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LARGE DEVIATIONS FOR IMPULSIVE PROCESSES IN THE SCHEME OF THE LÉVY APPROXIMATION

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Abstract. Asymptotic analysis of the large deviations problem for impulsive processes in the scheme of the Lévy approximation is realized. Large deviations for impulsive processes in the scheme of the Lévy approximation are defined by the exponential generator for a jump process with independent increments.

1. Introduction

An asymptotic analysis of the large deviations problem for impulsive processes in the scheme of the Lévy approximation (see [7, Chapter 9]) is realized in this paper.

An asymptotic analysis for impulsive processes in the scheme of the Lévy approximation is realized in the papers [8, 9].

An effective method is developed in the monograph [2] to study the large deviations problem with the help of the theory of convergence of exponential (nonlinear) operators. The exponential operator in the scheme of series with respect to a small parameter $\varepsilon \to 0$, $\varepsilon > 0$, is given by

$$\mathbb{H}^\varepsilon \varphi(x) := e^{-\varphi(x)/\varepsilon} \varepsilon \mathbb{L}^\varepsilon e^{\varphi(x)/\varepsilon}$$

(see [5, 6]), where $\mathbb{L}^\varepsilon$, $\varepsilon > 0$, are the operators defining the Markov processes $x^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$, in the scheme of series.

The impulsive processes are defined as follows:

$$\xi(t) = \xi_0 + \sum_{k=1}^{\nu(t)} \alpha_k(x_{k-1}), \qquad t \geq 0,\ \xi_0 \in \mathbb{R} \tag{1}$$

(see [7, Chapter 2]). Here the random variables $\alpha_k(x)$, $k \geq 1$, $x \in E$, are independent and identically distributed with the distribution function

$$G_x(dv) = \mathsf{P}(\alpha_k(x) \in dv).$$

The switching process $x(t)$, $t \geq 0$, is a Markov jump process with a standard phase space $(\mathcal{E}, E)$. Assume that this process is determined by the generator

$$Q\varphi(x) = q(x) \int_{\mathbb{E}} [\varphi(y) - \varphi(x)]\, P(x, dy), \qquad x \in E. \tag{2}$$

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The semi-Markov kernel

$$Q(x,B,t) = P(x,B)\left(1 - e^{-q(x)t}\right), \qquad x \in E,\ B \in \mathcal{E},\ t \geq 0,$$

defines an associated Markov renewal process (x_k, τ_k), $k \geq 0$, where x_k, $k \geq 0$, is the embedded Markov chain with the stochastic kernel

$$P(x,B) = \mathsf{P}(x_{k+1} \in B \mid x_k = x)$$

and where τ_k, $k \geq 0$, is the point process of jump moments corresponding to the distribution function

$$\mathsf{P}(\theta_{k+1} \leq t \mid x_k = x) = 1 - e^{-q(x)t}$$

of the sojourn time $\theta_{k+1} = \tau_{k+1} - \tau_k$, $k \geq 0$.

Finally, $\nu(t)$ in (1) is the counting jump process: $\nu(t) = \max\{k\colon \tau_k \leq t\}$.

The following is the main assumption concerning the switching Markov process:

C1: The Markov process $x(t)$, $t \geq 0$, is uniformly ergodic with the stationary distribution $\pi(A)$, $A \in \mathcal{E}$.

Remark 1.1. If $x(t)$, $t \geq 0$, has a stationary distribution $\pi(x)$, then x_n, $n \geq 1$, also has a stationary distribution $\rho(x)$. Moreover,

$$\pi(dx)q(x) = q\,\rho(dx), \qquad q := \int_E \pi(dx)q(x).$$

We denote by Π the projector onto the subspace of zeros of the reducibly-invertible operator Q defined by equality (2):

$$\Pi\varphi(x) = \int_E \pi(dx)\varphi(x).$$

It is clear that

$$Q\Pi = \Pi Q = 0.$$

The potential R_0 is such that

$$QR_0 = R_0 Q = \Pi - I$$

(see [7, Chapter 1]).

