left\{ \sup_{a \leq t, s \leq b, |t-s| \leq h} |w(s) - w(t)| > x \sqrt{h} \right\} \leq C(b-a) \exp \left\{ -\frac{x^2}{4} \right\}
\]
for all $0 \leq a < b < \infty$, $h \leq b-a$, and for an arbitrary $x > 0$.

Thus we get, for another Wiener process $\tilde{w}(t) = w(c^{k}t)/\sqrt{c^{k}}$, that
\[
P \left\{ \sup_{t \in [0,1], s \in [t, c^{k+1}]} |z_{k}(t) - z_{k}(s)| \geq \frac{\varepsilon}{2} \right\} 
= P \left\{ \sup_{t \in [0,1], s \in [t, c^{k+1}]} \left| \int_{t}^{s} \frac{d\tilde{w}(u)}{1 + \beta \text{sgn} \eta(c^{k}u)} \right| \geq \frac{\phi (c^{k}) \varepsilon}{2} \right\}.
\]

Next we make a random change of time. Consider the function
\[
\tau(u) = \int_{0}^{u} \frac{ds}{(1 + \beta \text{sgn} \eta(c^{k}s))^{2}}.
\]
Let $\gamma(u)$ be the inverse function to $\tau(u)$. It is clear that $\gamma(u)$ and $\tau(u)$ are increasing functions and that $\gamma(0) = \tau(0) = 0$. Moreover, the derivatives
\[
\tau'(u) = \frac{1}{(1 + \beta \text{sgn} \eta(c^{k}u))^{2}}, \quad \gamma'(u) = \frac{1}{\tau'(\gamma(u))} = \left(1 + \beta \text{sgn} \eta(c^{k} \gamma(u))\right)^{2}
\]
exist almost surely. Letting $P_{1} = (1 - |\beta|)^{2}$ and $P_{2} = (1 + |\beta|)^{2}$, we prove that
\[
P_{1} u \leq \gamma(u) \leq P_{2} u, \quad \frac{u}{P_{2}} \leq \tau(u) \leq \frac{u}{P_{1}}.
\]

According to the change of time made above, we get, for yet another Wiener process $\hat{w}(t)$, that
\[
\int_{t}^{s} \frac{d\hat{w}(u)}{1 + \beta \text{sgn} \eta(c^{k}u)} = \hat{w}(\gamma(s)) - \hat{w}(\gamma(t)).
\]

Further,
\[
P \left\{ \sup_{t \in [0,1], s \in [t, c^{k+1}]} \left| \int_{t}^{s} \frac{d\tilde{w}(u)}{1 + \beta \text{sgn} \eta(c^{k}u)} \right| \geq \frac{\phi (c^{k}) \varepsilon}{2} \right\} 
= P \left\{ \sup_{t \in [0,1], s \in [t, c^{k+1}]} |\hat{w}(\gamma(s)) - \hat{w}(\gamma(t))| \geq \frac{\phi (c^{k}) \varepsilon}{2} \right\}
= P \left\{ \sup_{\gamma(t) \in [\gamma(0), \gamma(1)], \gamma(s) \in [\gamma(t), \gamma(c^{k+1})]} |\hat{w}(\gamma(s)) - \hat{w}(\gamma(t))| \geq \frac{\phi (c^{k}) \varepsilon}{2} \right\}
\leq P \left\{ \sup_{u, v \in [0, P_{2}], |v-u| \leq P_{2} \frac{\varepsilon}{2}} |\hat{w}(v) - \hat{w}(u)| \geq \frac{\phi (c^{k}) \varepsilon}{2} \right\}.
\]
In view of the result of [1] mentioned above,

\[
P \left\{ \sup_{t \in [0,1], s \in [t, ct \wedge 1]} |z_k(t) - z_k(s)| \geq \frac{\varepsilon}{2} \right\}
\leq P \left\{ \sup_{u,v \in [0,P_2]} |\hat{w}(v) - \hat{w}(u)| \geq \frac{\phi (c^k) \varepsilon \sqrt{P_2(c-1)}}{2 \sqrt{P_2(c-1) \sqrt{c}}} \right\}
\leq \frac{2CP_2 \sqrt{c}}{\phi (c^k) \varepsilon \sqrt{P_2(c-1)}} \exp \left\{ - \frac{\phi^2 (c^k) \varepsilon^2 c}{16P_2(c-1)} \right\}.
\]

Here we used the property

\[
|v - u| \leq \gamma (ct \wedge 1) - \gamma (t) = \int_t^{ct \wedge 1} \gamma' (x) \, dx \leq (1 + |\beta|)^2 \int_t^{ct \wedge 1} \, dx \leq P_2 \frac{c - 1}{c},
\]

since \( ct \wedge 1 - t \leq (c - 1)/c \) under the assumptions of the theorem.

Choosing

\[
c_\varepsilon = 1 + \frac{\varepsilon^2}{8(1 + |\beta|)^2G^2(\phi)}
\]

we get

\[
\exp \left\{ - \frac{\phi^2 (c^k) \varepsilon^2 c}{16P_2(c-1)} \right\} \leq \exp \left\{ - \frac{\phi^2 (c^k)}{2}G^2(\phi) \right\}
\]

for \( c \in (1, c_\varepsilon) \). Next, for every positive constant \( C_1 \), there exists a positive integer \( k_0 \) such that

\[
\frac{2CP_2 \sqrt{c}}{\phi (c^k) \varepsilon \sqrt{P_2(c-1)}} \leq C_1
\]

for all \( k \geq k_0 \). Thus

\[
P \left\{ \sup_{t \in [0,1], s \in [t, ct \wedge 1]} |z_k(t) - z_k(s)| \geq \frac{\varepsilon}{2} \right\} \leq C_1 \exp \left\{ - \frac{\phi^2 (c^k)}{2}G^2(\phi) \right\}.
\]

In a similar way we prove that, for any positive constant \( C_2 \), there exists a positive integer \( k_0 \) such that

\[
P \left\{ \sup_{t \in [0,1], s \in [t/c, t]} |z_{k+1}(t) - z_{k+1}(s)| \geq \frac{\varepsilon}{2} \right\} \leq C_2 \exp \left\{ - \frac{\phi^2 (c^k)}{2}G^2(\phi) \right\}
\]

for all \( k \geq k_0 \). Hence (16), (15), and Borel–Cantelli imply (14).

