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A REFINEMENT OF CONDITIONS FOR THE ALMOST SURE CONVERGENCE OF SERIES OF MULTIDIMENSIONAL REGRESSION SEQUENCES

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Abstract. We obtain a general criterion for the almost sure convergence of a series whose terms are elements of a multidimensional autoregressive sequence with arbitrary matrix coefficients. In particular, the case of degenerate matrices is also considered. This result extends an earlier result by Buldygin and Runovska who obtained necessary and sufficient conditions for the almost sure convergence of a random series whose terms are elements of a multidimensional Gaussian Markov sequence with nondegenerate matrix coefficients.

1. Introduction

Let $\mathbb{R}^d$, $d \geq 1$, be a finite dimensional Euclidean space equipped with a scalar product (X, Y) and norm $\|X\| = \sqrt{(X, X)}$, $X, Y \in \mathbb{R}^d$. In the space $\mathbb{R}^d$, consider a multidimensional sequence of random vectors $(X_k) = (X_k, k \geq 1)$, satisfying a first order recurrence relation

$$X_1 = V_1, \qquad X_k = C_k X_{k-1} + V_k, \quad k \geq 2, \tag{1}$$

where (C_k) is a sequence of $d \times d$ nonrandom matrices, (V_k) a sequence of jointly independent symmetric random vectors belonging to the space $\mathbb{R}^d$. Throughout the paper, we refer to (X_k) as a *regression sequence*.

Given a sequence (X_k), consider the random series

$$\sum_{k=1}^{\infty} X_k. \tag{2}$$

Necessary and sufficient conditions for the almost sure convergence of series (2) are studied in papers by Buldygin and Runovska [3]–[8] for both one-dimensional and multidimensional cases. A full collection of those results is presented in the monograph [6]. A general criterion for the almost sure convergence of series whose terms are elements of a regression sequence with arbitrary coefficients (C_k) is found in [7, 6] for the case of $d = 1$. A similar criterion for the case of $d > 1$ is known only for regression sequences with nondegenerate matrices (C_k) (see [8, 5, 6]). This restriction can, in the first place, be explained by some limitation of the method used in [8, 5, 6] for the one-dimensional case.

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A perturbation of zero coefficients being the crucial ingredient of this method becomes technically complicated and unfeasible in the case of matrices.

The case of nondegenerate matrix coefficients is rather restrictive, since degenerate matrices (C_k) of the regression scheme (1) appear quite naturally in several related problems. This observation motivated the author to look for a different proof which is not based on the perturbation of the coefficients of a sequence used in [6, 7]. The aim of this paper is to prove a general criterion for the almost sure convergence of series (2) where sequence (1) is constructed from general coefficients (C_k) including the case where the coefficients are degenerate.

The paper is organized as follows. Section 2 introduces definitions, notation, and a preliminary result on the conditions for the almost sure convergence of series whose terms are elements of a regression sequence with nondegenerate coefficients. Section 3 contains the criterion for the almost sure convergence of series (2) for sequence (1) with general (degenerate, in particular) matrix coefficients (C_k). Note that a new proof of the sufficiency of conditions for the almost sure convergence of series (2) allows us to improve the main result of [5, 6]. Using the result of Section 3, we improve the conditions for the almost sure convergence of series of m-regression sequences in Section 4.

2. Notation and preliminary result

Below we recall necessary and sufficient conditions for the almost sure convergence of series (2) for regression sequences with nondegenerate matrix coefficients obtained in the papers [5, 6]. First we introduce some notation that will be used throughout this paper. For $n \geq 1$, let

$$Q(n,k) = \begin{cases} I + \sum_{l=1}^{n-k} \left(\prod_{j=k+l}^{k+1} C_j \right), & 1 \leq k \leq n-1, \\ I, & k = n, \\ \mathcal{O}, & k > n, \end{cases} \tag{3}$$

where $\mathcal{O}$ is the $d \times d$ zero matrix, I is the $d \times d$ unit matrix, and

$$\prod_{j=k+l}^{k+1} C_j = C_{k+l} C_{k+l-1} \dots C_{k+1}, \qquad l \geq 1.$$

Consider the random series

$$\sum_{l=1}^{\infty} \Big(\prod_{j=k+l}^{k+1} C_j \Big) V_k, \qquad k \geq 1, \tag{4}$$

and put

$$V(\infty, k) = V_k + \sum_{l=1}^{\infty} \Big(\prod_{j=k+l}^{k+1} C_j \Big) V_k$$

provided the series on the right-hand side converges almost surely in the norm of the space $\mathbb{R}^d$. By $\mathfrak{R}^\infty$, we denote the class of all monotonic sequences of positive integer numbers that tend to infinity.

