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LIMIT BEHAVIOR OF FUNCTIONALS OF SOLUTIONS OF DIFFUSION TYPE EQUATIONS

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Abstract. The asymptotic behavior as $T \to \infty$ of the functionals $I(tT)$ with an appropriate normalizing factor is studied, where $I(t) = F(\xi(t)) + \int_0^t g(\xi(s))\,dW(s)$, $t \geq 0$, F is a continuous function, g is a locally square integrable function, ξ is an unstable solution of the Itô stochastic differential equation $d\xi(t) = a(\xi(t))\,dt + dW(t)$, and a is a measurable and bounded function. We find the normalizing factor for the weak convergence of stochastic processes $I(tT)$, $t \geq 0$, for certain classes of these equations. The explicit form of the limit processes is established.

1. Introduction

Consider the one-dimensional Itô stochastic differential equation

$$d\xi(t) = a(\xi(t))\,dt + dW(t), \qquad t \geq 0,\ \xi(0) = x_0, \tag{1}$$

where $a = a(x)\colon \mathbb{R} \to \mathbb{R}$ is a measurable bounded function and $W = \{W(t), t \geq 0\}$ is a standard Wiener process defined in a complete probability space $(\Omega, \Im, \mathsf{P})$.

It is shown in the paper [1] that equation (1) possesses a unique strong pathwise solution $\xi = \{\xi(t), t \geq 0\}$ and that this solution is a homogeneous strong Markov process (see [2, §15]).

Definition 1.1. A solution ξ of equation (1) is called unstable if

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \mathsf{P}\{|\xi(s)| < N\}\,ds = 0$$

for an arbitrary constant $N > 0$.

It is known [2] that the asymptotic behavior of solutions of stochastic differential equations is closely related to the behavior as $|x| \to \infty$ of the function

$$f(x) = \int_0^x \exp\left\{-2\int_0^u a(v)\,dv\right\} du. \tag{2}$$

Lemma 4.2 below provides a sufficient condition for a solution ξ of equation (1) to be unstable. This condition is related to the behavior at infinity of the function f defined by equality (2).

We introduce two classes of equations (1), K_1 and K_2, described in terms of the behavior at infinity of the coefficient a,

$$K_1\colon \quad \left|\int_0^x a(u)\,du\right| \leq C \quad \text{for all } x \in \mathbb{R}, \qquad \lim_{|x|\to\infty} \left[\frac{1}{f(x)} \int_0^x \frac{du}{f'(u)} - \frac{1}{\bar{\sigma}^2(x)}\right] = 0,$$

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where

$$\bar{\sigma}(x) = \begin{cases} \sigma_1, & x > 0, \\ \sigma_2, & x < 0, \end{cases} \qquad 0 < \sigma_i < \infty,$$

$$K_2\colon \quad |xa(x)| \le C \quad \text{for all } x \in \mathbb{R}, \qquad \lim_{|x|\to\infty} \frac{1}{x}\int_0^x va(v)\,dv = c_0 > -\frac{1}{2}.$$

If a solution ξ of equation (1) is in the class K_1, then there are constants δ and C such that $0 < \delta < f'(x) < C$ for all x. Thus a solution ξ of equation (1) of the class K_1 is unstable by Lemma 4.2. Also, it is proved in [3] that solutions of equations (1) of the class K_2 are unstable.

It is also known (see [3]) that the process $\zeta_T(t) = f(\xi(tT))/\sqrt{T}$, $T > 0$, weakly converges as $T \to \infty$ to the process $\zeta = \{\zeta(t), t \ge 0\}$ being a unique solution of the equation

$$\zeta(t) = \int_0^t \bar{\sigma}(\zeta(s))\,dW(s), \tag{3}$$

where ξ is a solution of the equation of the class K_1. Moreover,

$$\int_0^t \mathsf{P}\{|\zeta(s)| = 0\}\,ds = 0.$$

On the other hand, if $\xi(t)$ is a solution of an equation of the class K_2, then the process $r_T(t) = |\xi(tT)|/\sqrt{T}$ weakly converges as $T \to \infty$ to the process $r = \{r(t) \ge 0, t \ge 0\}$ being a unique strong solution of the Itô equation

$$r^2(t) = (2c_0 + 1)t + 2\int_0^t r(s)\,d\widehat{W}(s). \tag{4}$$

In addition, the transient densities of the Markov processes ζ and r are known; the explicit expressions for them can be found in [9].

Remark 1.1. Here and throughout the paper the weak convergence of the processes means the weak convergence in the uniform topology of the space of continuous functions.

In the current paper, we consider the asymptotic behavior as $t \to \infty$ of the distributions of the functionals

$$I(t) = F(\xi(t)) + \int_0^t g(\xi(s))\,dW(s), \tag{5}$$

where $F = F(x)\colon \mathbb{R} \to \mathbb{R}$ is a real valued and continuous function, $g = g(x)\colon \mathbb{R} \to \mathbb{R}$ is a locally square integrable function, the processes ξ and W are related to each other via equation (1), and equation (1) belongs either to the class K_1 or K_2.

This paper is a continuation of [4]–[9]. The behavior of the distributions of functionals

$$\beta^{(1)}(t) = \int_0^t g(\xi(s))\,ds, \qquad \beta^{(2)}(t) = \int_0^t g(\xi(s))\,dW(s)$$

is studied in [4]–[7] for equations (1) of the class K_1. A similar problem for the functionals $\beta^{(1)}(t)$ and $\beta^{(2)}(t)$ is considered in [8, 9] for equations (1) of the class K_2. The behavior of the distributions of the functional $I(t)$ for a Wiener process ($\xi = W$) is studied in [10]. A detailed survey of the literature can be found in [8, 9].

We prove in Lemma 4.1 below that $\beta^{(1)}(t)$ and $\beta^{(2)}(t)$, as well as

$$\beta(t) = \int_0^t g(\xi(s))\,d\xi(s),$$

are particular cases of the functional $I(t)$.

The paper is organized as follows. Section 2 contains the statements of the main results. The proof of results of Section 2 is given in Section 3. Some examples are included in Section 3 as well. Auxiliary results are collected in Section 4.

Throughout the paper, Ψ denotes the class of functions $\psi = \psi(x) > 0$, $x \geq 0$, being nondecreasing and regularly varying at infinity of order $\alpha \geq 0$, that is,

$$\lim_{T\to\infty} \psi(|x|T)/\psi(T) = |x|^\alpha$$

for all $x \neq 0$.