**Step 3.** To complete the proof of Theorem 2 it is sufficient to prove that if \( G^2(\phi) \leq \infty \), then every function \( f \in K_G \) such that \( 2J(f) = h^2 < G^2(\phi) \) is a limit point of the sequence \( \{z_k(t)\} \). Therefore it is sufficient to prove that, for every function \( f : 2J(f) = h^2 \), there exists a number \( c > 1 \) such that the random events

\[
B_k = \left\{ \omega : \sup_{t \in [0,1]} |z_k(t) - f(t)| < \delta \right\}
\]

occur infinitely often for every \( \delta > 0 \). This means that

\[
P \left\{ \limsup_{k \to \infty} B_k \right\} = 1.
\]
We use the Borel–Cantelli–Lévy lemma [5] to prove relation (18). Introduce the family of \( \sigma \)-algebras \( \mathcal{F}_j = \sigma\{\eta(s), s \leq c^j\} \). Put
\[
A_k = \left\{ \omega : \sup_{t \in [0, 1/c]} |z_k(t) - f(t)| < \delta \right\} 
\]
and
\[
D_k = \left\{ \omega : \sup_{t \in [1/c, 1]} |z_k(t) - f(t)| < \delta \right\}.
\]

Note that \( B_k = A_k \cap D_k \) and \( D_k \prec \mathcal{F}_{k-1} \). Since
\[
z_k(t) = z_{k-1}(tc) \frac{\psi(k-1)}{\psi(k)},
\]
z\(_k(t) \prec \mathcal{F}_{k-1}\) for \( t \in [0, 1/c] \). Then \( A_k \prec \mathcal{F}_{k-1} \) and
\[
P(B_k | \mathcal{F}_{k-1}) = I(A_k) \cdot P(D_k | \mathcal{F}_{k-1}).
\]
It follows from the Borel–Cantelli–Lévy lemma that relation (18) holds if
\[
\sum_k I(A_k) \cdot P(D_k | \mathcal{F}_{k-1}) = \infty.
\]

We construct a partition of the interval \([1/c, 1]\) consisting of smaller intervals of length \( \Delta \) as follows: let \( \Delta \) be a sufficiently small positive number such that \( n(\Delta) = \frac{c-1}{c\Delta} \) is a positive integer number. Then the members of the partition of the interval \([1/c, 1]\) are
\[
\Delta_i = [d_i, d_{i+1}], \quad d_i = \frac{1}{c} + i\Delta, \quad i = 0, \ldots, n(\Delta) - 1.
\]
In what follows we construct all the partitions of the interval \([1/c, 1]\) in the way described above.

Now we consider the set
\[
\overline{D}_k = \left\{ \sup_{t \in [1/c, 1]} |z_k(t) - f(t)| \geq \delta \right\}
\]

\[
\subseteq \left\{ \sup_{i \in \Delta_i} \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3} \right\} \cup \left\{ \sup_{i} |z_k(d_i) - f(d_i)| \geq \frac{\delta}{3} \right\}
\]

\[
\cup \left\{ \sup_{i \in \Delta_i} \sup_{t \in \Delta_i} |f(d_i) - f(t)| \geq \frac{\delta}{3} \right\}.
\]

By the Cauchy–Bunyakovskii inequality,
\[
|f(t) - f(d_i)|^2 = \left| \int_{d_i}^{t} f(s) \, ds \right|^2 \leq (t - d_i) \int_{d_i}^{t} |f(s)|^2 \, ds \leq \Delta h^2.
\]
If \( \Delta < \Delta_* = \delta^2/(9h^2) \), then
\[
\left\{ \sup_{i \in \Delta_i} \sup_{t \in \Delta_i} |f(d_i) - f(t)| \geq \frac{\delta}{3} \right\}
\]
is an empty set. For such a number \( \Delta \),
\[
P(D_k | \mathcal{F}_{k-1}) \geq P\left\{ \sup_{i} |u(c^k d_i) - f(d_i)\psi(c^k)\psi(c^k)| \mathcal{F}_{k-1} \right\}
\]
\[
- P\left\{ \sup_{i \in \Delta_i} \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3} \mathcal{F}_{k-1} \right\}.
\]
Now we make use of several auxiliary results stated below. Lemma 2 (see Section 4) implies that there exists a constant $c > 1$ such that

$$I_{A_k}(\omega) = 1$$

almost surely for all sufficiently large $k$.

Now Lemma 3 (see Section 4) implies that, for a fixed $c > 1$ and all $\delta > 0$ and $Q > 0$, there exists a partition of the interval $[1/c, 1]$ consisting of smaller intervals of length $\Delta_{**}$ such that

$$P\left\{ \sup_i \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3} |I_{k-1}^{**} | \right\} \leq 2n(\Delta_{**}) \exp \left\{ -\phi^2 \left( \frac{c^k}{2} \right) Q \right\}$$

almost surely. Lemma 8 (see Section 4) implies that

$$P\left\{ \sup_i \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3} \right\} \leq 2n(\Delta) \exp \left\{ -\phi^2 \left( \frac{c^k}{2} \right) \right\}$$

almost surely for the constant $c$ defined in Lemma 2 and for an arbitrary $q > 0$ if $k$ is sufficiently large.

Now we turn back to the proof of the theorem. We pick up a number $c > 1$ such that equality (22) holds. Then we choose $Q = \frac{G^2(\phi)}{2} - q + 1$ in inequality (23), where the constant $q$ is the same as in (24), and a partition of the interval $[1/c, 1]$ with $\Delta < \min(\Delta_*, \Delta_{**})$. If the number $k$ is sufficiently large, namely, if

$$8n(\Delta) \leq \exp \left\{ \phi^2 \left( \frac{c^k}{2} \right) \right\} ,$$

then

$$P\left\{ \sup_i \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \frac{\delta}{3} |I_{k-1}^{**} | \right\} \leq 2n(\Delta) \exp \left\{ -\phi^2 \left( \frac{c^k}{2} \right) \right\} \leq \frac{1}{4} \exp \left\{ -\phi^2 \left( \frac{c^k}{2} \right) \right\} \leq \frac{1}{4} \exp \left\{ -\phi^2 \left( \frac{c^k}{2} \right) \right\}$$

almost surely, whence

$$P(D_k | I_{k-1}) \geq \frac{1}{4} \exp \left\{ -\phi^2 \left( \frac{c^k}{2} \right) \right\}$$

almost surely by (24) and (21) for sufficiently large $k$ and some $q > 0$.