Proposition 2.1. *The random series $\sum_{k=1}^{\infty} X_k$ converges almost surely if and, in the case of nondegenerate matrices C_k, $k \geq 1$, only if*

1) *random series (4) converges almost surely for all $k \geq 1$;*
2) *random series $\sum_{k=1}^{\infty} V(\infty, k)$ converges almost surely;*

3) *for every sequence* (m_j) *of the class* $\mathfrak{R}^\infty$,

$$\lim_{j\to\infty}\left\|\sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)V_k\right\| = 0 \quad \textit{almost surely.}$$

3. Criterion for the almost sure convergence of series of multidimensional regression sequences

The following theorem generalizes Proposition 2.1 to the case of general matrix coefficients (C_k). The proof of the sufficiency in Theorem 3.1 differs from that in Proposition 2.1 (see [5, 6]) and this allows us to conclude that condition 2) follows from condition 3) in Proposition 2.1.

Theorem 3.1. *Random series* (2) *converges almost surely if and only if*

I) *series* (4) *converges almost surely for every* $k \geq 1$;
II) *for every sequence* (m_j) *of the class* $\mathfrak{R}^\infty$,

$$\lim_{j\to\infty}\left\|\sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k)V_k\right\| = 0 \quad \textit{almost surely.}$$

The proof of Theorem 3.1 is based on a contraction principle in the scheme of series in the space of convergent sequences. Thus, we briefly discuss this principle before starting the proof.

Let $(\mathbb{R}^d)^\infty$ be the space of sequences whose elements belong to the space $\mathbb{R}^d$, and let $c(\mathbb{R}^d)$ be the space of all convergent sequences of elements of the space $\mathbb{R}^d$. Recall that $c(\mathbb{R}^d)$ is a separable Banach space with respect to the norm $\|(x_k)\|_\infty = \sup_{k\geq 1}\|x_k\|$, where $(x_k) \in c(\mathbb{R}^d)$. The following contraction principle holds in the scheme of series for the space $c(\mathbb{R}^d)$ (see [1, 2]).

Proposition 3.1. *Let* $(Y_{n,k}; n,k \geq 1)$ *be an array of d-dimensional random vectors depending on two indices and such that*

a) *the series* $\sum_{k=1}^\infty Y_{n,k}$ *converges almost surely in the norm of the space* $\mathbb{R}^d$ *for all* $n \geq 1$;
b) $W_k = (Y_{n,k}, n \geq 1)$, $k \geq 1$, *are sequences of independent and symmetric elements belonging to the space* $(\mathbb{R}^d)^\infty$.

Let $Z_n = \sum_{k=1}^\infty Y_{n,k}$, $n \geq 1$. *Then the following statements are equivalent*:

A) $(Z_n, n \geq 1) \in c(\mathbb{R}^d)$ *almost surely*;
B) $W_k \in c(\mathbb{R}^d)$ *almost surely for every* $k \geq 1$, *and the series* $\sum_{k=1}^\infty W_k$ *converges almost surely in the norm of the space* $c(\mathbb{R}^d)$;
C) $W_k \in c(\mathbb{R}^d)$ *almost surely for every* $k \geq 1$, *and*

$$\lim_{m\to\infty}\sup_{M\geq m}\sup_{n\geq 1}\left\|\sum_{k=m}^{M} Y_{n,k}\right\| = 0 \quad \textit{almost surely.}$$

Proof of Theorem 3.1. Consider the sequence (S_n) of partial sums of series (2),

$$S_n = \sum_{k=1}^n X_k, \qquad n \geq 1.$$

Recurrence relations (1) imply that the sequence (S_n) admits the following representation

$$\begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \\ \vdots \end{pmatrix} = \begin{pmatrix} Q(1,1) \\ Q(2,1) \\ \vdots \\ Q(n,1) \\ \vdots \end{pmatrix} V_1 + \begin{pmatrix} \mathbf{0} \\ Q(2,2) \\ \vdots \\ Q(n,2) \\ \vdots \end{pmatrix} V_2 + \dots + \begin{pmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ Q(n,n) \\ \vdots \end{pmatrix} V_n + \dots, \tag{5}$$

whence

$$S_n = \sum_{k=1}^{n} Q(n,k) V_k, \qquad n \geq 1.$$

We represent series (5) as follows:

$$\overrightarrow{S} = \sum_{k=1}^{\infty} \overrightarrow{Q}_k V_k, \tag{6}$$

where

$$\overrightarrow{S} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \\ \vdots \end{pmatrix}, \qquad \overrightarrow{Q}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ Q(k,k) \\ Q(k+1,k) \\ \vdots \end{pmatrix}, \qquad k \geq 1.$$

Note that each coordinate of series (6) converges. Hence, the sequence of partial sums (S_n) can be written in the form of series on the right-hand side of (6) with independent symmetric terms in the space $(\mathbb{R}^d)^\infty$.

