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WEAK CONVERGENCE OF INTEGRAL FUNCTIONALS CONSTRUCTED FROM SOLUTIONS OF ITÔ'S STOCHASTIC DIFFERENTIAL EQUATIONS WITH NON-REGULAR DEPENDENCE ON A PARAMETER

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Abstract. The weak convergence of the functionals $\int_0^t g_T(\xi_T(s))\,dW_T(s)$, $t\ge 0$, is studied as $T\to\infty$, where $\xi_T(t)$ is a strong solution of the stochastic differential equation $d\xi_T(t)=a_T(\xi_T(t))\,dt+dW_T(t)$ and $T>0$ is a parameter. Here $a_T(x)$, $x\in\mathbb{R}$, are some real-valued measurable functions such that $|a_T(x)|\le C_T$ for all x, $W_T(t)$ are standard Wiener processes, and $g_T(x)$ are real-valued measurable locally bounded non-random functions. The explicit form of the limit processes is found in the case where both $g_T(x)$ and $a_T(x)$ depend on the parameter in a non-regular way.

1. Introduction

Consider the Itô stochastic differential equation

$$d\xi_T(t)=a_T\big(\xi_T(t)\big)\,dt+dW_T(t),\qquad t\ge 0,\ \xi_T(0)=x_0, \tag{1}$$

where $T>0$ is a parameter; $a_T(x)$, $x\in\mathbb{R}$, are real-valued functions such that $|a_T(x)|\le L_T$ for some constants $L_T>0$ and all $x\in\mathbb{R}$; and $W_T=\{W_T(t),t\ge 0\}$ is a family of standard Wiener processes defined on a complete probability space $(\Omega,\mathfrak{F},\mathsf{P})$.

It is known [14] that equation (1) has a strong and pathwise unique solution $\xi_T=\{\xi_T(t),t\ge 0\}$ whatever parameter T and initial value x_0 are. Moreover, this solution is a homogeneous Markov process.

We study the weak convergence as $T\to\infty$ of the functionals

$$\int_0^t g_T\big(\xi_T(s)\big)\,dW_T(s),$$

where $g_T(x)$ are measurable locally bounded non-random functions; the processes ξ_T and W_T are related to each other via equation (1). The case where the functions $a_T(x)$ and $g_T(x)$ depend on the parameter T in a non-regular way is also considered. This means that a_T and g_T may not have limits as $T\to\infty$, or they may have infinite limits, or they may have a degeneracy of a different type.

The limit distributions as $T\to\infty$ of functionals

$$\beta_T^{(1)}(t)=\int_0^t a_T\big(\xi_T(s)\big)\,ds$$

of solutions $\xi_T(t)$ of equation (1) are studied in the papers [3,4] for the case of $a_T(x)=\sqrt{T}a\big(x\sqrt{T}\,\big)$, where $a(x)$ is a function, absolutely integrable in the whole axis and such

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that

$$\int_{-\infty}^{\infty} a(x)\,dx = \lambda$$

($a_T(x)$ is a family of functions being similar at the point $x = 0$ to the δ function with the weight λ).

The convergence in probability of $\beta_T^{(1)}(t)$ to zero as $T \to \infty$ is obtained in the paper [3] for every $t > 0$ if $\lambda = 0$. If $\lambda \neq 0$, the convergence of distributions of functionals $\beta_T^{(1)}(t)$ to the distribution of a certain functional $\beta^{(1)}(t)$ constructed from a solution $\zeta(t)$ of the Itô stochastic differential equation $d\zeta(t) = \bar{\sigma}\big(\zeta(t)\big)\,dW(t)$ follows from the paper [4], where the function $\bar{\sigma}$ is such that $\bar{\sigma}(x) = \sigma_1$ for $x > 0$, $\bar{\sigma}(x) = \sigma_2$ for $x \le 0$, $\sigma_i = e^{-2\lambda_i}$,

$$\lambda_1 = \int_0^{\infty} a(x)\,dx, \qquad \lambda_2 = \int_0^{-\infty} a(x)\,dx.$$

In addition, the explicit form of the transient density is obtained in [4] for the Markov process $\zeta(t)$. Moreover, it follows from [5] that

$$\beta^{(1)}(t) = 2\left[\int_0^{\zeta(t)} \bar{b}(u)\,du - \int_0^t \bar{b}\big(\zeta(s)\big)\,d\zeta(s)\right], \qquad x_0 = 0,$$

where $\bar{b}(x) = \lambda_1\sigma_1^{-2}$ for $x > 0$ and $\bar{b}(x) = \lambda_2\sigma_2^{-2}$ for $x \le 0$. In particular, $\beta^{(1)}(t) \equiv 0$ for $\lambda_1 = \lambda_2$ and $\beta^{(1)}(t) = 2c_0 L_\zeta(t, 0)$ for $\lambda_2\sigma_2^{-2} = -\lambda_1\sigma_1^{-2} = c_0$, where $L_\zeta(t, 0)$ is the local time of the process $\zeta(t)$ at the point 0 in the interval $[0, t]$. It also follows from the results of [5] that the distributions of the functional

$$\beta_T^{(2)}(t) = \int_0^t \sqrt{|a_T(\xi_T(s))|}\,dW_T(s)$$

converge as $T \to \infty$ to those of the process $W^*\left(\beta^{(1)}(t)\right)$, where $\xi_T(t)$ and $W_T(t)$ are related to each other via equation (1) with $x_0 = 0$, $W^*(t)$ a Wiener process, and $W^*(t)$ and $\beta^{(1)}(t)$ independent. This means that a new process $W^*(t)$ is involved in the description of limit distributions, and this process is independent of the solution $\xi_T(t)$. Analogous limit distributions of the Wiener process $\zeta(t) = W(t)$ are obtained in the monograph [13, Chapter 5, §5] for additive functionals constructed from a random walk.