Taking into account equalities (22) and (19) together with the definition of $G^2(\phi)$ we obtain (20). The proof of Step 3 is complete and thus Theorem 2 is proved. \hfill \Box

4. PROOF OF THEOREM 1 AND FURTHER AUXILIARY RESULTS

We start with the auxiliary results.

**Lemma 2.** For all $\delta > 0$ and all $h < \infty$, there exist a constant $c > 1$ and a positive integer number $k_0$ such that

$$\sup_{t \in [0, 1/c]} |z_k(t) - g(t)| < \delta$$

almost surely for all $k > k_0$ and $g \in K_h$. 
Proof. According to Step 1 in the proof of Theorem 2, for all \( c > 1 \) and an arbitrary \( \delta > 0 \) there exists a number \( k_0 \) such that

\[
\inf_{g \in K_h} \sup_{t \in [0, 1/c]} |z_k(t) - g(t)| < \frac{\delta}{3}
\]

almost surely for all \( k > k_0 \).

On the other hand, for every \( g \in K_h \),

\[
|g(t)|^2 = \left| \int_0^t \dot{g}(s) \, ds \right|^2 \leq 2th^2.
\]

Let \( c > \max(1, 18h^2/\delta^2) \); then

\[
\sup_{t \in [0, 1/c]} |g(t)| < \frac{\delta}{3}.
\]

We deduce from (26) and (27) that

\[
\sup_{t \in [0, 1/c]} |z_k(t)| < \frac{2\delta}{3}
\]

almost surely. The latter inequality together with (27) proves (25), which completes the proof of Lemma 2. \( \square \)

The following result uses the partition of the interval \([1/c, 1]\) consisting of smaller intervals of length \( \Delta \) described above.

**Lemma 3.** Let \( c > 1 \) be fixed. Then, for all \( \delta > 0 \) and an arbitrary \( Q > 0 \), there exists a partition of the interval \([1/c, 1]\) consisting of smaller intervals of length \( \Delta \) such that

\[
P \left\{ \sup_{i} \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \delta \left| \mathbb{I}_{k-1} \right| \right\} \leq 2n(\Delta) \exp \left\{ -\phi^2 \left( c^k \right) Q \right\}
\]

almost surely.

**Proof.** Consider the \( \sigma \)-algebras \( G_{c^k d_i} = \sigma\{\eta(s), s \leq c^k d_i\} \). Then

\[
P \left\{ \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \delta \left| G_{c^k d_i} \right| \right\} \leq 2 \exp \left\{ -\phi^2 \left( c^k \right) Q \right\}
\]

almost surely. The latter bound is proved similarly to the proof of Theorem 5 in [3, p. 172].

Since \( \mathbb{I}_{k-1} \subset \mathbb{G}_{c^k d_i} \) for \( i = 0, 1, \ldots, n(\Delta) - 1 \),

\[
P \left\{ \sup_{t \in \Delta_i} |z_k(t) - z_k(d_i)| \geq \delta \left| \mathbb{I}_{k-1} \right| \right\} \leq 2 \exp \left\{ -\phi^2 \left( c^k \right) Q \right\}
\]

almost surely.

This implies inequality (28) and completes the proof of Lemma 3. \( \square \)

**Lemma 4.** Let \( h(x) \) be a positive increasing function for \( x \geq 0 \). Then

\[
E \zeta I_{\{\xi > a\}} \leq \frac{1}{h(a)} E \zeta h(|\xi|)
\]

for \( \zeta \geq 0 \) and \( a > 0 \).
Lemma 5. Let

\[ \mathbb{E} \zeta I_{(\xi| > a)} = \int_{(\omega: h(\xi| > h(a))} \zeta \mathbb{P}(d\omega) \]

almost surely. Lemma 5 is proved.

\[ \square \]

Proof. Since

\[ M_k(f; x) = \frac{1}{\phi^2(c^k)} \ln \left\{ \mathbb{E} \left\{ \exp \left[ \frac{\phi(c^k)}{\sqrt{c^k}} \int_{c^k-1}^{c^k} f \left( \frac{s}{c^k} \right) 1 + \beta \text{sgn} \eta(s) \right] \eta(c^{k-1}) = x \right\} \right\} \]

Lemma 4 is proved.

\[ \square \]

Put

\[ M_k(f; x) = \frac{1}{\phi^2(c^k)} \ln \left\{ \mathbb{E} \left\{ \exp \left[ \frac{\phi(c^k)}{\sqrt{c^k}} \int_{c^k-1}^{c^k} f \left( \frac{s}{c^k} \right) 1 + \beta \text{sgn} \eta(s) \right] \eta(c^{k-1}) = x \right\} \right\} . \]

Lemma 5. Let \(|\beta| < 1\) and let \(c > 1\) be fixed. Then

\[ M_k(f; x) \leq \frac{1}{2(1 - |\beta|)^2} \int_{1/c}^{1} f^2(s) \, ds \]

almost surely for \(f \in C[0, 1]\).

Proof. Since

\[ \frac{\phi(c^k)}{\sqrt{c^k}} \int_{c^k-1}^{c^k} f \left( \frac{s}{c^k} \right) 1 + \beta \text{sgn} \eta(s) \, ds = \frac{\phi^2(c^k)}{2c^k} \int_{c^k-1}^{c^k} f^2 \left( \frac{s}{c^k} \right) \, ds \]

and

\[ \frac{1}{(1 + \beta \text{sgn} \eta(s))^2} \leq \frac{1}{(1 - |\beta|)^2}, \]

the Girsanov theorem implies

\[ M_k(f; x) \leq \frac{1}{\phi^2(c^k)} \ln \left\{ \mathbb{E} \left\{ \exp \left[ \frac{\phi^2(c^k)}{2(1 - |\beta|)^2c^k} \int_{c^k-1}^{c^k} f^2 \left( \frac{s}{c^k} \right) \, ds \right] \right\} \right\} \]

almost surely. Lemma 5 is proved.

\[ \square \]