2. Statement of the main results

Theorem 2.1. *Let ξ be a solution of equation* (1) *of the class K_1. Let two functions F and g define the functional $I(t)$ by equality* (5) *and let $\psi \in \Psi$ be a regularly varying function of order $\alpha \geq 0$. Assume that*

$$(A_1) \quad \lim_{|x|\to\infty} \left[\frac{F(x)}{f(x)\psi(|f(x)|)} - \bar{a}(x)\right] = 0, \qquad \bar{a}(x) = \begin{cases} a_1, & x > 0, \\ a_2, & x < 0; \end{cases}$$

$$(A_2) \quad \left|\frac{g(x)}{f'(x)\psi(|f(x)|)}\right| \chi_{\{|x|\leq N\}} \leq K_N$$

for an arbitrary $N > 0$ and

$$\lim_{|x|\to\infty} \left[\frac{g(x)}{f'(x)\psi(|f(x)|)} - \bar{b}(x)\right] = 0, \qquad \bar{b}(x) = \begin{cases} b_1, & x > 0, \\ b_2, & x < 0, \end{cases}$$

for some constants a_i and b_i, $i = 1, 2$.

Then the stochastic process

$$I_T(t) = \frac{I(tT)}{\sqrt{T}\psi(\sqrt{T})}$$

weakly converges as $T \to \infty$ to the process

$$I_0(t) = \bar{a}(\zeta(t))\zeta(t)|\zeta(t)|^\alpha + \int_0^t \bar{b}(\zeta(s))|\zeta(s)|^\alpha \, d\zeta(s),$$

where $\zeta = \zeta(t)$ is a solution of equation (3).

Remark 2.1. 1) If $\alpha = 0$, $\bar{a}(x) = a_0 \operatorname{sign} x$, and $\bar{b}(x) = -a_0 \operatorname{sign} x$, then

$$I_0(t) = a_0 \left[|\zeta(t)| - \int_0^t \operatorname{sign} \zeta(s) \, d\zeta(s)\right] = a_0 L_\zeta(0, t),$$

where $L_\zeta(0, t)$ denotes the local time at point 0 in the interval $[0, t]$ (see [11]) for the process ζ.

2) If $\alpha = 1$, $\bar{a}(x) = a_0 \operatorname{sign} x$, and $\bar{b}(x) = -2a_0 \operatorname{sign} x$, then the Itô formula for the process $\zeta^2(t)$ yields

$$I_0(t) = a_0 \int_0^t \bar{\sigma}^2 (\zeta(s)) \, ds,$$

and $I_0(t) = a_0\sigma_0^2 t$ for $\sigma_1 = \sigma_2 = \sigma_0$.

Theorem 2.2. *Let ξ be a solution of equation* (1) *of the class K_1. Let two functions F and g define the functional $I(t)$ by equality* (5) *and let $\psi_i \in \Psi$ be a regularly varying*

function of order $\alpha_i \geq 0$, $i = 1, 2$. *Assume that*

$$(A_3) \quad \lim_{|x|\to\infty} \left[\frac{F(x)}{\sqrt{|f(x)|\,\psi_1(|f(x)|)}} - \bar{a}(x) \right] = 0, \qquad \bar{a}(x) = \begin{cases} a_1, & x > 0, \\ a_2, & x < 0; \end{cases}$$

$$(A_4) \quad \lim_{|x|\to\infty} \left[\frac{1}{\psi_1(|f(x)|)} \int_0^x \frac{g^2(|v|)}{f'(v)}\, dv - \bar{b}(x) \right] = 0,$$

$$\bar{b}(x) = \begin{cases} b_1, & x > 0, \\ b_2, & x < 0, \end{cases} \qquad b_1 \geq 0,\ b_2 \leq 0;$$

$$(A_5) \quad \left| \frac{1}{\psi_2(|f(x)|)} \int_0^x \frac{|g(v)|}{f'(v)}\, dv \right| \leq C, \qquad \lim_{T\to\infty} \frac{\psi_2(\sqrt{T})}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}} = 0$$

for some constants a_i *and* b_i, $i = 1, 2$.

Then the stochastic process

$$I_T(t) = \frac{I(tT)}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}}$$

weakly converges as $T \to \infty$ *to the process*

$$I_0^*(t) = \bar{a}(\zeta(t))|\zeta(t)|^{\frac{1+\alpha_1}{2}} + W^*\left(\bar{\beta}^{(1)}(t)\right),$$

$$\bar{\beta}^{(1)}(t) = 2\left[\int_0^{\zeta(t)} \bar{b}(x)|x|^{\alpha_1} dx - \int_0^t \bar{b}(\zeta(s))|\zeta(s)|^{\alpha_1} d\zeta(s) \right],$$

where ζ *is a solution of equation* (3), $W^* = \{W^*(t), t \geq 0\}$ *is a Wiener process, and the processes* W^* *and* ζ *are independent.*

It follows from the proof of Theorem 2.2 (see relation (16)) that the process $\bar{\beta}^{(1)}(t)$ is nonnegative.

Remark 2.2. 1) If $\alpha_1 = 0$, $\bar{b}(x) = b_0 \operatorname{sign} x$, and $b_0 \geq 0$, then

$$I_0^*(t) = \bar{a}(\zeta(t))\sqrt{|\zeta(t)|} + W^*\left(2b_0 L_\zeta(0,t)\right),$$

where $L_\zeta(0,t)$ is the local time at point 0 in the interval $[0, t]$ for the process ζ, W^* is a Wiener process, and the processes W^* and ζ are independent.

2) If $\alpha_1 = 1$, $\bar{b}(x) = b_0 \operatorname{sign} x$, and $b_0 \geq 0$, then the Itô formula yields

$$I_0^*(t) = \bar{a}(\zeta(t))|\zeta(t)| + W^*\left(b_0 \int_0^t \bar{\sigma}^2\left(\zeta(s)\right) ds\right),$$

where the processes W^* and ζ are independent. In particular, if $\sigma_1 = \sigma_2 = \sigma_0$, then

$$I_0^*(t) = \bar{a}(\zeta(t))|\zeta(t)| + W^*\left(b_0\sigma_0^2 t\right).$$

3) If $\bar{b}(x) \equiv 0$, then $I_0^*(t) = \bar{a}(\zeta(t))|\zeta(t)|^{\frac{1+\alpha_1}{2}}$, $\alpha_1 \geq 0$.