Remark 1.2. The latter relation yields that if the equation

$$\Pi\psi = 0$$

is solvable, then the Poisson equation

$$Q\varphi = \psi$$

possesses a unique solution

$$\varphi = R_0\psi$$

with $\Pi\varphi = 0$.

The impulsive process defined by (1) is determined by the generator of the two component Markov process $\xi(t)$, $x(t)$, $t \geq 0$,

$$\mathbb{L}\varphi(u,x) = q(x)\int_E P(x,dy)\int_{\mathbb{R}} G_y(dv)[\varphi(u+v,y) - \varphi(u,x)]$$

(see [7, Chapter 2]).

Remark 1.3. The latter generator can be rewritten as follows:

$$\mathbb{L}\varphi(u,x) = [Q + Q_0 G_x]\varphi(u,x),$$

where

$$Q_0\varphi(x) := q(x)\int_E P(x,dy)\varphi(y),$$

$$G_x\varphi(u) := \int_{\mathbb{R}} G_x(dv)[\varphi(u+v) - \varphi(u)].$$

Remark 1.4. The limit properties of Markov processes are usually studied with the help of their martingale characterization. Namely, consider the martingale

$$\mu_t = \varphi(x(t)) - \varphi(x(0)) - \int_0^t \mathbb{L}\varphi(x(s))\,ds, \tag{3}$$

where $\mathbb{L}$ is the generator defining the Markov process $x(t)$, $t \geq 0$, in a standard phase space $(E, \mathcal{E})$. Assume that the domain $\mathcal{D}(\mathbb{L}) \subseteq \mathcal{B}_E$ of the generator $\mathbb{L}$ is dense and contains continuously differentiable functions. Here $\mathcal{B}_E$ is the Banach space of real valued bounded test functions $\varphi(x) \in E$ equipped with the norm $\|\varphi\| := \sup_{x\in E} |\varphi(x)|$.

The large deviations theory is based on the exponential martingale characterization, that is, on the fact that

$$\widetilde{\mu}_t = \exp\left\{\varphi(x(t)) - \varphi(x(0)) - \int_0^t \mathbb{H}\varphi(x(s))\,ds\right\} \tag{4}$$

is a martingale (see [2, Chapter 1]).

Here $\mathbb{H}\varphi(x)$ is the exponential nonlinear operator,

$$\mathbb{H}\varphi(x) := e^{-\varphi(x)}\mathbb{L}e^{\varphi(x)}, \qquad \varphi(x) \in \mathcal{B}_E.$$

Equalities (3) and (4) are equivalent according to the following result.

Proposition 1 ([1, p. 66])**.** *Let*

$$\mu(t) = x(t) - \int_0^t y(s)\,ds.$$

Then $\mu(t)$ is a martingale if and only if

$$\widetilde{\mu}(t) = x(t)\exp\left\{-\int_0^t \frac{y(s)}{x(s)}\,ds\right\}$$

is a martingale.

Remark 1.5. A standard solution of the large deviations problem consists of the following four steps (see [2, Chapter 2]):

1) The evaluation of the limit exponential (nonlinear) operator that determines the large deviations.
2) The proof of the exponential compactness.
3) The proof of the comparison principle for the limit operator.
4) The construction of the variation representation of the action functional that determines the large deviations.

Steps 2)–4) are described in the monograph [2] for exponential generators that correspond to different types of random walks.

The classical approach in the large deviations problem uses the cumulant of the process [3]. A relationship between the cumulant and exponential generator is described, for example, in [10].

In order to solve the large deviations problem for the Lévy approximation one considers two small parameters $\varepsilon, \delta \to 0$ in the scheme of series such that $\varepsilon^{-1}\delta \to 1$. The parameters are used for the normalization of the impulsive process defined by (1):

$$\xi^{\varepsilon,\delta}(t) = \xi_0^{\varepsilon,\delta} + \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^3)} \alpha_k^\delta(x_{k-1}), \qquad t \geq 0,$$

$$\mathbb{L}^{\varepsilon,\delta}\varphi(u,x) = \varepsilon^{-3} q(x) \int_E P(x,dy) \int_{\mathbb{R}} G_y^\delta(dv)[\varphi(u+\varepsilon v, y) - \varphi(u,x)],$$

where the kernel $G_x^\delta(v)$ satisfies the assumptions of the Lévy approximation.