Put

\[ C_k = \left\{ \sup_i |u(c^kd_i) - f(d_i)\psi(c^k)| < \frac{\delta}{3} \psi(c^k) \right\}; \]

\[ C_k(i) = \left\{ |u(c^kd_i) - f(d_i)\psi(c^k)| < \frac{\delta}{3} \psi(c^k) \right\}, \quad i = 0, 1, \ldots, n(\Delta) - 1; \]

\[ J_c(f) = \frac{1}{2} \int_{1/c}^{1} (1 + \beta \text{sgn} f(s))^2 f^2(s) \, ds. \]

For \(|\beta| < 1\), we choose the constants \(l, m,\) and \(p\) such that

\[ A_1. \quad 0 < m < \frac{(1 - |\beta|)^2}{(1 + |\beta|)^2}. \]

\[ A_2. \quad \text{If } \beta \neq 0, \text{ then } \sqrt{m \frac{1 - |\beta|}{1 + |\beta|}} < l < \sqrt{m - \frac{m^2}{4|\beta|}} (1 - |\beta|); \text{ otherwise } l = m. \]

\[ A_3. \quad p = \frac{(1 - |\beta|)^2}{l} \left( \frac{c^2 - m^2 + m}{(1 + |\beta|)^2} + 1 \right). \]
Put
\[ K_1 = \frac{l^2 - m^2 + m}{(1 + |\beta|)^2} - \frac{l^2}{(1 - |\beta|)^2}. \]

**Remark 2.** It is not hard to check that the following properties hold:

1. Condition \( A_1 \) implies that \( 0 < m < 1 \), that is, the expression under the square root in condition \( A_2 \) is positive.
2. Condition \( A_2 \) implies that \( K_1 > 0 \).
3. The set of numbers \( l \) satisfying the inequality in condition \( A_2 \) is nonempty, since this inequality is equivalent to
\[
\frac{1}{1 + |\beta|} < \sqrt{\frac{1 - m}{4|\beta|}};
\]
the latter inequality holds by condition \( A_1 \).

For the constants \( l, m, \) and \( p \) put
\[ \rho_k(l, m) = \exp \left\{ \frac{l}{\sqrt{c^k}} \int_{c^{k-1}}^{c^k} \frac{1 + \beta \sgn f \left( \frac{s}{c^k} \right)}{1 + \beta \sgn \eta(s)} \int_{c^{k-1}}^{c^k} \frac{1 + \beta \sgn f \left( \frac{s}{c^k} \right)}{(1 + \beta \sgn \eta(s))^2} f^2 \left( \frac{s}{c^k} \right) \frac{(1 + \beta \sgn f \left( \frac{s}{c^k} \right))}{2c^k} \right\} \]
and
\[ L_{k,p}(\delta) = \left\{ \int_{c^{k-1}}^{c^k} \frac{1 + \beta \sgn f \left( \frac{s}{c^k} \right)}{1 + \beta \sgn \eta(s)} \int_{c^{k-1}}^{c^k} f^2 \left( \frac{s}{c^k} \right) dw(s) \right\} \]
\[ - \frac{p}{2\sqrt{c^k(1 - |\beta|)^2}} \left\{ \int_{c^{k-1}}^{c^k} \left( 1 + \beta \sgn f \left( \frac{s}{c^k} \right) \right)^2 \frac{(1 + \beta \sgn f \left( \frac{s}{c^k} \right))}{2c^k} \int_{c^{k-1}}^{c^k} \frac{1 + \beta \sgn f \left( \frac{s}{c^k} \right)}{(1 + \beta \sgn \eta(s))^2} f^2 \left( \frac{s}{c^k} \right) \right\} \]
\[ \leq \frac{\delta \psi(c^k)}{(1 - |\beta|)^2} J_c(f) \]

**Lemma 6.** Let \( |\beta| < 1 \). Then, for the constants \( l \) and \( m \) chosen above, there exists a constant \( c > 1 \) such that
\[
P \{ \rho_k(l, m)I_{\Omega \setminus C_k(i)}(\omega) | \exists_{k-1} \} \leq \exp \left\{ \phi^2(c^k) \frac{l^2 - m^2}{(1 + |\beta|)^2} J_c(f) \right\} a_k(i)
\]
almost surely, where the numbers \( a_k(i) \) do not depend on \( \theta \) and \( \Delta \) and are such that
\[
\lim_{k \to \infty} a_k(i) = 0, \quad i = 0, 1, \ldots, n(\Delta) - 1.
\]

**Proof.** Let \( \theta \prec \exists_{k-1} \) be an arbitrary positive bounded random variable. We apply Lemma \[4\] to the function
\[
h(x) = \exp \left\{ \frac{\phi(c^k) N x}{\sqrt{c^k}} \right\}
\]
with some constant \( N \) to be specified later. Then

\[
E \theta_{\rho_k}(l, m) I_{\Omega \setminus C_k(i)}(\omega)
\]

\[
\leq E \theta_{\rho_k}(l, m) \exp \left\{ \frac{N \phi(c^k)}{\sqrt{c^k}} |u(c^kd_i) - f(d_i)\psi(c^k)| - \frac{\delta}{3} N \phi^2(c^k) \right\}
\]

\[
\leq E \theta_{\rho_k}(l, m) \exp \left\{ \frac{N \phi(c^k)}{\sqrt{c^k}} (u(c^kd_i) - f(d_i)\psi(c^k)) - \frac{\delta}{3} N \phi^2(c^k) \right\}
\]

\[
+ E \theta_{\rho_k}(l, m) \exp \left\{ -\frac{N \phi(c^k)}{\sqrt{c^k}} (u(c^kd_i) - f(d_i)\psi(c^k)) - \frac{\delta}{3} N \phi^2(c^k) \right\}
\]

\[
= J^1_k(i) + J^2_k(i).
\]