The current paper is a generalization of [8] and [9], where the weak convergence of such types of functionals of the solution ξ_T is studied for equation (1) in the case of a special type of dependence of the shift parameter $a_T(x) = \sqrt{T}a\big(x\sqrt{T}\big)$ on the parameter T. The weak convergence as $T \to \infty$ is studied in [7] by using the probabilistic method for the following functionals of a solution ξ_T of equation (1) (the equation belongs to the class $K(G_T)$):

$$\beta_T^{(1)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\,ds, \qquad \beta_T^{(2)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\,dW_T(s),$$

$$I_T(t) = F_T\big(\xi_T(t)\big) + \int_0^t g_T\big(\xi_T(s)\big)\,dW_T(s), \qquad \beta_T(t) = \int_0^t g_T\big(\xi_T(s)\big)\,d\xi_T(s),$$

where the processes ξ_T and W_T are related to each other via equation (1), $g_T(x)$ is a family of measurable locally bounded real-valued functions, and $F_T(x)$ are real-valued continuous functions. Our results hold for wider classes of convergent functionals $\beta_T^{(2)}(t)$ (Theorems 3.1 and 3.2). Sufficient conditions for $\beta_T^{(1)}(t)$ to weakly converge to the Wiener process are also studied here (Theorem 3.3). A detailed survey of known results in this direction is given in the paper [8].

The paper is organized as follows. Section 2 contains the basic definitions and some remarks. Statements of the main results are given in Section 3. Section 4 is devoted to the proof of the main results. Some auxiliary results are placed in Section 5. Section 6 complements the main results with a number of examples.

2. Basic definitions and some remarks

Throughout the paper, C, N, and C_N denote some constants that do not depend on T. We also use the notation

$$f_T(x) = \int_0^x \exp\left\{-2\int_0^u a_T(v)\,dv\right\} du. \tag{2}$$

Definition 2.1. We say that equation (1) belongs to the class $K(G_T)$ if

(1) there exists a family of continuous functions $G_T(x)$, $x \in \mathbb{R}$, that have continuous derivatives $G'_T(x)$ and almost everywhere (with respect to Lebesgue measure) have locally integrable second derivatives $G''_T(x)$ such that

$$\textbf{(A}_\textbf{1}\textbf{)}\quad \left[G'_T(x)a_T(x) + \frac{1}{2}\,G''_T(x)\right]^2 + \left[G'_T(x)\right]^2 \le C\left[1 + |G_T(x)|^2\right], \qquad |G_T(x_0)| \le C$$

for all $T > 0$, $x \in \mathbb{R}$, and some constant $C > 0$;

(2) there exist constants $C > 0$ and $\alpha > 0$ such that $|G_T(x)| \ge C|x|^\alpha$ for all $x \in \mathbb{R}$;

(3) there exist a bounded function $\psi(x)$, $x \ge 0$, and a constant $m \ge 0$ such that $\psi(x) \to 0$ as $x \to 0$; and

$$\textbf{(A}_\textbf{2}\textbf{)}\quad \int_0^x f'_T(u)\left(\int_0^u \frac{\chi_B\left(G_T(v)\right)}{f'_T(v)}\,dv\right) du \le \psi\big(\lambda(B)\big)\left[1 + |x|^m\right]$$

for an arbitrary measurable bounded set B, where $\chi_B(v)$ and $\lambda(B)$ denote the indicator and Lebesgue measure of a set B, respectively, and $f'_T(x)$ is the derivative of the function $f_T(x)$ defined by equality (2).

In what follows we assume that, for some locally bounded functions $q_T(x)$,

$$\textbf{(A}_\textbf{3}\textbf{)}\quad \lim_{T\to\infty}\sup_{|x|\le N} f'_T(x)\left|\int_0^x \frac{q_T(v)}{f'_T(v)}\,dv\right| = 0$$

for an arbitrary constant $N > 0$.

Definition 2.2. We say that the family of processes $\zeta_T = \{\zeta_T(t), t \ge 0\}$ weakly converges as $T \to \infty$ to the process $\zeta = \{\zeta(t), t \ge 0\}$ if, for an arbitrary $L > 0$, the measures $\mu_T[0, L]$ generated by the processes $\zeta_T(\cdot)$ in the interval $[0, L]$ weakly converge to the measure $\mu[0, L]$ generated by the process $\zeta(\cdot)$ in the interval $[0, L]$.

Remark 2.1. If the processes ζ_T and ζ are continuous with probability one, then Definition 2.2 is, in fact, the definition of the weak convergence of the processes ζ_T as $T \to \infty$ to the process ζ in the uniform topology of the space of continuous functions (see [2, Chapter IX, §1]).

Remark 2.2. We often use the Itô formula for the process $\Phi(\xi_T(t))$, where ξ_T is a solution of equation (1), and the function $\Phi(x)$ has the continuous derivative $\Phi'(x)$ and almost everywhere (with respect to the Lebesgue measure) locally integrable second derivative $\Phi''(x)$. It follows from the results of [10] that

$$\Phi\big(\xi_T(t)\big) = \Phi(x_0) + \int_0^t \left[\Phi'\big(\xi_T(s)\big)a_T\big(\xi_T(s)\big) + \frac{1}{2}\Phi''\big(\xi_T(s)\big)\right] ds + \int_0^t \Phi'\big(\xi_T(s)\big)\,dW_T(s)$$

with probability one for all $t \ge 0$ (this is a version of the Itô formula we use below).