Then we apply Proposition 3.1 with

$$(Y_{n,k}; n,k \geq 1) = (Q(n,k)V_k; n,k \geq 1), \qquad W_k = \overrightarrow{Q}_k V_k, \qquad k \geq 1.$$

Note also that the array $(Y_{n,k}; n,k \geq 1)$ satisfies conditions a) and b) of Proposition 3.1. Moreover,

$$S_n = \sum_{k=1}^{\infty} Y_{n,k}, \qquad n \geq 1,$$

that is, $Z_n = S_n$, $n \geq 1$, in Proposition 3.1.

Now we prove the necessity. Assume that the sequence of partial sums (S_n) converges almost surely, that is, $(S_n) \in c(\mathbb{R}^d)$ almost surely. This means that condition A) of Proposition 3.1 holds. Thus, condition B) of Proposition 3.1 also holds, whence

$$\overrightarrow{Q}_k V_k \in c(\mathbb{R}^d)$$

almost surely for all $k \geq 1$, and

$$\lim_{m\to\infty} \sup_{M \geq m} \sup_{n \geq 1} \left\| \sum_{k=m}^{M} Y_{n,k} \right\| = 0 \quad \text{almost surely.}$$

Therefore, series (4) converges almost surely in the norm of the space $\mathbb{R}^d$ for all $k \geq 1$, that is, condition I) of Theorem 3.1 holds. Moreover,

$$\left\| \sum_{k=m_j+1}^{m_{j+1}} Q(m_{j+1},k) V_k \right\| \underset{j\to\infty}{\longrightarrow} 0 \quad \text{almost surely}$$

for every sequence (m_j) of the class $\mathfrak{R}^\infty$, that is, condition II) of Theorem 3.1 also holds. The necessity of conditions I) and II) in Theorem 3.1 is proved.

Now we prove the sufficiency. Let conditions I) and II) of Theorem 3.1 hold. We are going to show that random series (2) converges almost surely. First we check condition B) of Proposition 3.1.

Condition I) of Theorem 3.1 implies $\overrightarrow{Q}_k V_k \in c(\mathbb{R}^d)$ almost surely for all $k \geq 1$.

Then we show that

$$\overline{\lim_{m\to\infty}} \sup_{M\geq m} \sup_{n\geq 1} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| = 0 \quad \text{almost surely.} \tag{7}$$

We prove relation (7) by contradiction. Assume, contrary to relation (7), that

$$\delta = \overline{\lim_{m\to\infty}} \sup_{M\geq m} \sup_{n\geq 1} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| > 0$$

with a positive probability. By the 0–1 law, δ is nonrandom, since the corresponding limit superior is measurable with respect to the tail σ-algebra constructed from the sequence of independent vectors (V_k).

According to definition (3),

$$\sup_{n\geq 1} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| = \sup_{n\geq m} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\|$$

for all $m \geq 1$. Since

$$\begin{aligned}
\delta &= \overline{\lim}\, m \to \infty \sup_{M\geq m} \sup_{n\geq 1} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| = \overline{\lim_{m\to\infty}} \sup_{M\geq m} \sup_{n\geq m} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| \\
&\leq \overline{\lim_{m\to\infty}} \sup_{M\geq m} \left(\sup_{m\leq n\leq M} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| + \sup_{n\geq M+1} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| \right) \\
&= \overline{\lim_{m\to\infty}} \sup_{M\geq m} \left(\sup_{m\leq n\leq M} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| \right. \\
&\qquad \left. + \sup_{n\geq M+1} \left\| \sum_{k=m}^{n} Q(n,k)V_k - \sum_{k=M+1}^{n} Q(n,k)V_k \right\| \right) \\
&\leq \overline{\lim_{m\to\infty}} \sup_{M\geq m} \left(\sup_{m\leq n\leq M} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| \right. \\
&\qquad \left. + \sup_{n\geq M+1} \left\| \sum_{k=m}^{n} Q(n,k)V_k \right\| + \sup_{n\geq M+1} \left\| \sum_{k=M+1}^{n} Q(n,k)V_k \right\| \right) \\
&\leq \overline{\lim_{m\to\infty}} \sup_{M\geq m} \left(\sup_{n\geq m} \left\| \sum_{k=m}^{M} Q(n,k)V_k \right\| \right. \\
&\qquad \left. + \sup_{n\geq m} \left\| \sum_{k=m}^{n} Q(n,k)V_k \right\| + \sup_{n\geq m} \left\| \sum_{k=M+1}^{n} Q(n,k)V_k \right\| \right) \\
&= \overline{\lim_{m\to\infty}} \sup_{M\geq m} \left(3 \sup_{n\geq m} \left\| \sum_{k=m}^{n} Q(n,k)V_k \right\| \right) = 3 \overline{\lim_{m\to\infty}} \sup_{n\geq m} \left\| \sum_{k=m}^{n} Q(n,k)V_k \right\|,
\end{aligned}$$