2. Main assumptions of the Lévy approximation

C2: *Lévy approximation.* The family of stochastic processes $\alpha_k^\delta(x)$, $k \geq 1$, $x \in E$, $t \geq 0$, is such that:

LA1: (Approximation of averages)

$$a_\delta(x) = \int_{\mathbb{R}} v\, G_x^\delta(dv) = \delta a_1(x) + \delta^2 \left[a(x) + \theta_a^\delta(x)\right]$$

and

$$c_\delta(x) = \int_{\mathbb{R}} v^2\, G_x^\delta(dv) = \delta^2 \left[c(x) + \theta_c^\delta(x)\right],$$

where

$$\sup_{x\in E} |a_1(x)| \leq a < +\infty, \qquad \sup_{x\in E} |a(x)| \leq a < +\infty,$$
$$\sup_{x\in E} |c(x)| \leq c < +\infty.$$

LA2: The intensity kernel admits the asymptotic representation

$$G_{g,x}^\delta = \int_{\mathbb{R}} g(v)\, G_x^\delta(dv) = \delta^2 \left[G_{g,x} + \theta_g^\delta(x)\right]$$

for all $g \in C_3(\mathbb{R})$, where $C_3(\mathbb{R})$ is a class of functions determining the measure (see [4, Chapter 7]) and where $G_{g,x}$ is a bounded kernel,

$$|G_{g,x}| \leq G_g.$$

Here G_g is a constant that depends on a function g. The kernel $G_x^0(dv)$ is defined at functions belonging to a class $C_3(\mathbb{R})$ determining the measure according to the rule

$$G_{g,x} = \int_{\mathbb{R}} g(v)\, G_x^0(dv), \qquad g \in C_3(\mathbb{R}).$$

Asymptotically negligible terms θ_a^δ, θ_c^δ, and θ_g^δ are such that

$$\sup_{x\in E} |\theta_\cdot^\delta(x)| \to 0, \qquad \delta \to 0.$$

LA3: (Balance condition)

$$q \int_E \rho(dx) a_1(x) = 0.$$

C3: *Uniform quadratic integrability*:

$$\lim_{c\to\infty} \sup_{x\in E} \int_{|v|>c} v^2\, G_x^0(dv) = 0.$$

C4: *Exponential boundedness*:

$$\int_{\mathbb{R}} e^{p|v|}\, G_x^{\delta}(dv) < \infty, \qquad \forall p \in \mathbb{R}.$$

3. Main result

Theorem 3.1. *Let*

$$\xi^{\varepsilon,\delta}(t) = \xi_0^{\varepsilon,\delta} + \varepsilon \sum_{k=1}^{\nu(t/\varepsilon^3)} \alpha_k^{\delta}(x_{k-1}), \qquad t \geq 0,$$

be the impulsive process defined by the generator of a two component Markov process $\xi^{\varepsilon,\delta}(t)$, $x(t/\varepsilon^3)$, $t \geq 0$,

$$\mathbb{L}^{\varepsilon,\delta}\varphi(u,x) = \varepsilon^{-3} q(x) \int_E P(x,dy) \int_{\mathbb{R}} G_y^{\delta}(dv)[\varphi(u+\varepsilon v, y) - \varphi(u,x)]$$

or

$$\mathbb{L}^{\varepsilon,\delta}\varphi(u,x) = \left[\varepsilon^{-3} Q + Q_0 G_x^{\varepsilon,\delta}\right] \varphi(u,x), \tag{5}$$

where

$$G_x^{\varepsilon,\delta}\varphi(u) := \varepsilon^{-3} \int_{\mathbb{R}} G_x^{\delta}(dv)[\varphi(u+\varepsilon v) - \varphi(u)]. \tag{6}$$