Remark 2.3. If ξ_T is a solution of equation (1) and assumption (1) of Definition 2.1 holds for the family of functions $G_T(x)$, then the processes $\zeta_T(t) = G_T(\xi_T(t))$ are weakly compact (this is shown in the paper [6]). We use the Itô formula

$$\zeta_T(t) = G_T(x_0) + \int_0^t \left[G'_T\big(\xi_T(s)\big)a_T\big(\xi_T(s)\big) + \frac{1}{2}G''_T\big(\xi_T(s)\big)\right] ds + \eta_T(t), \tag{3}$$
$$\eta_T(t) = \int_0^t G'_T\big(\xi_T(s)\big)\, dW_T(s)$$

to prove the weak compactness. Following the standard argument (see [1, Chapter 2, §6, Theorem 4]), we obtain the inequalities

$$\mathsf{E} \sup_{0\le t\le L} |\zeta_T(t)|^k \le C_k, \qquad \mathsf{E}\,|\zeta_T(t_2) - \zeta_T(t_1)|^4 \le C|t_2 - t_1|^2 \tag{4}$$

for all $k > 0$ and some constants C_k and C. These inequalities imply the convergence

$$\begin{gathered}\lim_{N\to\infty} \overline{\lim_{T\to\infty}} \sup_{0\le t\le L} \mathsf{P}\left\{|\zeta_T(t)| > N\right\} = 0,\\ \lim_{h\to 0} \overline{\lim_{T\to\infty}} \sup_{|t_1-t_2|\le h;\, t_i\le L} \mathsf{P}\left\{|\zeta_T(t_2) - \zeta_T(t_1)| > \varepsilon\right\} = 0\end{gathered} \tag{5}$$

for all constants $L > 0$ and $\varepsilon > 0$. It is also shown in the paper [6] that inequalities (4) and convergence (5) hold for the processes $\eta_T(t)$, as well. It is clear that inequalities (4) and convergence (5) hold for the processes W_T, as well.

Remark 2.4. Let ξ_T be a solution of equation (1) of the class $K(G_T)$ and let $G_T(x_0) \to y_0$ as $T \to \infty$. Assume that there exist measurable and locally bounded functions $a_0(x)$ and $\sigma_0(x)$ such that

(A_4) (1) condition (A_3) holds for the functions

$$q_T^{(1)}(x) = G'_T(x)\, a_T(x) + \frac{1}{2}\, G''_T(x) - a_0\big(G_T(x)\big),$$
$$q_T^{(2)}(x) = \left[G'_T(x)\right]^2 - \sigma_0^2\big(G_T(x)\big);$$

(2) a unique weak solution $\big(\zeta(t), \widehat{W}(t)\big)$ exists for the Itô stochastic differential equation

$$\zeta(t) = y_0 + \int_0^t a_0\big(\zeta(s)\big)\, ds + \int_0^t \sigma_0\big(\zeta(s)\big)\, d\widehat{W}(s). \tag{6}$$

Then Theorem 2.1 of [7] claims that the process $\zeta_T = \zeta_T(t) = G_T(\xi_T(t))$ weakly converges as $T \to \infty$ to a solution ζ of equation (6).

3. Main results

Theorem 3.1. *Let ξ_T be a solution of equation* (1) *belonging to the class $K(G_T)$ and let condition* (A_4) *hold. Moreover, let $g_T(x)$ be measurable locally bounded functions and let there exist measurable locally bounded functions $\hat{g}_T(x)$ and $g_0(x)$ such that condition* (A_3) *holds for the functions*

$$Q_T^{(1)}(x) = \left[g_T(x) - \hat{g}_T\big(G_T(x)\big)\right]^2,$$
$$Q_T^{(2)}(x) = \hat{g}_T^2\big(G_T(x)\big) - g_0^2\big(G_T(x)\big),$$

and

(A_5) *condition* (A_3) *holds for the functions* $Q_T^{(3)}(x) = \left|\hat{g}_T\big(G_T(x)\big)\right|$ *and, in addition,*

$$\left|\int_0^x f'_T(u) \int_0^u \frac{Q_T^{(3)}(v)}{f'_T(v)}\, dv\, du\right| \le C\,(1 + |x|^\alpha)$$

for some constants $C > 0$ *and* $\alpha \ge 0$.

Then the stochastic process

$$\beta_T^{(2)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\, dW_T(s)$$

weakly converges as $T \to \infty$ *to the process* $\beta^{(2)}(t) = W^*\left(\beta^{(1)}(t)\right)$, *where*

$$\beta^{(1)}(t) = \int_0^t g_0^2\big(\zeta(s)\big)\, ds.$$

Here ζ *is a solution of equation* (6), *and* $W^* = \{W^*(t), t \ge 0\}$ *is a Wiener process such that* W^* *and* $\beta^{(1)}(t)$ *are independent.*