we conclude that

$$\delta_1 = \overline{\lim_{m\to\infty}} \sup_{n\geq m} \left\| \sum_{k=m}^{n} Q(n,k)V_k \right\| \geq \frac{\delta}{3}$$

with a positive probability. Note that δ_1 also is nonrandom, since the corresponding limit superior is measurable with respect to the tail σ-algebra constructed from the sequence of independent random vectors (V_k), that is,

$$\overline{\lim_{m\to\infty}} \sup_{n\geq m} \left\| \sum_{k=m}^{n} Q(n,k)V_k \right\| \geq \frac{\delta}{3} \quad \text{almost surely.}$$

This implies that there exist a number $\varepsilon \in (0,\delta_1)$ and an increasing subsequence (m_j^*) of positive integer numbers such that

$$\sup_{n\geq m_j^*} \left\| \sum_{k=m_j^*}^{n} Q(n,k)V_k \right\| > \varepsilon.$$

Then, for every fixed m_j^*, there exists a positive integer number $n_j^* \geq m_j^*$ such that

$$\left\| \sum_{k=m_j^*}^{n_j^*} Q(n_j^*,k)V_k \right\| > \varepsilon. \tag{8}$$

Consider a new sequence (l_j) formed by alternating the sequences (m_j^*-1) and (n_j^*) in such a way that the sequence (l_j) is increasing. Without loss of generality, one can assume that the sequence (l_j) is such that

$$m_1^*-1, \quad n_1^*, \quad m_2^*-1, \quad n_2^*, \quad m_3^*-1, \quad n_3^*, \quad \ldots .$$

Otherwise, one can choose a subsequence of the sequence (l_j) with such a property. It is clear that the sequence (l_j) belongs to the class $\mathfrak{R}^\infty$. Then inequality (8) yields

$$\left\| \sum_{k=l_{2j-1}+1}^{l_{2j}} Q(l_{2j},k)V_k \right\| > \varepsilon$$

for all $j \geq 1$.

On the other hand, condition II) of Theorem 3.1 implies that

$$\left\| \sum_{k=m_i+1}^{m_{i+1}} Q(m_{i+1},k)V_k \right\| \underset{i\to\infty}{\longrightarrow} 0 \quad \text{almost surely}$$

for an arbitrary sequence (m_i) of the class $\mathfrak{R}^\infty$. Thus, the sequence (l_j) contradicts condition II) of Theorem 3.1.

Therefore, relation (7) holds, whence condition B) of Proposition 3.1 follows. According to this result, the limit $\lim_{n\to\infty} S_n$ exists almost surely, that is, series (2) converges almost surely. The proof of Theorem 3.1 is complete. □

Remark 3.1. Assumption 2) in Proposition 2.1 contains a requirement on the almost sure convergence of the series $\sum_{k=1}^{\infty} V(\infty,k)$. This assumption, however, is not used in the proof of sufficiency of Theorem 3.1. Repeating the reasoning similar to that used in the proof of Theorem 3.1 one can show that assumption 2) follows from assumption 3) of Proposition 2.1. The crucial point in the proof is that the matrix $(Y_{n,k}; n,k \geq 1)$ is triangular in our case, since $Q(n,k) = \mathcal{O}$ for $k > n$.

Remark 3.2. Since assumption 3) of Proposition 2.1 is more involved as compared to assumption 2), a natural question arises on whether or not assumption 3) follows from assumption 2) in Proposition 2.1.