Theorem 2.3. *Let* ξ *be a solution of equation* (1) *of the class* K_2. *Let two functions* F *and* g *define the functional* $I(t)$ *by equality* (5) *and let* $\psi \in \Psi$ *be a regularly varying function of order* $\alpha > 0$. *Assume that*

$$(B_1) \quad \lim_{|x|\to\infty} \frac{F(x)}{|x|\psi(|x|)} = a_0;$$

$$(B_2) \quad \textit{the function} \quad \frac{g(x)}{\psi(|x|)} \quad \textit{is bounded and} \quad \lim_{|x|\to\infty} \left[\frac{g(x)}{\psi(|x|)} - b_0 \operatorname{sign} x \right] = 0,$$

where a_0 *and* b_0 *are some constants.*

Then the stochastic process

$$I_T(t) = \frac{I(tT)}{\sqrt{T}\psi(\sqrt{T})}$$

weakly converges as $T \to \infty$ *to the process*

$$I_0(t) = a_0 r^{\alpha+1}(t) + b_0 \int_0^t r^{\alpha}(s)\, d\widehat{W}(s),$$

where r *and the Wiener process* $\widehat{W} = \{\widehat{W}(t), t \geq 0\}$ *are related to each other via equation* (4).

Remark 2.3. If $\alpha = 1$, then

$$I_0(t) = \left(a_0 + \frac{b_0}{2}\right) r^2(t) - \frac{b_0}{2}\,(2c_0 + 1)\, t.$$

In particular, $I_0(t) = a_0(2c_0 + 1)t$ for $a_0 = -b_0/2$.

Theorem 2.4. *Let* ξ *be a solution of equation* (1) *of the class* K_2. *Let two functions* F *and* g *define the functional* $I(t)$ *by equality* (5). *Assume that there are two regularly varying functions* $\psi_i \in \Psi$ *of order* $\alpha_i > 0$, $i = 1, 2$, *and an odd locally square integrable function* $\hat{g} = \hat{g}(x)\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$(B_3) \qquad \lim_{|x|\to\infty} \frac{F(x)}{\sqrt{|x|\psi_1(|x|)}} = a_0,$$

$$(B_4) \qquad \lim_{|x|\to\infty} \left[\frac{f'(x)}{\psi_1(|x|)} \int_0^x \frac{\hat{g}^2(u)}{f'(u)}\, du - b_0 \operatorname{sign} x\right] = 0,$$

$$(B_5) \qquad \lim_{|x|\to\infty} \left[\frac{f'(x)}{\psi_1(|x|)} \int_0^x \frac{[g(u) - \hat{g}(u)]^2}{f'(u)}\, du\right] = 0,$$

$$(B_6) \qquad \left|\frac{f'(x)}{\psi_2(|x|)} \int_0^x \frac{|g(u)|}{f'(u)}\, du\right| \leq C, \qquad \lim_{T\to\infty} \frac{\psi_2(\sqrt{T})}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}} = 0$$

for some constants a_0 *and* b_0.

Then the stochastic process

$$I_T(t) = \frac{I(tT)}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}}$$

weakly converges as $T \to \infty$ *to the process*

$$I_0^*(t) = a_0\, [r(t)]^{\frac{1+\alpha_1}{2}} + W^*\left(\hat{\beta}^{(1)}(t)\right),$$

where

$$\hat{\beta}^{(1)}(t) = 2b_0 \left[\frac{r^{\alpha_1+1}(t)}{\alpha_1 + 1} - \int_0^t r^{\alpha_1}(s)\, d\widehat{W}(s)\right],$$

r *is a solution of equation* (4), W^* *is a Wiener process, and the processes* W^* *and* $\widehat{W}$ *are independent.*

Remark 2.4. Condition (B_6) in Theorem 2.4 is used only to prove that W^* and $\widehat{W}$ are independent. If $\alpha_1 = 1$ and $a_0 = 0$, then one can omit condition (B_6) and get $I_0^*(t) = \sqrt{b_0\,(2c_0 + 1)}W^*(t)$.

3. Proof of the main results

Let $T > 0$ be a parameter. Throughout the rest of this paper we use this parameter for the following notation:

$$\zeta_T(t) = \frac{f(\xi(tT))}{\sqrt{T}},$$

$$W_T(t) = \frac{W(tT)}{\sqrt{T}}, \qquad \widehat{W}_T(t) = \int_0^t \operatorname{sign} \xi(sT)\, dW_T(s),$$

$$r_T(t) = \frac{|\xi(tT)|}{\sqrt{T}}, \qquad t \geq 0,$$

where the function $f(x)$ is defined by (2) and $\varphi(x)$ is the inverse function to $f(x)$.

Note that $W_T(t)$ and $\widehat{W}_T(t)$ are Wiener processes for every T (see, for example, [8]).

Proof of Theorem 2.1. It is obvious that the derivative $f'(x)$ of the function $f(x)$ is continuous and that the second derivative $f''(x)$ exists almost everywhere (with respect to the Lebesgue measure) and is locally integrable. Thus one may apply the Itô formula to the process $f(\xi(t))$, where ξ is a solution of equation (1) (see [13, Chapter II, §10]). With the equality $f'(x)a(x) + \frac{1}{2} f''(x) = 0$ being true almost everywhere (with respect to the Lebesgue measure), we obtain the equation

$$d\zeta_T(t) = f'(\xi(tT))\, dW_T(t).$$

Then the equality $\xi(tT) = \varphi(\zeta_T(t)\sqrt{T})$ implies the following representation:

$$I_T(t) = F_T(\zeta_T(t)) + \int_0^t g_T(\zeta_T(s))\, d\zeta_T(s),$$

where

$$F_T(x) = \frac{F\big(\varphi(x\sqrt{T})\big)}{\sqrt{T}\psi(\sqrt{T})}, \qquad g_T(x) = \frac{g(\varphi(x\sqrt{T}))}{\psi(\sqrt{T}) f'(\varphi(x\sqrt{T}))}.$$

Now we prove the convergence

$$\begin{aligned} &\lim_{T\to\infty} \sup_{|x|\leq N} |F_T(x) - F_0(x)| = 0, \\ &\lim_{T\to\infty} \int_{-N}^{N} |g_T(x) - g_0(x)|^2\, dx = 0 \end{aligned} \tag{6}$$

for an arbitrary number $N > 0$ if conditions (A_1)–(A_2) hold for an equation of the class K_1, where $F_0(x) = \bar{a}(x)x|x|^\alpha$ and $g_0(x) = \bar{b}(x)|x|^\alpha$.