Theorem 3.2. *Let* ξ_T *be a solution of equation* (1) *belonging to the class* $K(G_T)$ *and let condition* (A_4) *hold. Further, let* $g_T(x)$ *be measurable locally bounded functions and let there exist measurable locally bounded functions* $\hat{g}_T(x)$ *and* $g_0(x)$ *such that condition* (A_5) *holds for the functions* $Q_T^{(3)}(x) = \left|\hat{g}_T\big(G_T(x)\big)\right|$ *and condition* (A_3) *is satisfied with functions* $Q_T^{(1)}(x) = \left[g_T(x) - \hat{g}_T\left(G_T(x)\right)\right]^2$. *In addition, let*

$$\lim_{T\to\infty} \sup_{|x|\le N} |J_T(x)| = 0 \tag{7}$$

for all $N > 0$, *where*

$$J_T(x) = f'_T(x) \int_0^x \frac{\hat{g}_T^2\big(G_T(v)\big)}{f'_T(v)}\, dv - g_0\big(G_T(x)\big) G'_T(x).$$

Then the stochastic process

$$\beta_T^{(2)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\, dW_T(s)$$

weakly converges as $T \to \infty$ *to the process* $\beta^{(2)}(t) = W^*\left(\beta^{(1)}(t)\right)$, *where*

$$\beta^{(1)}(t) = 2\left[\int_{y_0}^{\zeta(t)} g_0(x)\, dx + \int_0^t g_0\big(\zeta(s)\big)\sigma_0\big(\zeta(s)\big)\, d\widehat{W}(s)\right],$$

$\big(\zeta(t), \widehat{W}(t)\big)$ *is a solution of equation* (6), *and* $W^* = \{W^*(t), t \ge 0\}$ *is a Wiener process such that* W^* *and* $\beta^{(1)}(t)$ *are independent.*

Remark 3.1. Condition (7) in Theorem 3.2 can be weakened for solutions of equation (1) belonging to the class $K(G_T)$, where $G_T(x) = f_T(x)$ if there are constants $\delta > 0$ and $C > 0$ such that $0 < \delta \le f'_T(x) \le C$ for all $x \in \mathbb{R}$. Instead one can use the conditions that $J_T(x) \to 0$ almost everywhere as $T \to \infty$ and that $|J_T(x)|\,\chi_{\{|x|\le N\}} \le C_N$ for all $N > 0$. This observation follows from the proof of Theorem 3 in [5] and Lemma 4.1 of [7].

Remark 3.2. Condition (A_5) in Theorems 3.1 and 3.2 is used only in the proof of the independence of the processes W^* and $\beta^{(1)}(t)$. Therefore if $\beta^{(1)}(t)$ is non-random, then condition (A_5) in Theorem 3.1 as well as in Theorem 3.2 can be omitted.

Theorem 3.3. *Let ξ_T be a solution of equation* (1) *belonging to the class $K(G_T)$. Assume that condition* (A_4) *holds. We also assume that condition* (A_3) *holds for the coefficient $a_T(x)$ of equation* (1). *Let $g_T(x)$ be measurable locally bounded functions and let there exist some constants c_0 and b_0 such that*

$$\lim_{T\to\infty} \sup_{|x|\le N} \left| \int_0^x \left[f'_T(u) \int_0^u \frac{g_T(v)}{f'_T(v)}\, dv - c_0 \right] du \right| = 0$$

for an arbitrary $N > 0$. If condition (A_3) *holds for the function*

$$Q_T(x) = \left[f'_T(x) \int_0^x \frac{g_T(v)}{f'_T(v)}\, dv - c_0 \right]^2 - b_0^2,$$

then the stochastic process

$$\beta_T^{(1)}(t) = \int_0^t g_T\big(\xi_T(s)\big)\, ds$$

weakly converges as $T \to \infty$ to the process $2b_0 W(t)$, where $W(t)$ is a standard Wiener process.

4. Proof of main results

Proof of Theorem 3.1. We rewrite equality (3) as

$$\zeta_T(t) = G_T(x_0) + \int_0^t a_0\big(\zeta_T(s)\big)\, ds + \alpha_T^{(1)}(t) + \eta_T(t), \tag{8}$$

where

$$\alpha_T^{(1)}(t) = \int_0^t q_T^{(1)}\big(\xi_T(s)\big)\, ds, \qquad q_T^{(1)}(x) = G'_T(x) a_T(x) + \frac{1}{2} G''_T(x) - a_0\big(G_T(x)\big).$$

Accordingly, the characteristics $\langle \eta_T \rangle(t)$ of the almost sure continuous martingales $\eta_T(t)$ are rewritten as

$$\langle \eta_T \rangle(t) = \int_0^t \big[G'_T\big(\xi_T(s)\big)\big]^2\, ds = \int_0^t \sigma_0^2\big(\zeta_T(s)\big)\, ds + \alpha_T^{(2)}(t), \tag{9}$$

where

$$\alpha_T^{(2)}(t) = \int_0^t q_T^{(2)}\big(\xi_T(s)\big)\, ds, \qquad q_T^{(2)}(x) = [G'_T(x)]^2 - \sigma_0^2\big(G_T(x)\big).$$

Assumptions of Lemma 5.1 hold for the functions $q_T^{(1)}(x)$ and $q_T^{(2)}(x)$. Hence, for an arbitrary $L > 0$,

$$\sup_{0\le t\le L} \left| \alpha_T^{(k)}(t) \right| \xrightarrow{\mathsf{P}} 0, \qquad k = 1, 2, \tag{10}$$

as $T \to \infty$.