It turns out that assumptions 2) and 3) are equivalent in some cases (see [3, 4, 6]). For example, if all coefficients (C_k) of a one-dimensional regression sequence are nonnegative, then series (2) converges almost surely if and only if assumptions 1) and 2) of Proposition 2.1 hold. Nevertheless, assumption 3) of Proposition 2.1 is more restrictive. A corresponding example where assumption 2) of Proposition 2.1 holds, while assumption 3) does not, is discussed in [6].

4. Series related to m-regression sequences

The results obtained for multidimensional regression sequences can be applied to study the almost sure convergence of series whose terms are determined by an m-regression sequence of random variables.

Consider a sequence of random variables (ξ_k) defined with the help of a system of recurrent relations of order m:

$$\xi_{1-m} = \cdots = \xi_{-1} = \xi_0 = 0,$$

$$\xi_k = b_1^{(k)}\xi_{k-1} + b_2^{(k)}\xi_{k-2} + \cdots + b_m^{(k)}\xi_{k-m} + \theta_k, \qquad k \geq 1, \tag{9}$$

where (θ_k) is a sequence of independent symmetric random variables such that

$$\mathsf{P}\{\theta_k = 0\} < 1, \qquad k \geq 1,$$

and $\big(b_j^{(k)}; 1 \leq j \leq m, k \geq 1\big)$ is an array of nonrandom numbers.

Using Frobenius matrices (see [2]) one can reduce the problem for an m-regression sequence (ξ_k) to the problem for a multidimensional sequence in the space $\mathbb{R}^m$:

$$X_1 = V_1, \qquad X_k = C_k X_{k-1} + V_k, \qquad k \geq 2, \tag{10}$$

by letting

$$X_k = \begin{pmatrix} \xi_k \\ \xi_{k-1} \\ \dots \\ \xi_{k-m+1} \end{pmatrix}, \qquad V_k = \begin{pmatrix} \theta_k \\ 0 \\ \dots \\ 0 \end{pmatrix}, \qquad C_k = \begin{pmatrix} b_1^{(k)} & b_2^{(k)} & \dots & b_{m-1}^{(k)} & b_m^{(k)} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix},$$

$$k \geq 2.$$

Then the random series $\sum_{k=1}^{\infty} \xi_k$ converges almost surely if and only if the series $\sum_{k=1}^{\infty} X_k$ converges almost surely in the space $\mathbb{R}^m$. Thus, the following result follows from Theorem 3.1.

Theorem 4.1. *The random series $\sum_{k=1}^{\infty} \xi_k$ converges almost surely if and only if*

i) *the nonrandom series $\sum_{l=0}^{\infty} u_{k+l}^{(k+1)}$ converges for all $k \geq 1$, where $\big(u_n^{(j)}, n \geq j\big)$ is a nonrandom recurrent sequence defined by*

$$u_{k-m+1}^{(k+1)} = u_{k-m+2}^{(k+1)} = \cdots = u_{k-1}^{(k+1)} = 0, \qquad u_k^{(k+1)} = 1,$$

$$u_n^{(k+1)} = b_1^{(n)} u_{n-1}^{(k+1)} + b_2^{(n)} u_{n-2}^{(k+1)} + \cdots + b_m^{(n)} u_{n-m}^{(k+1)}, \qquad n \geq k+1;$$

ii) *for all sequences (m_j) of the class $\mathfrak{R}^{\infty}$,*

$$\Bigg\| \sum_{k=m_j+1}^{m_{j+1}} \Bigg(\sum_{l=0}^{m_{j+1}-k} u_{k+l}^{(k+1)} \Bigg) \theta_k \Bigg\| \underset{j\to\infty}{\longrightarrow} 0 \quad \textit{almost surely.}$$

Remark 4.1. The papers [8, 5, 6] deal with m-regression sequences such that $b_m^{(k)} \neq 0$, $k \geq 1$. Theorem 4.1 extends sufficient conditions for the almost sure convergence of series of m-regression sequences obtained in [8, 5, 6] to the case of general coefficients $\big(b_j^{(k)}; 1 \leq j \leq m, k \geq 1\big)$.

5. Concluding remarks

We found necessary and sufficient conditions for the almost sure convergence of series of multidimensional regression sequences for the case of general matrix coefficients. This allows us to provide necessary and sufficient conditions for the almost sure convergence of series whose terms are elements of an m-regression sequence of random variables with arbitrary coefficients. The results obtained in this paper can be applied to the Marcinkiewicz–Zygmund strong law of large numbers for sums of elements of regression sequences.

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