It is clear that $F_T(x) - \alpha_T^{(1)}(x) = \alpha_T^{(2)}(x)$, where

$$\alpha_T^{(1)}(x) = \bar{a}(x\sqrt{T})\, x \frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})},$$

$$\alpha_T^{(2)}(x) = \left[\frac{F\big(\varphi(x\sqrt{T})\big)}{\sqrt{T}\psi(\sqrt{T})} - \bar{a}(x\sqrt{T})\, x \frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} \right].$$

Condition (A_1) implies that, for an arbitrary $\varepsilon > 0$, there exists a constant C_ε such that

$$\left| \frac{F\big(\varphi(x\sqrt{T})\big)}{x\sqrt{T}\psi(|x|\sqrt{T})} - \bar{a}(x\sqrt{T}) \right| < \varepsilon$$

for $|x\sqrt{T}| > C_\varepsilon$. Hence

$$\begin{aligned}\sup_{|x|\le N}\left|\alpha_T^{(2)}(x)\right| &\le \sup_{|x|\le N}\left|\frac{F\big(\varphi(x\sqrt{T})\big)}{\sqrt{T}\psi(\sqrt{T})} - \bar{a}(x\sqrt{T})x\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})}\right|\chi_{\{|x\sqrt{T}|\le C_\varepsilon\}}\\ &\quad + \sup_{|x|\le N}\left|\frac{F\big(\varphi(x\sqrt{T})\big)}{x\sqrt{T}\psi(|x|\sqrt{T})} - \bar{a}(x\sqrt{T})\right||x|\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})}\chi_{\{|x\sqrt{T}|>C_\varepsilon\}}\\ &\le \frac{K_\varepsilon}{\sqrt{T}\psi(\sqrt{T})} + \frac{CNC_N}{\sqrt{T}} + \varepsilon NC_N,\end{aligned} \tag{7}$$

where the constants K_ε and C are such that

$$\left|F\big(\varphi(x\sqrt{T})\big)\right|\chi_{\{|x\sqrt{T}|\le C_\varepsilon\}} \le K_\varepsilon, \qquad |\bar{a}(x\sqrt{T})| \le C,$$

and where the constant C_N is such that

$$\sup_{|x|\le N}\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} \le C_N.$$

To get the result, we multiplied and divided the second term $F\big(\varphi(x\sqrt{T})\big)/\big(\sqrt{T}\psi(\sqrt{T})\big)$ by $x\psi(|x|\sqrt{T})$ and then took out a common multiplier.

Note that the latter property is valid for any function ψ of the class Ψ (see [8, Lemma 3.1]).

Now estimate (7) implies

$$\varlimsup_{T\to\infty}\sup_{|x|\le N}\left|\alpha_T^{(2)}(x)\right| \le \varepsilon NC_N.$$

Since $\varepsilon > 0$ is arbitrary,

$$\lim_{T\to\infty}\sup_{|x|\le N}\left|\alpha_T^{(2)}(x)\right| = 0.$$

Moreover, the estimate

$$\begin{aligned}&\sup_{|x|\le N}\left|\alpha_T^{(1)}(x) - \bar{a}(x)x|x|^\alpha\right|\\ &\quad\le \max\left(|a_1|,|a_2|\right)\sup_{|x|\le N}|x|\left|\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^\alpha\right|\\ &\quad\le \max\left(|a_1|,|a_2|\right)\left\{\delta\sup_{|x|\le\delta}\left|\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^\alpha\right| + N\sup_{0<\delta\le|x|\le N}\left|\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^\alpha\right|\right\}\end{aligned}$$

yields $\sup_{|x|\le N}\left|\alpha_T^{(1)}(x) - \bar{a}(x)x|x|^\alpha\right| \to 0$ as $T\to\infty$.

Now we derive from the inequality

$$\sup_{|x|\le N}\left|F_T(x) - \bar{a}(x)x|x|^\alpha\right| \le \left|\alpha_T^{(2)}(x)\right| + \left|\alpha_T^{(1)}(x) - \bar{a}(x)x|x|^\alpha\right|$$

that

$$\sup_{|x|\le N}\left|F_T(x) - \bar{a}(x)x|x|^\alpha\right| \to 0, \qquad T\to\infty. \tag{8}$$

Moreover, the inequality

$$\int_{-N}^{N} |g_T(x) - g_0(x)|^2\,dx \le 2\int_{-N}^{N} \left| \frac{g\big(\varphi(x\sqrt{T})\big)}{\psi(|x|\sqrt{T})f'\big(\varphi(x\sqrt{T})\big)} - \bar{b}(x) \right|^2 \left(\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} \right)^2 dx$$
$$+ 2C^2 \int_{-N}^{N} \left[\frac{\psi(|x|\sqrt{T})}{\psi(\sqrt{T})} - |x|^\alpha \right]^2 dx,$$

where $C = \max|\bar{b}(x)|$, together with condition (A_2) implies the second convergence in (6).

Using the convergence in (6) we conclude that the stochastic process $I_T(t)$ weakly converges as $T \to \infty$ in the uniform topology of the space of continuous functions to the process

$$I_0(t) = F_0(\zeta(t)) + \int_0^t g_0(\zeta(s))\,d\zeta(s)$$

(see [7]). On the other hand, conditions (6) are necessary for the weak convergence of $I_T(t)$ to $I_0(t)$ (see [7]). □

Example 3.1. Consider equation (1) with $a(x) = \frac{x}{1+x^4}$. The solution of equation (3) is given by $\zeta(t) = \sigma_0 W(t)$, where $\sigma_0 = \exp\{-2\int_0^\infty a(v)\,dv\} = e^{-\pi/2}$.