It is clear that

$$\beta_T^{(2)}(t) = \int_0^t \hat{g}_T\big(\zeta_T(s)\big)\, dW_T(s) + \gamma_T(t), \tag{11}$$

where

$$\gamma_T(t) = \int_0^t q_T\big(\xi_T(s)\big)\, dW_T(s), \qquad q_T(x) = g_T(x) - \hat{g}_T\big(G_T(x)\big).$$

Since condition (A_3) holds for the functions $q_T^2(x)$,

$$\int_0^L q_T^2\big(\xi_T(s)\big)\, ds \xrightarrow{\mathsf{P}} 0$$

as $T \to \infty$ for an arbitrary constant $L > 0$ by Lemma 5.1.

In view of Theorem 2 in [1, Chapter 1, §3]),

$$\mathsf{P}\left\{\sup_{0\le t\le L}|\gamma_T(t)|>\varepsilon\right\}\le\delta+\mathsf{P}\left\{\int_0^L q_T^2\big(\xi_T(s)\big)\,ds>\varepsilon^2\delta\right\}$$

for arbitrary constants $\varepsilon>0$, $\delta>0$, and $L>0$, whence we derive the convergence

$$\sup_{0\le t\le L}|\gamma_T(t)|\xrightarrow{\mathsf{P}}0 \tag{12}$$

as $T\to\infty$ if $L>0$ is an arbitrary constant.

Since relation (5) holds for the processes $\zeta_T(t)$, $\eta_T(t)$, and $W_T(t)$ and since relations (5) hold for the processes $\alpha_T^{(k)}(t)$, $k=1,2$, and $\gamma_T(t)$, one can apply Skorokhod's principle for the process

$$\Big(\zeta_T(t),\eta_T(t),W_T(t),\alpha_T^{(1)}(t),\alpha_T^{(2)}(t),\gamma_T(t)\Big)$$

in view of (10) and (12) (see [12, §6]). According to this principle, given an arbitrary sequence $T_n'\to\infty$, there are a subsequence $T_n\to\infty$, a probability space $\big(\widetilde\Omega,\widetilde{\mathfrak F},\widetilde{\mathsf P}\big)$, a stochastic process $\big(\tilde\zeta_{T_n}(t),\tilde\eta_{T_n}(t),\widetilde W_{T_n}(t),\tilde\alpha_{T_n}^{(1)}(t),\tilde\alpha_{T_n}^{(2)}(t),\tilde\gamma_{T_n}(t)\big)$ defined on this space whose finite dimensional distributions coincide with the corresponding finite dimensional distributions of the stochastic process $\big(\zeta_{T_n}(t),\eta_{T_n}(t),W_{T_n}(t),\alpha_{T_n}^{(1)}(t),\alpha_{T_n}^{(2)}(t),\gamma_{T_n}(t)\big)$, and moreover

$$\tilde\zeta_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\zeta(t),\qquad \tilde\eta_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\eta(t),\qquad \widetilde W_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\widetilde W(t),$$
$$\tilde\alpha_{T_n}^{(1)}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\alpha^{(1)}(t),\qquad \tilde\alpha_{T_n}^{(2)}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\alpha^{(2)}(t),\qquad \tilde\gamma_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\gamma(t)$$

for all $0\le t\le L$, where $\tilde\zeta(t)$, $\tilde\eta(t)$, $\widetilde W(t)$, $\tilde\alpha^{(1)}(t)$, $\tilde\alpha^{(2)}(t)$, and $\tilde\gamma(t)$ are some stochastic processes. In view of convergence (10), $\tilde\alpha^{(k)}(t)\equiv0$, $k=1,2$, with probability one. Similarly, $\tilde\gamma(t)\equiv0$ with probability one in view of convergence (12).

Inequalities (4) imply that the processes $\tilde\zeta(t)$, $\tilde\eta(t)$, and $\widetilde W(t)$ are continuous with probability one. In addition, $\tilde\eta(t)$ is a martingale, and $\widetilde W(t)$ is a Wiener process. Moreover, we obtain from Lemma 5.2 and equalities (8), (9), and (11) that

$$\tilde\zeta_{T_n}(t)=G_{T_n}(x_0)+\int_0^t a_0\left(\tilde\zeta_{T_n}(s)\right)ds+\tilde\alpha_{T_n}^{(1)}(t)+\tilde\eta_{T_n}(t),$$
$$\langle\tilde\eta_{T_n}\rangle(t)=\int_0^t\sigma_0^2\left(\tilde\zeta_{T_n}(s)\right)ds+\tilde\alpha_{T_n}^{(2)}(t),$$
$$\tilde\beta_{T_n}^{(2)}(t)=\int_0^t\hat g_{T_n}\left(\tilde\zeta_{T_n}(s)\right)d\widetilde W_{T_n}(s)+\tilde\gamma_{T_n}(t),$$

where

$$\tilde\zeta_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\zeta(t),\qquad \tilde\eta_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\tilde\eta(t),\qquad \widetilde W_{T_n}(t)\xrightarrow{\widetilde{\mathsf P}}\widetilde W(t),$$
$$\sup_{0\le t\le L}\left|\tilde\alpha_{T_n}^{(k)}(t)\right|\xrightarrow{\widetilde{\mathsf P}}0,\quad k=1,2,\qquad \sup_{0\le t\le L}|\tilde\gamma_{T_n}(t)|\xrightarrow{\widetilde{\mathsf P}}0\quad\text{as }T_n\to\infty.$$

Now we derive from [6] that

$$\lim_{h\to0}\varlimsup_{T_n\to\infty}\widetilde{\mathsf P}\left\{\sup_{|t_1-t_2|\le h;\,t_i\le L}|\lambda_{T_n}(t_2)-\lambda_{T_n}(t_1)|>\varepsilon\right\}=0 \tag{13}$$

for arbitrary constants $L>0$ and $\varepsilon>0$ where $\lambda_{T_n}(t)=\tilde\zeta_{T_n}(t)$, $\lambda_{T_n}(t)=\tilde\eta_{T_n}(t)$, and $\lambda_{T_n}(t)=\widetilde W_{T_n}(t)$.