First we consider the term \( J^1_k(i) \). Using equality \( 30 \) together with

\[
\frac{1}{(1 + \beta \text{sgn} \eta(s c^k))^2} \geq \frac{1}{(1 + |\beta|)^2},
\]

we get

\[
J^1_k(i) = \exp \left\{ -\phi^2(c^k) \left[ N f(d_i) + N \frac{\delta}{3} + \frac{m}{2} \int_{1/c}^1 (1 + \beta \text{sgn} f(s))^2 \int_{1/c}^1 \frac{1}{1 + \beta \text{sgn} \eta(s c^k)} ds ds \right] \right\}
\]

\[
\times E \theta \exp \left\{ \frac{\phi(c^k)}{\sqrt{c^k}} \left( Nu(c^kd_i) + \int_{c^k-1}^{c^k} \left[ 1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right) \right] \frac{1}{1 + \beta \text{sgn} \eta(s)} \right) \right\}
\]

\[
\leq \exp \left\{ -\phi^2(c^k) \left[ N f(d_i) + N \frac{\delta}{3} + \frac{m}{(1 + |\beta|)^2} J_c(f) \right] \right\}
\]

\[
\times E \left\{ \theta E \left\{ \exp \left[ \frac{\phi(c^k)}{\sqrt{c^k}} \left( Nu(c^{k-1}) + \int_{c^k-1}^{c^k} \left[ \frac{1}{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right) \left( \frac{\rho}{c^k} \right) \left( \frac{s}{c^k} \right) \left( \frac{\rho}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \right) \right) \right] \right\} \right\}.
\]

The Markov property of the process \( \eta(t) \) implies that

\[
E \left\{ \exp \left[ \frac{\phi(c^k)}{\sqrt{c^k}} \int_{c^k-1}^{c^k} \left[ \frac{1}{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right) \left( \frac{\rho}{c^k} \right) \left( \frac{s}{c^k} \right) \left( \frac{\rho}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \right) \right] \right\} \right\}
\]

\[
= \exp \left\{ \phi^2(c^k) M_k \left( l(1 + \beta \text{sgn} f) \dot{f} + NI_{[c^{k-1}, c^{k-1}]}(s) \eta(c^{k-1}) \right) \right\}.
\]
Applying Lemma 5 we obtain
\[ J_k^i(i) \leq \exp \left\{ -\phi^2(c^k) \left[ N \frac{\delta}{3} + \frac{m}{(1 + |\beta|)^2} J_c(f) \right] \right\} \times \mathbb{E} \theta \exp \left\{ \frac{\phi(c^k)}{\sqrt{c^k}} N \left( u(c^{k-1}) - f(d_i) \psi(c^k) \right) \right\} \times \exp \left\{ \phi^2(c^k) M_k \left( J(1 + |\beta| f) \dot{f} + NI_{[c^k, c^k]}(\cdot; \eta(c^{k-1})) \right) \right\} \leq \exp \left\{ -\phi^2(c^k) \left[ N \frac{\delta}{3} + \frac{m}{(1 + |\beta|)^2} J_c(f) \right] \right\} \times \mathbb{E} \theta \exp \left\{ \frac{\phi(c^k)}{\sqrt{c^k}} N \left( u(c^{k-1}) - f(1/c) \psi(c^k) \right) \right\} \times \exp \left\{ \frac{\phi^2(c^k)}{2(1 - |\beta|)^2} \int_{1/c}^{d_i} \left( J(1 + \beta \text{sgn} f(s)) \dot{f}(s) + NI_{[1/c, d_i]}(s) \right)^2 ds \right\}.

By Lemma 2 there exists a constant \( c > 1 \) such that
\[ \text{(31)} \quad \exp \left\{ \frac{\phi(c^k)}{\sqrt{c^k}} N \left( u(c^{k-1}) - f(1/c) \psi(c^k) \right) \right\} \leq \exp \left\{ \frac{\delta}{6} N \phi^2(c^k) \right\} \]
almost surely for sufficiently large \( k \). Then we estimate
\[ \int_{1/c}^{1} \left( J(1 + \beta \text{sgn} f(s)) \dot{f}(s) + NI_{[1/c, d_i]}(s) \right)^2 ds \]
\[ \leq 2l^2 J_c(f) + 2NI \int_{1/c}^{d_i} (1 + \beta \text{sgn} f(s)) \dot{f}(s) ds + N^2(1 - 1/c). \text{(32)} \]

Denote the right hand side of inequality (32) by \( A_c(J_c, N, l) \). Then (31) and (32) imply
\[ J_k^i(i) \leq \exp \left\{ -\phi^2(c^k) \left[ N \frac{\delta}{6} + \frac{m}{(1 + |\beta|)^2} J_c(f) \right] \right\} \exp \left\{ \frac{\phi^2(c^k)}{2(1 - |\beta|)^2} A_c(J_c, N, l) \right\} \times \exp \left\{ \phi^2(c^k) N (f(1/c) - f(d_i)) \right\} \mathbb{E} \theta \]
\[ = \exp \left\{ -\phi^2(c^k) \left[ N \frac{\delta}{6} + \frac{m}{(1 + |\beta|)^2} J_c(f) - \frac{l^2}{(1 - |\beta|)^2} J_c(f) \right] \right\} \times \exp \left\{ \phi^2(c^k) N \int_{1/c}^{d_i} \left( \frac{(1 + \beta \text{sgn} f(s))}{(1 - |\beta|)^2} - 1 \right) \dot{f}(s) ds \right\} \mathbb{E} \theta. \text{(33)} \]