Then the solution of the large deviations problem for $\xi^{\varepsilon,\delta}(t)$ *is determined by the exponential generator*

$$H^0\varphi(u) = (\widetilde{a} - \widetilde{a}_0)\varphi'(u) + \frac{1}{2}\sigma^2(x)(\varphi'(u))^2 + \int_{\mathbb{R}} \left[e^{v\varphi'(u)} - 1\right] \widetilde{G}^0(dv), \tag{7}$$

where

$$\widetilde{a} = \Pi a(x) := q \int_E \rho(dx) a(x), \qquad \widetilde{a}_0 = \Pi a_0(x) := q \int_E \rho(dx) a_0(x),$$

$$a_0(x) = \int_{\mathbb{R}} v G^0(dv), \qquad \widetilde{c} = \Pi c(x) := q \int_E \rho(dx) c(x),$$

$$\widetilde{c}_0 = \Pi c_0(x) := q \int_E \rho(dx) c_0(x), \qquad c_0(x) = \int_{\mathbf{R}} v^2 G^0(dv),$$

$$\sigma^2 = (\widetilde{c} - \widetilde{c}_0) + 2q \int_E \rho(dx) a_1(x) R_0 a_1(x), \qquad \widetilde{G}^0(v) = \Pi G^0(v) := q \int_E \rho(dx) G^0(v).$$

The averaging is done with respect to the stationary measure of the embedded Markov chain of the switching Markov process.

Remark 3.1. The limit generator in the Euclidean space $\mathbb{R}^d$, $d > 1$, is given by

$$H^0\varphi(u) = \sum_{k=1}^{d} (\widetilde{a}_k - \widetilde{a}_k^0)\varphi_k' + \frac{1}{2} \sum_{k,r=1}^{d} \sigma_{k,r}\varphi_k'\varphi_r' + \int_{\mathbb{R}^d} \left[e^{v\varphi'(u)} - 1\right] \widetilde{G}^0(dv),$$

$$\varphi_k' := \partial\varphi(u)/\partial u_k, \qquad 1 \leq k \leq d.$$

Here $\sigma^2 = [\sigma_{kr}; 1 \leq k, r \leq d]$ is the variation matrix.

Moreover, the latter exponential generator can be extended to the space of absolutely continuous functions

$$C_b^1(\mathbb{R}^d) = \left\{\varphi\colon \exists \lim_{|u|\to\infty} \varphi(u) = \varphi(\infty),\ \lim_{|u|\to\infty} \varphi'(u) = 0\right\}$$

(see [2]).

Proof. Passing to the limit for the exponential nonlinear operator can be justified for test functions

$$\varphi_\varepsilon^\delta(u,x) = \varphi(u) + \varepsilon \ln\left[1 + \delta\varphi_1(u,x) + \delta^2\varphi_2(u,x)\right],$$

where $\varphi(u) \in C^3(\mathbb{R})$ ($C^3(\mathbb{R})$ is the space of continuous bounded functions with continuous bounded derivatives up to the third order).

Considering the generator in (5) we obtain

$$\begin{aligned}\mathbb{H}^{\varepsilon,\delta}\varphi_\varepsilon^\delta &= e^{-\varphi_\varepsilon^\delta/\varepsilon}\varepsilon\mathbb{L}^{\varepsilon,\delta}e^{\varphi_\varepsilon^\delta/\varepsilon} = e^{-\varphi_\varepsilon^\delta/\varepsilon}\left[\varepsilon^{-2}Q + \varepsilon Q_0 G_x^{\varepsilon,\delta}\right]e^{\varphi_\varepsilon^\delta/\varepsilon}\\ &= e^{-\varphi/\varepsilon}\left[1+\delta\varphi_1+\delta^2\varphi_2\right]^{-1}\left[\varepsilon^{-2}Q + \varepsilon Q_0 G_x^{\varepsilon,\delta}\right]e^{\varphi/\varepsilon}\left[1+\delta\varphi_1+\delta^2\varphi_2\right].\end{aligned}$$

The further proof is based on the following auxiliary results.