I) Let $g(x) \to g_0$ as $|x| \to \infty$. Using the relationship between the functionals $\beta(t) = \int_0^t g(\xi(s))\,d\xi(s)$ and $I(t)$ (see Lemma 4.1) one can easily show that all the assumptions of Theorem 2.1 hold for $\psi(|x|) = 1$, $\bar{a}(x) = 2\bar{\alpha}(x)$, and $\bar{b}(x) = g_0/\sigma_0 - 2\bar{\alpha}(x)$, where

$$\bar{\alpha}(x) = \begin{cases} \alpha_1, & x > 0, \\ \alpha_2, & x < 0, \end{cases} \qquad \alpha_1 = \int_0^\infty \frac{g(v)a(v)}{f'(v)}\,dv, \qquad \alpha_2 = \int_0^{-\infty} \frac{g(v)a(v)}{f'(v)}\,dv.$$

Thus $\beta_T(t) = T^{-1/2}\int_0^{tT} g(\xi(s))\,d\xi(s)$ weakly converges as $T \to \infty$ to the process

$$I_0(t) = 2\left[\bar{\alpha}(\zeta(t))\zeta(t) - \int_0^t \bar{\alpha}(\zeta(s))\,d\zeta(s)\right] + \frac{g_0}{\sigma_0}\zeta(t)$$
$$= 2\sigma_0\left[\bar{\alpha}(W(t))W(t) - \int_0^t \bar{\alpha}(W(s))\,dW(s)\right] + g_0 W(t).$$

In particular,

(1) if $\alpha_0 = \alpha_1 = -\alpha_2$, then $I_0(t) = 2\sigma_0\alpha_0 L_W(0,t) + g_0 W(t)$;
(2) if $\alpha_0 = \alpha_1 = \alpha_2$, then $I_0(t) = g_0 W(t)$.

II) Let $I(t) = L_\xi(0,t) = |\xi(t)| - \int_0^t \operatorname{sign}\xi(s)\,d\xi(s)$ be the local time of a solution ξ at point 0 in the interval $[0, t]$. Then Lemma 4.1 implies that $g(x) = -\operatorname{sign} x + \Phi'(x)$ and $F(x) = |x| - \Phi(x) + \Phi(x_0)$, where

$$\Phi(x) = 2\int_0^x f'(u)\left(\int_0^u \frac{a(v)\operatorname{sign} v}{f'(v)}\,dv\right)du.$$

All the assumptions of Theorem 2.1 with $\psi(|x|) = 1$, $\bar{a}(x) = \operatorname{sign} x$, and $\bar{b}(x) = -\operatorname{sign} x$ hold for this functional.

Hence $T^{-1/2}L_\xi(0,tT)$ weakly converges as $T \to \infty$ to the process

$$I_0(t) = L_\zeta(0,t) = \sigma_0 L_W(0,t).$$

Proof of Theorem 2.2. We use the following representation:

$$
\begin{aligned}
I_T(t) &= \frac{I(tT)}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}} = \frac{F\left(\xi(tT)\right)}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}} + \frac{1}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}} \int_0^{tT} g(\xi(s))\, dW(s) \\
&= F_T\left(\zeta_T(t)\right) + \int_0^t g_T\left(\zeta_T(t)\right) dW_T(s),
\end{aligned} \tag{9}
$$

where

$$
F_T(x) = \frac{F\big(\varphi(x\sqrt{T})\big)}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}}, \qquad g_T(x) = \frac{\sqrt[4]{T}}{\sqrt{\psi_1(\sqrt{T})}} g(\varphi(x\sqrt{T})).
$$

It is known that

$$
\begin{gathered}
\lim_{N\to\infty} \overline{\lim_{T\to\infty}} \sup_{0\le t\le L} \mathsf{P}\left\{|\zeta_T(t)| > N\right\} = 0, \\
\lim_{h\to 0} \overline{\lim_{T\to\infty}} \sup_{|t_1-t_2|\le h;\, t_i\le L} \mathsf{P}\left\{|\zeta_T(t_2) - \zeta_T(t_1)| > \varepsilon\right\} = 0
\end{gathered} \tag{10}
$$

for all constants $L > 0$ and $\varepsilon > 0$ (see [3]).

It is clear that similar relations hold for $W_T(t)$ as well. Hence (see [12, Chapter I, §6]) one can use Skorokhod's convergent subsequence principle for the processes $\zeta_T(t)$ and $W_T(t)$. According to this principle, given an arbitrary sequence $T_n' \to \infty$, one can choose a subsequence $T_n \to \infty$, probability space $(\tilde{\Omega}, \tilde{\Im}, \tilde{P})$, and stochastic processes $\big(\tilde{\zeta}_{T_n}(t), \tilde{W}_{T_n}(t)\big)$ and $\big(\tilde{\zeta}(t), \tilde{W}(t)\big)$ defined in this space such that the finite-dimensional distributions of the stochastic processes $\big(\tilde{\zeta}_{T_n}(t), \tilde{W}_{T_n}(t)\big)$ coincide with those of the processes $(\zeta_{T_n}(t), W_{T_n}(t))$, and moreover

$$
\tilde{\zeta}_{T_n}(t) \xrightarrow{\tilde{P}} \tilde{\zeta}(t) \quad \text{and} \quad \tilde{W}_{T_n}(t) \xrightarrow{\tilde{P}} \tilde{W}(t)
$$

for all $t \ge 0$. Following a similar reasoning (see [13, Chapter II, §6]) one can show that the finite-dimensional distributions of the stochastic process $I_{T_n}(t)$ coincide with those of the process

$$
\tilde{I}_{T_n}(t) = \frac{F\left(\varphi\left(\tilde{\zeta}_{T_n}(t)\sqrt{T_n}\right)\right)}{\sqrt{\sqrt{T_n}\psi_1(\sqrt{T_n})}} + \frac{\sqrt[4]{T_n}}{\sqrt{\psi_1(\sqrt{T_n})}} \int_0^t g\left(\varphi\left(\tilde{\zeta}_{T_n}(s)\sqrt{T_n}\right)\right) d\tilde{W}_{T_n}(s). \tag{11}
$$

This allows us to assume without loss of generality that $\xi_{T_n}(t) \xrightarrow{\mathsf{P}} \xi(t)$ and $W_{T_n}(t) \xrightarrow{\mathsf{P}} W(t)$ for all $t \ge 0$ and for an arbitrary sequence $T_n \to \infty$ in (9).

The same reasoning as that used to prove convergence (8) yields

$$
\sup_{|x|\le N} \left| \frac{F\left(\varphi\left(x\sqrt{T_n}\right)\right)}{\sqrt{\sqrt{T_n}\psi_1(\sqrt{T_n})}} - \bar{a}(x)|x|^{\frac{\alpha_1+1}{2}} \right| \to 0 \tag{12}
$$

as $T_n \to \infty$ for all $N > 0$.