The expression written in the parentheses in the integral in (33) does not exceed
\[ \frac{l(1 + |\beta|)}{(1 - |\beta|)^2}, \]
while
\[ \int_{1/c}^{d_i} \dot{f}(s) ds \leq \left| \int_{1/c}^{d_i} \frac{1 + \beta \text{sgn} f}{1 + \beta \text{sgn} f} \dot{f}(s) ds \right| \leq \sqrt{2} J_c(f) \sqrt{1 - 1/c}. \]
Thus we deduce from inequality (33) that
\[
J^1_k(i) \leq \exp\left\{ -\phi^2\left(c^k\right) \left[ \frac{N\delta}{6} + J_c(f) \left( \frac{m}{(1+|\beta|)^2} - \frac{l^2}{(1-|\beta|)^2} \right) - \frac{N^2(1-1/c)}{2(1-|\beta|)^2} \right. \right.
\]
\[
\left. \left. - \frac{1N(1+|\beta|)}{(1-|\beta|)^3} \sqrt{2J_c(f)(1-1/c)} \right] \right\} \theta
\]
\[
= \exp\left\{ \phi^2\left(c^k\right) \frac{l^2 - m^2}{(1+|\beta|)^2} J_c(f) \right\}
\]
\[
\times \exp\left\{ -\phi^2\left(c^k\right) \left[ \frac{N\delta}{6} - \frac{N^2(1-1/c)}{2(1-|\beta|)^2} 
\right.ight.
\]
\[
\left. \left. + J_c(f) \left( \frac{l^2 - m^2}{(1+|\beta|)^2} + \frac{m}{(1+|\beta|)^2} - \frac{l^2}{(1-|\beta|)^2} \right) \right.ight.
\]
\[
\left. \left. - \frac{1N(1+|\beta|)}{(1-|\beta|)^3} \sqrt{2J_c(f)(1-1/c)} \right] \right\} \theta.
\]
Taking into account equality (24) we put
\[
\hat{a}_k(i) = \exp\left\{ -\phi^2\left(c^k\right) \left[ \frac{N\delta}{6} - \frac{N^2(1-1/c)}{2(1-|\beta|)^2} + K_1 J_c(f) - \frac{1N(1+|\beta|)}{(1-|\beta|)^3} \sqrt{2J_c(f)(1-1/c)} \right] \right\}.
\]
Since $K_1 > 0$, the expression in the square brackets is positive for some $N > 0$. For such a number $N$,
\[
\lim_{k \to \infty} \hat{a}_k(i) = 0
\]
and
\[
J^1_k(i) \leq \exp\left\{ \phi^2\left(c^k\right) \frac{l^2 - m^2}{(1+|\beta|)^2} J_c(f) \right\} \hat{a}_k(i) \theta.
\]
Similarly
\[
J^2_k(i) \leq \exp\left\{ \phi^2\left(c^k\right) \frac{l^2 - m^2}{(1+|\beta|)^2} J_c(f) \right\} \hat{a}_k(i) \theta,
\]
where
\[
\lim_{k \to \infty} \hat{a}_k(i) = 0.
\]
Now Lemma 6 follows from bounds (34) and (35) with $a_k(i) = \hat{a}_k(i) + \tilde{a}_k(i)$. \qed

Lemma 7. Let $|\beta| < 1$. Then
\[
P \{ p_k(l, m)I_{\Omega \setminus \Lambda_{k, \rho}(\delta)}(\omega)|\mathcal{F}_{k-1} \} \leq \exp\left\{ \phi^2\left(c^k\right) \frac{l^2 - m^2}{(1+|\beta|)^2} J_c(f) \right\} b_k(\delta)
\]
almost surely for the constants $l$, $m$, and $p$ chosen above and for all $\delta > 0$, where $b_k(\delta)$ does not depend on $\theta$ and is such that $\lim_{k \to \infty} b_k(\delta) = 0$.

Proof. Let $\theta \prec \mathcal{F}_{k-1}$ be an arbitrary positive bounded random variable. We use Lemma 4 with the function
\[
h(x) = \exp\left\{ \frac{\phi\left(c^k\right) N}{\sqrt{c^k}} x \right\}
\]
and with some constant $0 < N < 1$ to be specified later. Then

$$
\mathbb{E} \theta \rho_k(l, m) \mathbb{I}_{\Omega \setminus L_k, p}(\delta)(\omega)
\leq \mathbb{E} \theta \rho_k(l, m)
\times \exp \left\{ \phi \left( \frac{c^k}{\sqrt{c^k}} \right) N \int_{c_{k-1}}^{c_k} \frac{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \dot{f} \left( \frac{s}{c^k} \right) \, dw(s)
- \frac{\rho \phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right)}{2(1 - |\beta|)^2 c^k} N \int_{c_{k-1}}^{c_k} \left( 1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right) \right) \dot{f}^2 \left( \frac{s}{c^k} \right) \, ds \right\}
\times \exp \left\{ -\phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right) N \frac{\delta}{(1 - |\beta|)^2} J_c(f) \right\}
\leq \mathbb{E} \theta \rho_k(l, m) \exp \left\{ \phi \left( \frac{c^k}{\sqrt{c^k}} \right) N \int_{c_{k-1}}^{c_k} \frac{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \dot{f} \left( \frac{s}{c^k} \right) \, dw(s)
- \frac{(\delta + p)\phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right) N}{(1 - |\beta|)^2} J_c(f) \right\}
+ \mathbb{E} \theta \rho_k(l, m) \exp \left\{ -\phi \left( \frac{c^k}{\sqrt{c^k}} \right) N \int_{c_{k-1}}^{c_k} \frac{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \dot{f} \left( \frac{s}{c^k} \right) \, dw(s)
- \frac{(\delta - p)\phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right) N}{(1 - |\beta|)^2} J_c(f) \right\}
= J_k^1(\delta) + J_k^2(\delta).
$$

Substituting $\rho_k(l, m)$, we consider the term $J_k^1(\delta)$. We see from the Markov property of the process $\eta(t)$ that

$$
J_k^1(\delta) = \mathbb{E} \theta \exp \left\{ \phi \left( \frac{c^k}{\sqrt{c^k}} \right) (N + l) \int_{c_{k-1}}^{c_k} \frac{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \dot{f} \left( \frac{s}{c^k} \right) \, dw(s)
- \phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right) \left( \frac{(\delta + p)N}{(1 - |\beta|)^2} J_c(f) + \frac{m}{2} \int_{1/c}^{1} \left( 1 + \beta \text{sgn} f(s) \right)^2 \dot{f}^2(s) \, ds \right) \right\}
\leq \exp \left\{ -\phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right) J_c(f) \left( \frac{(\delta + p)N}{(1 - |\beta|)^2} + \frac{m}{(1 + |\beta|)^2} \right) \right\}
\times \mathbb{E} \theta \mathbb{E} \left\{ \exp \left\{ \phi \left( \frac{c^k}{\sqrt{c^k}} \right) (N + l) \int_{c_{k-1}}^{c_k} \frac{1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right)}{1 + \beta \text{sgn} \eta(s)} \dot{f} \left( \frac{s}{c^k} \right) \, dw(s) \right\} \mathbb{I}_{k - 1} \right\}
= \exp \left\{ -\phi^2 \left( \frac{c^k}{\sqrt{c^k}} \right) J_c(f) \left( \frac{(\delta + p)N}{(1 - |\beta|)^2} + \frac{m}{(1 + |\beta|)^2} \right) \right\} \mathbb{E} \theta \mathbb{E} \left\{ \phi \left( \frac{c^k}{\sqrt{c^k}} \right) M_k \left( (l + N)(1 + \beta \text{sgn} f) \dot{f}; \eta \left( c^{k-1} \right) \right) \right\}.
$$