Lemma 3.1. *The exponential generator*

$$H_Q^\varepsilon\varphi_\varepsilon^\delta(u,x) = e^{-\varphi_\varepsilon^\delta/\varepsilon}\varepsilon^{-2}Qe^{\varphi_\varepsilon^\delta/\varepsilon} \tag{8}$$

admits the representation

$$H_Q^\varepsilon\varphi_\varepsilon^\delta = \varepsilon^{-1}Q\varphi_1 + Q\varphi_2 - \varphi_1 Q\varphi_1 + \theta_Q^{\varepsilon,\delta}(x), \tag{9}$$

where $\sup_{x\in E}\left|\theta_Q^{\varepsilon,\delta}(x)\right| \to 0$ *as* $\varepsilon,\delta\to 0$.

Proof. We have

$$\begin{aligned}H_Q^\varepsilon\varphi_\varepsilon^\delta &= e^{-\varphi/\varepsilon}\left[1+\delta\varphi_1+\delta^2\varphi_2\right]^{-1}\varepsilon^{-2}Qe^{\varphi/\varepsilon}\left[1+\delta\varphi_1+\delta^2\varphi_2\right]\\ &= \left[1-\delta\varphi_1+\delta^2\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}\right]\left[\delta\varepsilon^{-2}Q\varphi_1+\delta^2\varepsilon^{-2}Q\varphi_2\right]\\ &= \delta\varepsilon^{-2}Q\varphi_1+\delta^2\varepsilon^{-2}Q\varphi_2-\delta^2\varepsilon^{-2}\varphi_1Q\varphi_1+\theta_Q^{\varepsilon,\delta}(x),\end{aligned}$$

where

$$\theta_Q^{\varepsilon,\delta}(x) = \delta^3\varepsilon^{-2}\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}[Q\varphi_1+\delta Q\varphi_2] - \delta^3\varepsilon^{-2}\varphi_1Q\varphi_2.$$

Now we get representation (9) in view of the limit condition $\varepsilon^{-1}\delta \to 1$ as $\varepsilon,\delta\to 0$. Lemma 3.1 is proved. □

Lemma 3.2. *The exponential generator*

$$H_G^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta(u,x) = e^{-\varphi_\varepsilon^\delta/\varepsilon}\varepsilon Q_0 G_x^{\varepsilon,\delta}e^{\varphi_\varepsilon^\delta/\varepsilon} \tag{10}$$

admits the asymptotic representation

$$H_G^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta = H_G(x)\varphi(u) + \varepsilon^{-1}a_1(x)\varphi'(u) + \theta_G^{\varepsilon,\delta}(x),$$

where

$$\begin{aligned}H_G(x)\varphi(u) &= Q_0\mathcal{H}_G(x)\\ &:= Q_0\Bigg[(a(x)-a_0(x))\varphi'(u) + \frac{1}{2}(c(x)-c_0(x))(\varphi'(u))^2\\ &\qquad + \int_{\mathbb{R}}\left[e^{v\varphi'(u)}-1\right]G^0(dv)\Bigg]\end{aligned} \tag{11}$$

and $\sup_{x\in E}|\theta_G^{\varepsilon,\delta}(x)| \to 0$ *as* $\varepsilon,\delta\to 0$.

Proof. We have

$$
\begin{aligned}
H_G^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta &= e^{-\varphi/\varepsilon}\left[1+\delta\varphi_1+\delta^2\varphi_2\right]^{-1}\varepsilon Q_0 G_x^{\varepsilon,\delta} e^{\varphi/\varepsilon}\left[1+\delta\varphi_1+\delta^2\varphi_2\right] \\
&= e^{-\varphi/\varepsilon}\left[1-\delta\varphi_1+\delta^2\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2}\right] \\
&\quad\times\left[\varepsilon Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}+\varepsilon\delta Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_1+\varepsilon\delta^2 Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_2\right] \\
&= H_G^{\varepsilon,\delta}(x)\varphi(u)+e^{-\varphi/\varepsilon}\varepsilon\delta\left[Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_1-\varphi_1 Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\right]+\widetilde{\theta}_\Gamma^{\varepsilon,\delta}(x),
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\widetilde{\theta}_\Gamma^{\varepsilon,\delta}(x) &= \varepsilon\delta^2\left[e^{-\varphi/\varepsilon}Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_2-e^{-\varphi/\varepsilon}\varphi_1 Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_1\right] \\
&\quad+\varepsilon\delta^2\frac{\varphi_1^2+\delta\varphi_1\varphi_2-\varphi_2}{1+\delta\varphi_1+\delta^2\varphi_2} \\
&\quad\times\left[e^{-\varphi/\varepsilon}Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}+e^{-\varphi/\varepsilon}\delta Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_1+e^{-\varphi/\varepsilon}\delta^2 Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_2\right] \\
&\quad-\varepsilon\delta^3 e^{-\varphi/\varepsilon}\varphi_1 Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}\varphi_2.
\end{aligned}
$$