Moreover,

$$
\sup_{0\le t\le L} |\zeta_{T_n}(t) - \zeta(t)| \xrightarrow{\mathsf{P}} 0
$$

as $T_n \to \infty$, $\int_0^t \mathsf{P}\left\{|\zeta(s)| = 0\right\} ds = 0$ for all $t \ge 0$, and

$$
\lim_{N\to\infty} \overline{\lim_{T\to\infty}} \mathsf{P}\left\{\sup_{0\le t\le L} |\zeta_T(t)| > N\right\} = 0
$$

for all constants $L > 0$ (see [3]).

The inequality

$$\begin{aligned}
&\mathsf{P}\left\{\sup_{0\le t\le L}\left|F_T\left(\zeta_T(t)\right)-\bar{a}(\zeta_T(t))\left|\zeta_T(t)\right|^{\frac{\alpha_1+1}{2}}\right|>\varepsilon\right\}\\
&\qquad\le \mathsf{P}\left\{\sup_{0\le t\le L}|\zeta_T(t)|>N\right\}\\
&\qquad\quad+\mathsf{P}\left\{\sup_{0\le t\le L}\left|F_T\left(\zeta_T(t)\right)-\bar{a}(\zeta_T(t))\left|\zeta_T(t)\right|^{\frac{\alpha_1+1}{2}}\right|\chi_{\{|\zeta_T(t)|\le N\}}>\frac{\varepsilon}{2}\right\}
\end{aligned}$$

holds for all $\varepsilon>0$, $L>0$, and $N>0$, where $F_T(x)$ is defined in (9). Thus convergence (12) implies

$$\sup_{0\le t\le L}\left|\frac{F\left(\varphi\left(\zeta_{T_n}(t)\sqrt{T_n}\right)\right)}{\sqrt{\sqrt{T_n}\psi_1(\sqrt{T_n})}}-\bar{a}(\zeta(t))|\zeta(t)|^{\frac{\alpha_1+1}{2}}\right|\xrightarrow{\mathsf{P}}0 \tag{13}$$

as $T_n\to\infty$.

We use a random change of time (see [2, §4]) and apply Lemma 4.3 for the stochastic integral in equality (9) with $T=T_n$. Then we obtain

$$\frac{\sqrt[4]{T_n}}{\sqrt{\psi_1(\sqrt{T_n})}}\int_0^t g\left(\varphi\left(\zeta_{T_n}(s)\sqrt{T_n}\right)\right)dW_{T_n}(s)=W^*_{T_n}\left(\beta^{(1)}_{T_n}(t)\right),$$

where $W^*_{T_n}(t)$ is a family of Wiener processes,

$$\beta^{(1)}_{T_n}(t)=\frac{\sqrt{T_n}}{\psi_1(\sqrt{T_n})}\int_0^t g^2\left(\varphi\left(\zeta_{T_n}(s)\sqrt{T_n}\right)\right)ds.$$

Therefore

$$I_{T_n}(t)=\bar{a}(\zeta(t))|\zeta(t)|^{\frac{\alpha_1+1}{2}}+W^*_{T_n}\left(\beta^{(1)}_{T_n}(t)\right)+\alpha_{T_n}(t), \tag{14}$$

where

$$\begin{aligned}
\alpha_{T_n}(t)=&\left[\frac{F\left(\varphi\left(\zeta_{T_n}(t)\sqrt{T_n}\right)\right)}{\sqrt{|\zeta_{T_n}(t)|\sqrt{T_n}\,\psi_1\left(|\zeta_{T_n}(t)|\sqrt{T_n}\right)}}-\bar{a}(\zeta_{T_n}(t))\right]\sqrt{|\zeta_{T_n}(t)|}\sqrt{\frac{\psi_1\left(|\zeta_{T_n}(t)|\sqrt{T_n}\right)}{\psi_1\left(\sqrt{T_n}\right)}}\\
&+\bar{a}(\zeta_{T_n}(t))\left[\sqrt{|\zeta_{T_n}(t)|}\sqrt{\frac{\psi_1\left(|\zeta_{T_n}(t)|\sqrt{T_n}\right)}{\psi_1\left(\sqrt{T_n}\right)}}-|\zeta(t)|^{\frac{\alpha_1+1}{2}}\right]\\
&+|\zeta(t)|^{\frac{\alpha_1+1}{2}}\left[\bar{a}(\zeta_{T_n}(t))-\bar{a}(\zeta(t))\right].
\end{aligned}$$

Now using convergence (13) we show that

$$\sup_{0\le t\le L}|\alpha_{T_n}(t)|\xrightarrow{\mathsf{P}}0 \tag{15}$$

as $T_n\to\infty$ by the property of regularly varying at infinity functions and by properties of the process ζ. The proof of Theorem 1 of [9] shows that condition (A_4) implies

$$\beta^{(1)}_{T_n}(t)\xrightarrow{\mathsf{P}}\bar{\beta}^{(1)}(t) \tag{16}$$

as $T_n\to\infty$, where

$$\bar{\beta}^{(1)}(t)=2\left[\int_0^{\zeta(t)}\bar{b}(x)|x|^{\alpha_1}dx-\int_0^t\bar{b}(\zeta(s))|\zeta(s)|^{\alpha_1}d\zeta(s)\right]$$

and ζ is a solution of equation (3). Moreover,

$$\lim_{h\to 0}\overline{\lim_{T\to\infty}}\,\mathsf{P}\left\{\sup_{|t_1-t_2|\le h;\, t_i\le L}\left|\bar\beta^{(1)}_{T_n}(t_2)-\bar\beta^{(1)}_{T_n}(t_1)\right|>\varepsilon\right\}=0 \tag{17}$$

for all $L>0$ and $\varepsilon>0$.