By Lemma [5] we get

$$
M_k \left( (l + N)(1 + \beta \text{sgn} f) \dot{f}; \eta \left( c^{k-1} \right) \right) \leq \frac{(l + N)^2}{2(1 - |\beta|)^2} \int_{1/c}^{1} \left( 1 + \beta \text{sgn} f \right)^2 \dot{f}^2 \, ds
= \frac{(l + N)^2}{(1 - |\beta|)^2} J_c(f)
$$
almost surely. Hence

\[
J_k^1(\delta) \leq \exp \left\{ -\phi^2 (c^k) J_c(f) \left[ (\delta + p)N + \frac{m}{(1 + |\beta|^2)} \right] - \frac{(l + N)^2}{(1 + |\beta|^2)} \right\} E \theta
\]

\[
= \exp \left\{ \phi^2 (c^k) J_c(f) \left[ \frac{l^2 - m^2}{(1 + |\beta|^2)} \right] - \frac{(\delta + p)N - (l + N)^2}{(1 - |\beta|^2)} + \frac{m + l^2 - m^2}{(1 + |\beta|^2)} \right\} E \theta
\]

(36)

\[
= \exp \left\{ \phi^2 (c^k) J_c(f) \left[ \frac{l^2 - m^2}{(1 + |\beta|^2)} J_c(f) \right] \right\} \times \exp \left\{ -\phi^2 (c^k) J_c(f) \left[ (\delta + p - 2l)N - N^2 \right] + K_1 \right\} E \theta
\]

\[
= \exp \left\{ \phi^2 (c^k) J_c(f) \left[ \frac{l^2 - m^2}{(1 + |\beta|^2)} J_c(f) \right] \right\} \cdot \tilde{b}_k(\delta) E \theta
\]

for

\[
\tilde{b}_k(\delta) = \exp \left\{ -\phi^2 (c^k) J_c(f) \left[ (\delta + p - 2l)N - N^2 \right] + K_1 \right\}.
\]

Since \( K_1 > 0 \), the expression in the square brackets on the right hand side of the definition of \( \tilde{b}_k(\delta) \) is positive for some \( N > 0 \). For such a number \( N \),

\[
\lim_{k \to \infty} \tilde{b}_k(\delta) = 0.
\]

Similarly,

\[
J_k^2(\delta) \leq \exp \left\{ \phi^2 (c^k) \frac{l^2 - m^2}{(1 + |\beta|^2)} J_c(f) \right\} \cdot \tilde{b}_k(\delta) E \theta,
\]

where

\[
\lim_{k \to \infty} \tilde{b}_k(\delta) = 0.
\]

Now Lemma holds with \( b_k(\delta) = \tilde{b}_k(\delta) + \hat{b}_k(\delta) \) for some \( N \).

\[ \square \]

**Lemma 8.** Let \( f \in K_G \) be an arbitrary function such that \( 2J(f) = h^2 < G^2 \). Then there are numbers \( c > 1 \) and \( v > 0 \) such that

\[
P(C_k|I_{k-1}) \geq \frac{1}{2} \exp \left\{ -\phi^2 (c^k) \left( \frac{G^2}{2} - v \right) \right\}
\]

almost surely for sufficiently large \( k \).

**Proof.** Let \( \theta < I_{k-1} \) be an arbitrary positive bounded random variable. Then

\[
E \theta I_{C_k}(\omega) = E \theta \rho_k(l, m) I_{C_k}(\omega)
\]

\[
\times \exp \left\{ -l \frac{\phi \left( c^k \right)}{\sqrt{c^k}} \int_{c^{k-1}}^{c^k} 1 + \beta \frac{\text{sgn} f \left( \frac{s}{c^k} \right)}{\sqrt{c^k}} \int \frac{f \left( \frac{s}{c^k} \right) dw(s)}{1 + \beta \text{sgn} \eta(s)} \int \frac{f \left( \frac{s}{c^k} \right) dw(s)}{1 + \beta \text{sgn} \eta(s)} \int \frac{f \left( \frac{s}{c^k} \right) dw(s)}{1 + \beta \text{sgn} \eta(s)} \right\}
\]

\[
+ \frac{m \phi^2 \left( c^k \right)}{2c^k} \int_{c^{k-1}}^{c^k} \frac{(1 + \beta \text{sgn} f \left( \frac{s}{c^k} \right))^2 f^2 \left( \frac{s}{c^k} \right) ds}{(1 + \beta \text{sgn} \eta(s))^2}
\]
\[
\geq \exp \left\{ \phi^2 (c^k) \left[ \frac{m J_c(f)}{(1 + |\beta|)^2} - \frac{pl J_c(f)}{(1 - |\beta|)^2} \right] \right\} \\
\times E \theta \rho_k(l, m) I_{C_k}(\omega)
\]
\[
\times \exp \left\{ -k \frac{\phi(c^k)}{\sqrt{c^k}} \int_{c_k-1}^{c_k} \frac{1 + \beta \text{sgn} f\left(\frac{s}{c^k}\right)}{1 + \beta \text{sgn} \eta(s)} f\left(\frac{s}{c^k}\right) dw(s) \\
+ pl \frac{\phi^2(c^k)}{2(1 - |\beta|)^2 c^k} \int_{c_k-1}^{c_k} \left(1 + \beta \text{sgn} f\left(\frac{s}{c^k}\right)\right)^2 f^2\left(\frac{s}{c^k}\right) ds \right\}
\]
\[
\geq \exp \left\{ \phi^2 (c^k) J_c(f) \left[ \frac{m}{(1 + |\beta|)^2} - \frac{pl}{(1 - |\beta|)^2} \right] \right\} \\
\times E \theta \rho_k(l, m) I_{C_k}(\omega) I_{L_k(\delta)}(\omega)
\]
\[
\times \exp \left\{ -k \frac{\phi(c^k)}{\sqrt{c^k}} \int_{c_k-1}^{c_k} \frac{1 + \beta \text{sgn} f\left(\frac{s}{c^k}\right)}{1 + \beta \text{sgn} \eta(s)} f\left(\frac{s}{c^k}\right) dw(s) \\
- pl \frac{\phi^2(c^k)}{2(1 - |\beta|)^2 c^k} \int_{c_k-1}^{c_k} \left(1 + \beta \text{sgn} f\left(\frac{s}{c^k}\right)\right)^2 f^2\left(\frac{s}{c^k}\right) ds \right\}
\]
\[
\geq \exp \left\{ \phi^2 (c^k) J_c(f) \left[ \frac{m}{(1 + |\beta|)^2} - \frac{pl}{(1 - |\beta|)^2} - \frac{\delta l}{(1 - |\beta|)^2} \right] \right\} \\
\times E \theta \rho_k(l, m) \left(1 - I_{\Omega \setminus C_k}(\omega) - I_{\Omega \setminus L_k(\delta)}(\omega)\right).
\]