Thus Lemma 1.11 of [11] implies that

$$\sup_{0\le t\le L}\left|\tilde\zeta_{T_n}(t)-\tilde\zeta(t)\right| \xrightarrow{\widetilde{\mathsf P}} 0,\qquad \sup_{0\le t\le L}\left|\tilde\eta_{T_n}(t)-\tilde\eta(t)\right| \xrightarrow{\widetilde{\mathsf P}} 0,$$

$$\sup_{0\le t\le L}\left|\widetilde W_{T_n}(t)-\widetilde W(t)\right| \xrightarrow{\widetilde{\mathsf P}} 0$$

as $T_n\to\infty$ for an arbitrary $L>0$.

Lemma 4.3 of [7] implies that

$$\tilde\eta(t)=\int_0^t \sigma_0\left(\tilde\zeta(s)\right)\,d\widehat W(s)$$

and that the process $\tilde\zeta(t)$ satisfies equation (6).

According to Lemma 5.2, the processes $\tilde\beta^{(2)}_{T_n}(t)$ and $\beta^{(2)}_{T_n}(t)$ are stochastically equivalent. Further, Lemma 5.3 allows one to use a random change of time in stochastic integrals (see [1, Chapter 1, §4]). Thus we obtain for an arbitrary $t\ge 0$ that

$$\tilde\beta^{(2)}_{T_n}(t)=W^*_{T_n}\left(\tilde\beta^{(1)}_{T_n}(t)\right)+\tilde\gamma_{T_n}(t)\tag{14}$$

with probability one where $W^*_{T_n}(t)$ is a family of Wiener processes,

$$\tilde\beta^{(1)}_{T_n}(t)=\int_0^t \hat g^2_{T_n}\left(\tilde\zeta_{T_n}(s)\right)\,ds.$$

Condition (A_3) holds for the function $\hat g^2_{T_n}(G_T(x))-g_0^2(G_T(x))$. Thus the convergence

$$\sup_{0\le t\le L}\left|\tilde\beta^{(1)}_{T_n}(t)-\tilde\beta^{(1)}(t)\right| \xrightarrow{\widetilde{\mathsf P}} 0$$

as $T_n\to\infty$ follows from Theorem 2.2 of [7], where $\tilde\beta^{(1)}(t)=\int_0^t g_0^2\big(\tilde\zeta(s)\big)\,ds$.

It is clear that

$$\begin{aligned}\widetilde{\mathsf P}&\left\{\sup_{0\le t\le L}\left|W^*_{T_n}\left(\tilde\beta^{(1)}_{T_n}(t)\right)-W^*_{T_n}\left(\tilde\beta^{(1)}(t)\right)\right|>\varepsilon\right\}\\ &\le \widetilde{\mathsf P}\left\{\tilde\beta^{(1)}_{T_n}(L)>N\right\}+\widetilde{\mathsf P}\left\{\tilde\beta^{(1)}(L)>N\right\}\\ &\quad+\widetilde{\mathsf P}\left\{\sup_{|t_1-t_2|\le\delta;\,t_i\le N}\left|W^*_{T_n}(t_2)-W^*_{T_n}(t_1)\right|>\varepsilon\right\}\\ &\quad+\widetilde{\mathsf P}\left\{\sup_{0\le t\le L}\left|\tilde\beta^{(1)}_{T_n}(t)-\tilde\beta^{(1)}(t)\right|>\delta\right\}\end{aligned}$$

for all $L>0$, $N>0$, $\varepsilon>0$, and $\delta>0$. It is also clear that an analogue of convergence (13) holds for the Wiener process $W^*_{T_n}(t)$, whence

$$\sup_{0\le t\le L}\left|W^*_{T_n}\left(\tilde\beta^{(1)}_{T_n}(t)\right)-W^*_{T_n}\left(\tilde\beta^{(1)}(t)\right)\right| \xrightarrow{\widetilde{\mathsf P}} 0$$

as $T_n\to\infty$. Then, in view of (14),

$$\sup_{0\le t\le L}\left|\tilde\beta^{(2)}_{T_n}(t)-W^*_{T_n}\left(\tilde\beta^{(1)}(t)\right)\right| \xrightarrow{\widetilde{\mathsf P}} 0\tag{15}$$

as $T_n \to \infty$. Using properties of stochastic integrals, we conclude that

$$\left|\mathsf{E}\, W^*_{T_n}(t)\widetilde{W}_{T_n}(t)\right| = \left|\mathsf{E}\int_0^{\tau_{T_n}(t)} \hat{g}_{T_n}\left(\tilde{\zeta}_{T_n}(s)\right) d\widetilde{W}_{T_n}(s)\, \widetilde{W}_{T_n}(t)\right|$$
$$= \left|\mathsf{E}\int_0^{\min(t,\tau_{T_n}(t))} \hat{g}_{T_n}\left(\tilde{\zeta}_{T_n}(s)\right) ds\right| \le \mathsf{E}\int_0^t \left|\hat{g}_{T_n}\left(\tilde{\zeta}_{T_n}(s)\right)\right| ds,$$

where $\tau_{T_n}(t) = \min\{s\colon \tilde{\beta}^{(1)}_{T_n}(s) = t\}$.