The rest of the proof is literally the same as that of Theorem 2.1 of [9] in view of convergence (16). The only difference is that we use condition (A_5) to prove the independence of the processes $W^*(t)$ and $\bar\beta^{(1)}(t)$. □

Proof of Theorem 2.3. We use the representation

$$\begin{aligned} I_T(t) &= \frac{I(tT)}{\sqrt{T}\psi(\sqrt{T})} \\ &= \frac{F\left(\xi(tT)\right)}{|\xi(tT)|\,\psi\left(|\xi(tT)|\right)}\frac{|\xi(tT)|\,\psi\left(|\xi(tT)|\right)}{\sqrt{T}\psi(\sqrt{T})}+\int_0^t\frac{g\left(\xi(sT)\right)}{\psi\left(|\xi(tT)|\right)}\frac{\psi\left(|\xi(tT)|\right)}{\psi(\sqrt{T})}\,dW_T(s) \\ &= a_0 r_T(t)\frac{\psi(r_T(t)\sqrt{T})}{\psi(\sqrt{T})}+b_0\int_0^t\frac{\psi(r_T(s)\sqrt{T})}{\psi(\sqrt{T})}\,d\widehat{W}_T(s)+\alpha_T^{(1)}(t)+\alpha_T^{(2)}(t), \end{aligned} \tag{18}$$

where

$$\alpha_T^{(1)}(t)=\left[\frac{F\left(\xi(tT)\right)}{|\xi(tT)|\,\psi\left(|\xi(tT)|\right)}-a_0\right]r_T(t)\frac{\psi(r_T(t)\sqrt{T})}{\psi(\sqrt{T})},$$

$$\alpha_T^{(2)}(t)=\int_0^t\left[\frac{g\left(\xi(sT)\right)}{\psi\left(|\xi(tT)|\right)}-b_0\,\operatorname{sign}\xi(sT)\right]\frac{\psi\left(|\xi(tT)|\right)}{\psi(\sqrt{T})}\,dW_T(s).$$

Similarly to the proof of convergence (13), one can easily show that conditions (B_1) and (B_2) imply

$$\sup_{0\le t\le L}\left|\alpha_T^{(i)}(t)\right|\xrightarrow{\mathsf{P}}0,\qquad i=1,2, \tag{19}$$

as $T\to\infty$ for an arbitrary constant $L>0$. Moreover (see [3]), convergence (10) holds for the processes $r_T(t)$ and $\widehat{W}_T(t)$. Now relation (19) implies convergence (10) for $\alpha_T^{(i)}(t)$, $i=1,2$. This allows us to apply Skorokhod's convergent subsequence principle to the process $\left(r_T(t),\widehat{W}_T(t),\alpha_T^{(1)}(t),\alpha_T^{(2)}(t)\right)$.

According to this principle, we assume without loss of generality that $r_{T_n}(t)\xrightarrow{\mathsf{P}}r(t)$ and $\widehat{W}_{T_n}(t)\xrightarrow{\mathsf{P}}\widehat{W}(t)$ in (18) for an arbitrary subsequence $T_n\to\infty$ and all $t\ge 0$.

Now we pass to the limit in (18) as $T_n\to\infty$ and obtain $I_{T_n}(t)\xrightarrow{\mathsf{P}}I_0(t)$, where

$$I_0(t)=a_0r^{\alpha+1}(t)+b_0\int_0^t r^{\alpha}(s)\,d\widehat{W}(s)$$

and $r(t)$ and $\widehat{W}(t)$ are related to each other via equation (4). Since a strong solution of equation (4) is unique, we obtain the convergence of finite-dimensional distributions of the process $I_T(t)$ to those of the process $I_0(t)$. It is clear that the processes $I_T(t)$ and $I_0(t)$ are continuous with probability one. Similarly to the proof of convergence (16) in the paper [8] we obtain equality (17) for the process $I_T(t)$ by using convergence (19).

Thus (see [14, Chapter IX, §2]) the process $I_T(t)$ weakly converges as $T\to\infty$ in the uniform topology of the space of continuous functions to the process $I_0(t)$. □

Proof of Theorem 2.4. We apply the representation

$$I_T(t)=\frac{I(tT)}{\sqrt{\sqrt{T}\psi_1(\sqrt{T})}}=\hat I_T(t)+\alpha_T^{(3)}(t)+\alpha_T^{(4)}(t), \tag{20}$$

where

$$\hat{I}_T(t) = a_0\sqrt{r_T(t)}\sqrt{\frac{\psi_1\left(r_T(t)\sqrt{T}\right)}{\psi_1(\sqrt{T})}} + \frac{\sqrt[4]{T}}{\sqrt{\psi_1(\sqrt{T})}}\int_0^t \hat{g}\left(r_T(s)\sqrt{T}\right)\,d\widehat{W}_T(s),$$

$$\alpha_T^{(3)}(t) = \left[\frac{F\left(\xi(tT)\right)}{\sqrt{|\xi(tT)|\,\psi_1\left(|\xi(tT)|\right)}} - a_0\right]\sqrt{r_T(t)}\sqrt{\frac{\psi_1\left(r_T(t)\sqrt{T}\right)}{\psi_1(\sqrt{T})}},$$

$$\alpha_T^{(4)}(t) = \frac{\sqrt[4]{T}}{\sqrt{\psi_1(\sqrt{T})}}\int_0^t \left[g\left(\xi(sT)\right) - \hat{g}\left(\xi(sT)\right)\right]\,dW_T(s).$$

Similarly to the proof of convergence (13) we prove relation (19) for $i = 3$, and thus (19) holds for $i = 4$ as $T \to \infty$, where $L > 0$ is an arbitrary constant. In doing so, we use Theorem 1 of the paper [5] to obtain the latter result.

It is shown in [3] that relation (10) holds for the processes $r_T(t)$ and $\widehat{W}_T(t)$. Since relation (19) holds for $\alpha_T^{(i)}(t)$ and $i = 3, 4$, these processes satisfy relation (10).

Now Skorokhod's convergent subsequence principle can be applied to the processes

$$\left(r_T(t),\, \widehat{W}_T(t),\, \alpha_T^{(3)}(t),\, \alpha_T^{(4)}(t)\right).$$

Thus we may assume without loss of generality that $r_{T_n}(t) \xrightarrow{\mathsf{P}} r(t)$, $\widehat{W}_{T_n}(t) \xrightarrow{\mathsf{P}} \widehat{W}(t)$, and $\alpha_{T_n}^{(i)}(t) \xrightarrow{\mathsf{P}} \alpha^{(i)}(t)$, $i = 3, 4$, in equality (20) for an arbitrary subsequence $T_n \to \infty$ and all $t \geq 0$. Moreover, the processes $r(t)$ and $\widehat{W}(t)$ are related to each other via equation (4) (see [3]), and convergence (19) holds as $T_n \to \infty$ for $\alpha_{T_n}^{(i)}(t)$, $i = 3, 4$. Similarly to the proof of Theorem 2.2 in the paper [9], we apply the random change of time in the stochastic integral $I_{T_n}(t)$. Then we obtain

$$\frac{\sqrt[4]{T_n}}{\sqrt{\psi_1(\sqrt{T_n})}}\int_0^t \hat{g}\left(r_{T_n}(s)\sqrt{T_n}\right)\,d\widehat{W}_{T_n}(s) = W^*_{T_n}\left(\hat{\beta}^{(1)}_{T_n}(t)\right),$$

where

$$\hat{\beta}^{(1)}_{T_n}(t) = \frac{\sqrt{T_n}}{\psi_1(\sqrt{T_n})}\int_0^t \hat{g}^2\left(r_{T_n}(s)\sqrt{T_n}\right)\,ds$$

and $W^*_{T_n}(t)$ is a family of Wiener processes.