In the above reasoning we used the inequalities \(1 \geq I_{L_k(\delta)}(\omega)\) and

\[
\exp\{-a\} I_{|a|<b} \geq \exp\{-b\} I_{|a|<b}.
\]

Since \(I_{C_k}(\omega) I_{L_k(\delta)}(\omega) \geq 1 - I_{\Omega \setminus C_k}(\omega) - I_{\Omega \setminus L_k(\delta)}(\omega)\), we obtain

\[
\mathbb{E} \theta I_{C_k}(\omega) \geq \mathbb{E} \theta \rho_k(l, m) \left(1 - I_{\Omega \setminus C_k}(\omega) - I_{\Omega \setminus L_k(\delta)}(\omega)\right).
\]

Then equality (30) implies that

\[
\mathbb{E} \theta \rho_k(l, m) = \mathbb{E} \{ \theta \mathbb{E} \{ \rho_k(l, m) \mid \mathcal{S}_{k-1} \}\}
= \mathbb{E} \left\{ \theta \mathbb{E} \left\{ \exp \left[ \frac{\phi^2 (c^k)}{2 c^k} \int_{c_k-1}^{c_k} \left(l^2 - m^2\right) \frac{1 + \beta \text{sgn} f^2}{1 + \beta \text{sgn} \eta^2} f^2 ds \mid \mathcal{S}_{k-1} \right] \right\} \right\}.
\]

It is clear that

\[
l^2 - m^2 > l^2 \frac{(1 + |\beta|)^2}{(1 - |\beta|)^2} - m.
\]

Considering the left hand side of property A2, we conclude that

\[
l^2 \frac{(1 + |\beta|)^2}{(1 - |\beta|)^2} - m > 0,
\]

whence

\[
l^2 - m^2 > 0.
\]

Hence

\[
\mathbb{E} \theta \rho_k(l, m) \geq \exp \left\{ \phi^2 (c^k) \frac{l^2 - m^2}{(1 + |\beta|)^2} J_c(f) \right\} \mathbb{E} \theta.
\]
We continue the proof by using the latter bound and applying Lemmas 6 and 7:

\[
\mathbb{E} \theta I_{C_k}(\omega) \geq \exp \left\{ \phi^2 \left( c^k \right) J_c(f) \left( \frac{m}{(1 + |\beta|)^2} - \frac{pl + \delta l}{(1 - |\beta|)^2} \right) \right\} 
\times \exp \left\{ \phi^2 \left( c^k \right) \frac{l^2 - m^2}{(1 + |\beta|)^2} J_c(f) \right\} \left( 1 - \sum_{i=1}^{n(\Delta)} a_k(i) - b_k(\delta) \right) \mathbb{E} \theta 
\geq \exp \left\{ -\phi^2 \left( c^k \right) J_c(f) \left( \frac{m^2 - l^2 - m}{(1 + |\beta|)^2} + \frac{pl + \delta l}{(1 - |\beta|)^2} \right) \right\} \mathbb{E} \theta.
\]

Then we use property A3:

\[
(37) \quad \mathbb{E} \theta I_{C_k}(\omega) \geq \exp \left\{ -\phi^2 \left( c^k \right) J_c(f) \left( 1 + \frac{\delta l}{(1 - |\beta|)^2} \right) \right\} \mathbb{E} \theta.
\]

It is clear that

\[
J_c(f) \left( 1 + \frac{\delta l}{(1 - |\beta|)^2} \right) \leq \left( 1 + \frac{\delta l}{(1 - |\beta|)^2} \right) \frac{h^2}{2}.
\]

Choose

\[
\delta < \frac{G^2 - h^2 (1 - |\beta|)^2}{3h^2 l}.
\]

The latter inequality implies that

\[
(38) \quad J_c(f) \left( 1 + \frac{\delta l}{(1 - |\beta|)^2} \right) \leq \frac{G^2}{2} - v,
\]

where \( v = \frac{1}{3}(G^2 - h^2) \). Now Lemma 8 follows from inequalities (37) and (38).

The Lipschitz property of the function \( \kappa \) (see definition (7)) yields the following result.

**Lemma 9.** Assume that

\[
P \left\{ \lim_{n \to \infty} \sup_{t \in [0,1]} |f_n(t) - g(t)| = 0 \right\} = 1
\]

for all one-dimensional functions \( \{f_n\} \) and \( g \). Then

\[
P \left\{ \lim_{n \to \infty} \sup_{t \in [0,1]} |\kappa(f_n(t)) - \kappa(g(t))| = 0 \right\} = 1,
\]

where the function \( \kappa \) is defined by (7).

**Proof of Theorem 1.** Using Theorem 2 we prove that, for an arbitrary function \( f \in \mathcal{K}_G \), there exists a subsequence \( \{T_m\} \) such that

\[
P \left\{ \lim_{T_m \to \infty} \sup_{t \in [0,1]} |\eta_{T_m}(t) - f(t)| = 0 \right\} = 1.
\]

Then Lemma 9 and relations (7)–(9) complete the proof of Theorem 1.

**Bibliography**


Department of Probability Theory and Mathematical Statistics, Institute for Applied Mathematics and Mechanics, National Academy of Science of Ukraine, Luxemburg Street, 74, Donetsk, 83114, Ukraine

E-mail address: ikrykun@iamm.ac.donetsk.ua

Received 09/NOV/2010

Translated by S. KVASKO