Now assumptions of Theorem 3.1 imposed on the function $Q^{(3)}_T(x)$ and Lemma 5.1 imply that

$$\mathsf{E}\int_0^t \left|\hat{g}_{T_n}\left(\tilde{\zeta}_{T_n}(s)\right)\right| ds \to 0$$

as $T_n \to \infty$ for all $t > 0$. Hence, $\mathsf{E}\, W^*_{T_n}(t)\widetilde{W}_{T_n}(t) \to 0$ as $T_n \to \infty$.

Since $W^*_{T_n}(t)$ and $\widetilde{W}_{T_n}(t)$ are asymptotically, as $T_n \to \infty$, non-correlated Wiener processes, $W^*_{T_n}(t)$ does not depend on $\widetilde{W}(t)$ asymptotically. It is clear that $\tilde{\beta}^{(1)}(t)$ is completely determined by $\tilde{\zeta}(s)$ for $s \le t$. Since a strong solution $\big(\xi_T(t), W_T(t)\big)$ of equation (1) is unique, the processes $\tilde{\zeta}(t)$ and $\widehat{W}(t)$ are measurable with respect to the σ-algebra $\sigma\big(\widetilde{W}(s), s \le t\big)$ generated by the Wiener process $\widetilde{W}(t)$ being the limit of $\widetilde{W}_{T_n}(t)$. Thus the process $W^*_{T_n}(t)$ does not depend on $\tilde{\beta}^{(1)}(t)$ asymptotically. The finite dimensional distributions of the process $W^*_{T_n}(t)$ do not depend on T_n, and hence the limit process is $W^*\big(\tilde{\beta}^{(1)}(t)\big)$, where $W^*(t)$ is a Wiener process being independent of the process $\tilde{\beta}^{(1)}(t)$. Considering (15) we conclude that

$$\sup_{0\le t\le L} \left|\tilde{\beta}^{(2)}_{T_n}(t) - W^*\left(\tilde{\beta}^{(1)}(t)\right)\right| \xrightarrow{\tilde{\mathsf{P}}} 0$$

as $T_n \to \infty$.

Therefore the process $\tilde{\beta}^{(2)}_{T_n}(t)$ weakly converges to $W^*\big(\tilde{\beta}^{(1)}(t)\big)$ as $T_n \to \infty$. This, in turn, means that the statement of Theorem 3.1 is valid for the process $\beta^{(2)}_{T_n}(t)$. Since the subsequence $T_n \to \infty$ is arbitrary, the uniqueness of the distributions of $W^*\big(\tilde{\beta}^{(1)}(t)\big)$ implies Theorem 3.1. □

Proof of Theorem 3.2. The proof of Theorem 3.2 is the same as that of Theorem 3.1 except the form of the process $\beta^{(1)}(t)$, which now is

$$\beta^{(1)}(t) = 2\left[\int_{y_0}^{\zeta(t)} g_0(x)\, dx + \int_0^t g_0\big(\zeta(s)\big)\sigma_0\big(\zeta(s)\big)\, d\widehat{W}(s)\right],$$

where $\big(\zeta(t), \widehat{W}(t)\big)$ is a solution of equation (6). Finally, one applies Theorem 2.3 of [7] to prove an analogue of convergence (15). □

Proof of Theorem 3.3. We apply Itô's formula to the process $\Phi_T(\xi_T(t))$, where

$$\Phi_T(x) = 2\int_0^x f'_T(u)\left(\int_0^u \frac{g_T(v)}{f'_T(v)}\, dv\right) du$$

and $\xi_T(t)$ is a solution of equation (1). Then we obtain

$$\beta^{(1)}_T(t) = 2c_0\int_0^t a_T\big(\xi_T(s)\big)\, ds + \alpha_T(t) + \eta^{(1)}_T(t),$$

where

$$\alpha_T(t) = 2\int_{x_0}^{\xi_T(t)} \left[f'_T(u)\int_0^u \frac{g_T(v)}{f'_T(v)}\,dv - c_0\right] du,$$

$$\eta_T^{(1)}(t) = -\int_0^t \left[\Phi'_T\big(\xi_T(s)\big) - 2c_0\right] dW_T(s).$$

Since condition (A_3) holds for the functions $a_T(x)$, Lemma 5.1 yields

$$\sup_{0\le t\le L}\left|\int_0^t a_T\big(\xi_T(s)\big)\,ds\right| \xrightarrow{\mathsf{P}} 0$$

as $T\to\infty$ for an arbitrary $L>0$. The obvious inequality

$$\begin{aligned}\mathsf{P}\left\{\sup_{0\le t\le L}|\alpha_T(t)|>\varepsilon\right\} &\le P_{NT} + \frac{1}{\varepsilon}\,\mathsf{E}\sup_{0\le t\le L}\left|\int_{x_0}^{\xi_T(t)}\left[\Phi'_T(u)-2c_0\right]du\right|\chi_{\{|\xi_T(t)|\le N\}}\\ &\le P_{NT} + \frac{2}{\varepsilon}\sup_{|x|\le N}\left|\int_{x_0}^{x}\left[f'_T(u)\int_0^u\frac{g_T(v)}{f'_T(v)}\,dv - c_0\right]du\right|\end{aligned}$$

being true for all $N>0$, $L>0$, and $\varepsilon>0$, where $P_{NT} = \mathsf{P}\left\{\sup_{0\le t\le L}|\xi_T(t)|>N\right\}$, combined with $\lim_{N\to\infty}\overline{\lim}_{T\to\infty} P_{NT}=0$, implies that