The proof of Theorem 2.2 of the paper [9] shows that

$$\hat{\beta}^{(1)}_{T_n}(t) \xrightarrow{\mathsf{P}} \hat{\beta}^{(1)}(t) = 2b_0\left[\frac{r^{\alpha_1+1}(t)}{\alpha_1+1} - \int_0^t r^{\alpha_1}(s)\,d\widehat{W}(s)\right].$$

To complete the proof of Theorem 2.4 we repeat the same reasoning as in the proof of Theorem 2.2 of [9]. In doing so, we take into account convergence (19) for $i = 3, 4$. □

Example 3.2. Consider equation (1) with $a(x) = \frac{x}{1+x^2}$. Then $f'(x) = \frac{1}{1+x^2}$. Let $g(x) = \sin x$. Using the relationship between the functionals

$$\beta(t) = \int_0^t \sin(\xi(s))\,d\xi(s)$$

and $I(t)$ (see Lemma 4.1), we show that conditions (B_3)–(B_5) hold for $\psi_1(|x|) = |x|$, $a_0 = 0$, $\alpha_1 = 1$, $\hat{g}(x) = \sin x$, and $b_0 = \frac{1}{6}$, $c_0 = 1$. According to Remark 2.4, the functional $\beta_T(t) = T^{-1/2}\int_0^{tT}\sin(\xi(s))\,d\xi(s)$ weakly converges as $T \to \infty$ to the process $I_0^*(t) = 2^{-1/2}W^*(t)$.

4. Auxiliary results

Below we prove Lemmas 4.1, 4.2, and 4.3 used in the proof of the main results of this paper.

Lemma 4.1. *The functionals*

$$\beta(t) = \int_0^t g(\xi(s))\, d\xi(s), \qquad \beta^{(1)}(t) = \int_0^t g(\xi(s))\, ds, \qquad \beta^{(2)}(t) = \int_0^t g(\xi(s))\, dW(s)$$

belong to the class of functionals $I(t)$.

Proof. It is clear that $\beta^{(2)}(t) = I(t)$ for $F(x) \equiv 0$.

Consider the function

$$\Phi(x) = 2\int_0^x f'(u) \left(\int_0^u \frac{g(v)}{f'(v)}\, dv \right) du,$$

where $f(x)$ is defined by (2). The latter function possesses the continuous derivative $\Phi'(x)$, and its second derivative $\Phi''(x)$ exists almost everywhere with respect to the Lebesgue measure. This allows us to apply the Itô formula to the process $\Phi(\xi(t))$,

$$\Phi(\xi(t)) - \Phi(x_0) = \int_0^t \left[\Phi'(\xi(s))a(\xi(s)) + \frac{1}{2}\,\Phi''(\xi(s)) \right] ds + \int_0^t \Phi'(\xi(s))\, dW(s)$$

with probability one for all $t > 0$ (see [13, Chapter II, §10]).

Next, the equality

$$\Phi'(x)a(x) + \frac{1}{2}\,\Phi''(x) = g(x)$$

holds almost everywhere with respect to the Lebesgue measure. Using the latter equality we conclude that

$$\beta^{(1)}(t) = \Phi(\xi(t)) - \Phi(x_0) - \int_0^t \Phi'(\xi(s))\, dW(s),$$

whence $\beta^{(1)}(t) = I(t)$ for $F(x) = \Phi(x) - \Phi(x_0)$ and $g(x) = \Phi'(x)$.

The obvious equality

$$\begin{aligned}
\beta(t) &= \int_0^t g(\xi(s))a(\xi(s))\, ds + \int_0^t g(\xi(s))\, dW(s) \\
&= \widehat{\Phi}(\xi(t)) - \widehat{\Phi}(x_0) - \int_0^t \widehat{\Phi}'(\xi(s))\, dW(s) + \int_0^t g(\xi(s))\, dW(s) \\
&= \widehat{\Phi}(\xi(t)) - \widehat{\Phi}(x_0) + \int_0^t \left[g(\xi(s)) - \widehat{\Phi}'(\xi(s)) \right] dW(s)
\end{aligned}$$

implies that $\beta(t)$ is reduced to a functional $I(t)$, as well, where

$$\widehat{\Phi}(x) = 2\int_0^x f'(u) \left(\int_0^u \frac{g(v)a(v)}{f'(v)}\, dv \right) du. \qquad \square$$

Lemma 4.2. *Let ξ be a solution of equation* (1)*. Assume that*

$$\lim_{t\to\infty} \frac{1}{t^2} \int_0^t \mathsf{E}\big(f'(\xi(s))\big)^2\, ds = 0.$$

Then ξ is an unstable solution.

Proof. Consider the function

$$\bar{\Phi}(x) = 2\int_0^x f'(u)\left(\int_0^u \frac{\chi_{\{|v|<N\}}}{f'(v)}\,dv\right)du.$$

Applying the Itô formula for the function $\Phi(x)$, we get

$$\begin{aligned}\frac{1}{t}\int_0^t \mathsf{P}\{|\xi(s)|<N\}\,ds &= \frac{1}{t}\,\mathsf{E}\left[\bar{\Phi}(\xi(t))-\bar{\Phi}(x_0)\right] \le C_N\frac{1}{t}\,\mathsf{E}\,|f(\xi(t))-f(x_0)| \\ &\le C_N\left(\frac{1}{t^2}\,\mathsf{E}\,(f(\xi(t))-f(x_0))^2\right)^{1/2} \\ &= C_N\left(\frac{1}{t^2}\int_0^t \mathsf{E}\big(f'(\xi(s))\big)^2\,ds\right)^{1/2},\end{aligned}$$

where we used the Itô formula for $f(\xi(t))$. The latter inequality proves Lemma 4.2. □

Lemma 4.3. *Let ξ be a homogeneous strong Markov process. Then, for every locally square integrable real valued function $g \not\equiv 0$,*

$$\int_0^\infty g^2\left(\xi(s)\right)\,ds = \infty$$

with probability one.

Proof. The proof is the same as that of Lemma 3.1 in [9]. □

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