To complete the proof we need the following two auxiliary results.

Lemma 3.3.

$$
Q_0 G_x^{\varepsilon,\delta}e^{\varphi(u)/\varepsilon}\varphi_1(u,x)=\varphi_1(u,x)Q_0 G_x^{\varepsilon,\delta}e^{\varphi(u)/\varepsilon}+(\varepsilon\delta)^{-1}\widehat{\theta}_\Gamma^{\varepsilon,\delta}(x)
$$

with the asymptotically negligible term

$$
\sup_{x\in E}\left|\widehat{\theta}_\Gamma^{\varepsilon,\delta}(x)\right|\to 0,\qquad \varepsilon,\delta\to 0.
$$

Proof. According to (6),

$$
\begin{aligned}
&Q_0 G_x^{\varepsilon,\delta}e^{\varphi(u)/\varepsilon}\varphi_1(u,x) \\
&\quad=\varepsilon^{-3}Q_0\int_{\mathbb{R}}\left[e^{\varphi(u+\varepsilon v)/\varepsilon}\varphi_1(u+\varepsilon v,x)-e^{\varphi(u)/\varepsilon}\varphi_1(u,x)\right]G_x^\delta(dv) \\
&\quad=\varphi_1(u,x)Q_0 G_x^{\varepsilon,\delta}e^{\varphi(u)/\varepsilon}+(\varepsilon\delta)^{-1}\left[\varphi_1'(u,x)\varepsilon^{-1}\delta\int_{\mathbb{R}}e^{\varphi(u+\varepsilon v)/\varepsilon}v\,G_x^\delta(dv)\right].
\end{aligned}
$$

We estimate the latter integral from above. Since the function $\varphi(u)$ is bounded, we get, for all fixed ε,

$$
\int_{\mathbb{R}}e^{\varphi(u+\varepsilon v)/\varepsilon}v\,G_x^\delta(dv)<e^C\int_{\mathbb{R}}v\,G_x^\delta(dv)=\delta e^C\left[a_1(x)+\delta a(x)+\delta\theta_a^\delta(x)\right].
$$

We see that the latter term is asymptotically negligible under the condition that $\varepsilon,\delta\to 0$.

Lemma 3.3 is proved. □

Lemma 3.4. *The exponential generator*

$$
H_G^{\varepsilon,\delta}(x)\varphi(u)=e^{-\varphi/\varepsilon}\varepsilon Q_0 G_x^{\varepsilon,\delta}e^{\varphi/\varepsilon}
\tag{13}
$$

admits the asymptotic representation

$$
H_G^{\varepsilon,\delta}(x)\varphi(u)=H_G(x)\varphi(u)+\varepsilon^{-1}a_1(x)\varphi'(u)+\theta^{\varepsilon,\delta}(x),
$$

where $\sup_{x\in E}|\theta^{\varepsilon,\delta}(x)|\to 0$ *as* $\varepsilon,\delta\to 0$.

Proof. We rewrite (13) by using the definition of a generator (6). We have

$$H_G^{\varepsilon,\delta}(x)\varphi(u) = \varepsilon^{-2}Q_0 \int_{\mathbb{R}} \left[e^{\Delta_\varepsilon \varphi(u)} - 1\right] G_x^\delta(dv),$$

where

$$\Delta_\varepsilon \varphi(u) := \varepsilon^{-1}[\varphi(u+\varepsilon v) - \varphi(u)].$$

The expression for the generator is rewritten as follows:

$$\begin{aligned} H_G^{\varepsilon,\delta}(x)\varphi(u) = \varepsilon^{-2}Q_0 \int_{\mathbb{R}} &\left[e^{\Delta_\varepsilon \varphi(u)} - 1 - \Delta_\varepsilon\varphi(u) - \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2\right] G_x^\delta(dv) \\ &+ \varepsilon^{-2}Q_0 \int_{\mathbb{R}} \left[\Delta_\varepsilon\varphi(u) + \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2\right] G_x^\delta(dv). \end{aligned}$$

It is easy to see that the function $\psi_u^\varepsilon(v) = e^{\Delta_\varepsilon\varphi(u)} - 1 - \Delta_\varepsilon\varphi(u) - \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2$ belongs to the class $C_3(\mathbb{R})$. Indeed,

$$\psi_u^\varepsilon(v)/v^2 \to 0, \qquad v \to 0.$$

Moreover, this function is continuous and bounded for all ε if the function $\varphi(u)$ is bounded. Further, the function $\psi_u^\varepsilon(v)$ is uniformly bounded with respect to u under assumptions **C3** and **C4** if the derivative $\varphi'(u)$ is bounded.

Thus

$$\begin{aligned} H_G^{\varepsilon,\delta}(x)\varphi(u) = \varepsilon^{-2}\delta^2 Q_0 \int_{\mathbb{R}} &\left[e^{\Delta_\varepsilon\varphi(u)} - 1 - \Delta_\varepsilon\varphi(u) - \frac{1}{2}(\Delta_\varepsilon\varphi(u))^2\right] G_x^0(dv) \\ &+ \varepsilon^{-2}Q_0 \int_{\mathbb{R}} \left[\Delta_\varepsilon\varphi(u) - v\varphi'(u) - \varepsilon\frac{v^2}{2}\varphi''(u)\right] G_x^\delta(dv) \\ &+ \varepsilon^{-2}\delta Q_0 a_1(x)\varphi'(u) + \varepsilon^{-2}\delta^2 Q_0 a(x)\varphi'(u) + \varepsilon^{-1}\delta^2 Q_0 c(x)\varphi''(u) \\ &+ \varepsilon^{-2}Q_0 \int_{\mathbb{R}} \left[\frac{1}{2}(\Delta_\varepsilon\varphi(u))^2 - \frac{v^2}{2}(\varphi'(u))^2\right] G_x^\delta(dv) \\ &+ \varepsilon^{-2}\delta^2\frac{1}{2}Q_0 c(x)(\varphi'(u))^2. \end{aligned}$$

Applying the Taylor formula to the test function $\varphi(u) \in C^3(\mathbb{R})$ and using condition **LA2** we get

$$\begin{aligned} H_G^{\varepsilon,\delta}(x)\varphi(u) = \varepsilon^{-2}\delta^2 Q_0 \int_{\mathbb{R}} &\left[e^{v\varphi'(u)} - 1 - v\varphi'(u) - \frac{v^2}{2}(\varphi'(u))^2\right] G_x^0(dv) \\ &+ \varepsilon^{-2}\delta^2 Q_0 \int_{\mathbb{R}} \left(e^{v\varphi'(u)}\varepsilon\frac{v^2}{2}\varphi''(\widetilde{u}) - \varepsilon\frac{v^2}{2}\varphi''(\widetilde{u}) - \varepsilon^2\frac{v^4}{8}(\varphi''(\widetilde{u}))^2\right) G_x^0(dv) \\ &+ \varepsilon^{-2}\delta^2 Q_0 \int_{\mathbb{R}} \varepsilon^2\frac{v^3}{3!}\varphi'''(\widetilde{u})\, G_x^0(dv) \\ &+ \varepsilon^{-2}\delta Q_0 a_1(x)\varphi'(u) + \varepsilon^{-2}\delta^2 Q_0 a(x)\varphi'(u) + \varepsilon^{-1}\delta^2 Q_0 c(x)\varphi''(u) \\ &+ \varepsilon^{-2}\delta^2 Q_0 \int_{\mathbb{R}} \varepsilon^2\frac{v^4}{4}(\varphi''(\widetilde{u}))^2\, G_x^0(dv) + \varepsilon^{-2}\delta^2\frac{1}{2}Q_0 c(x)(\varphi'(u))^2. \end{aligned}$$

Taking into account the limit condition $\varepsilon^{-1}\delta \to 1$ we finally get

$$H_G^{\varepsilon,\delta}(x)\varphi(u) = H_G(x)\varphi(u) + \theta_G^{\varepsilon,\delta}(x),$$

where $\sup_{x\in E}\left|\theta_G^{\varepsilon,\delta}(x)\right| \to 0$ as $\varepsilon, \delta \to 0$.

Lemma 3.4 is proved. □

Applying the relations proved in Lemmas 3.3 and 3.4 in equality (12) we complete the proof of Lemma 3.2.

Lemma 3.2 is proved. □

We see from (8) and (10) that

$$\mathbb{H}^{\varepsilon,\delta}\varphi_\varepsilon^\delta = H_Q^\varepsilon\varphi_\varepsilon^\delta(u,x) + H_G^{\varepsilon,\delta}(x)\varphi_\varepsilon^\delta(u,x).$$

Now we derive the following asymptotic representation from Lemmas 3.1 and 3.2:

$$\mathbb{H}^{\varepsilon,\delta}\varphi_\varepsilon^\delta = \varepsilon^{-1}[Q\varphi_1 + a_1(x)\varphi'(u)] + Q\varphi_2 - \varphi_1 Q\varphi_1 + H_G(x)\varphi(u) + h^{\varepsilon,\delta}(x),$$

where $h^{\varepsilon,\delta}(x) = \theta_Q^{\varepsilon,\delta}(x) + \theta_G^{\varepsilon,\delta}(x)$.

Now we are in a position to use the solution of the problem of singular perturbation to the reducibly-invertible operator Q (see [7, Chapter 1]):

$$Q\varphi_1 + a_1(x)\varphi'(u) = 0,$$
$$Q\varphi_2 - \varphi_1 Q\varphi_1 + H_G(x)\varphi(u) = H^0\varphi(u).$$

The first equation implies that

$$Q\varphi_1(u,x) = -a_1(x)\varphi'(u), \qquad \varphi_1(u,x) = R_0 a_1(x)\varphi'(u)$$

in view of the balance condition **LA3**. Substituting the result to the second equation we get

$$Q\varphi_2 + a_1(x)R_0 a_1(x)(\varphi'(u))^2 + H_G(x)\varphi(u) = H^0\varphi(u).$$

The solvability condition implies that

$$H^0\varphi(u) = \Pi H_G(x)\Pi\varphi(u) + \Pi a_1(x)R_0 a_1(x)(\varphi'(u))^2.$$

Using (11) and Remark 1.1 the right hand side of the latter equality can be rewritten as

$$H^0\varphi(u) = \int_E \pi(dx)q(x)\int_E P(x,dy)\mathcal{H}_G(y)\varphi(u) = q\int_E \rho(dx)\mathcal{H}_G(x)\varphi(u),$$

whence we derive equality (7):

$$H^0\varphi(u) = (\widetilde{a} - \widetilde{a}_0)\varphi'(u) + \frac{1}{2}\sigma^2(x)(\varphi'(u))^2 + \int_{\mathbb{R}}\left[e^{v\varphi'(u)} - 1\right]\widetilde{G}^0(dv).$$

The remainder term $h^{\varepsilon,\delta}(x)$ can be evaluated explicitly by using the solution of the Poisson equation

$$\varphi_1(u,x) = R_0\widetilde{H}(x)\varphi(u), \qquad \widetilde{H}(x) := H_G(x) - H^0$$

(see Remark 1.2; more detail is given in [7]).

The lemma is proved. □

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