$$\sup_{0\le t\le L}|\alpha_T(t)| \xrightarrow{\mathsf{P}} 0$$

as $T\to\infty$. The convergence of P_{NT} follows from the inequalities $|G_T(x)|\ge C|x|^\alpha$ and (4). Indeed,

$$\sup_{0\le t\le L}|\xi_T(t)| \le \sup_{0\le t\le L}\left(\frac{|\zeta_T(t)|}{C}\right)^{\frac{1}{\alpha}}$$

and

$$\begin{aligned}P_{NT} &\le \mathsf{P}\left\{\sup_{0\le t\le L}\left(\frac{|\zeta_T(t)|}{C}\right)^{\frac{1}{\alpha}} > N\right\} \le \mathsf{P}\left\{\sup_{0\le t\le L}|\zeta_T(t)|^{\frac{1}{\alpha}} > C^{\frac{1}{\alpha}}N\right\}\\ &\le \frac{1}{C^{\frac{1}{\alpha}}N}\,\mathsf{E}\left\{\sup_{0\le t\le L}|\zeta_T(t)|^{\frac{1}{\alpha}}\right\}.\end{aligned}$$

Thus

$$\sup_{0\le t\le L}\left|\beta_T^{(1)}(t) - \eta_T^{(1)}(t)\right| \xrightarrow{\mathsf{P}} 0$$

as $T\to\infty$.

It is clear that $\eta_T^{(1)}(t)$ is an almost surely continuous martingale with characteristics

$$\left\langle \eta_T^{(1)}\right\rangle(t) = 4b_0^2 t + \int_0^t q_T\big(\xi_T(s)\big)\,ds,$$

where $q_T(x) = \left[\Phi'_T(x) - 2c_0\right]^2 - 4b_0^2$. Condition ($A_3$) holds for the function $q_T(x)$, and thus

$$\sup_{0\le t\le L}\left|\left\langle \eta_T^{(1)}\right\rangle(t) - 4b_0^2 t\right| \xrightarrow{\mathsf{P}} 0$$

as $T\to\infty$ for an arbitrary $L>0$ by Lemma 5.1.

Now we use the random change of time, that is, $\eta_T^{(1)}(t) = W_T^*\big(\langle\eta_T^{(1)}\rangle(t)\big)$, where $W_T^*(t)$ is a Wiener process. Similarly to the proof of convergence (15) we obtain

$$\sup_{0\le t\le L}\left|\beta_T^{(1)}(t) - W_T^*\left(4b_0^2 t\right)\right| \xrightarrow{\mathsf{P}} 0$$

as $T \to \infty$. Therefore the process $\beta_T^{(1)}(t)$ weakly converges as $T \to \infty$ to the process $2b_0 W(t)$. □

5. Auxiliary results

Lemma 5.1. *Let ξ_T be a solution of equation* (1) *belonging to the class $K(G_T)$. If condition* (A_3) *holds for measurable locally bounded functions $q_T(x)$, then*

$$\sup_{0\le t\le L}\left|\int_0^t q_T\big(\xi_T(s)\big)\,ds\right| \xrightarrow{\mathsf{P}} 0$$

as $T \to \infty$ for an arbitrary $L > 0$.

Lemma 5.2. *Let ξ_T be a solution of equation* (1) *belonging to the class $K(G_T)$. Let*

$$\zeta_T(t) = G_T(\xi_T(t)), \qquad \eta_T(t) = \int_0^t G_T'(\xi_T(s))\,dW_T(s).$$

Assume that the process $\big(\zeta_T(t), \eta_T(t)\big)$ is stochastically equivalent to $\big(\tilde\zeta_T(t),\ \tilde\eta_T(t)\big)$. If $g(x)$ and $q(x)$ are measurable and locally bounded functions, then the process

$$S_T(t) = \int_0^t g\big(\zeta_T(s)\big)\,ds + \int_0^t q\big(\zeta_T(s)\big)\,d\eta_T(s)$$

is stochastically equivalent to

$$\widetilde S_T(t) = \int_0^t g\left(\tilde\zeta_T(s)\right)ds + \int_0^t q\left(\tilde\zeta_T(s)\right)d\tilde\eta_T(s).$$

Lemma 5.3. *Let ξ_T be a solution of equation* (1). *Let $g \neq 0$ be a locally square integrable real function defined in a bounded measurable set B of a positive Lebesgue measure. Then*

$$\int_0^\infty g^2\big(\xi_T(s)\big)\,ds = \infty$$

with probability one for all $T > 0$.

The proof of Lemmas 5.1 and 5.2 is given in the paper [7]. Lemma 5.3 with $B \subseteq [-1, 0]$ is proved in [8, Lemma 3.1].

6. Examples

Denote by $\{b_T\}$ a family of constants such that $b_T > 1$ and $b_T \uparrow \infty$ as $T \to \infty$.

Example 6.1. Let $a_T(x) \equiv 0$ in equation (1). Consider

$$g_T(x) = \sqrt{\frac{b_T}{1 + b_T^2 x^2}}.$$

It is clear that $g_T^2(x)$ is a δ-shaped family at the point $x = 0$ with weight π. Assumptions of Remark 3.1 hold with $g_0(x) = \frac{\pi}{2}\operatorname{sign} x$ and $G_T(x) = x$. Thus Theorem 3.2 implies that the process

$$\beta_T^{(2)}(t) = \int_0^t \sqrt{\frac{b_T}{1 + b_T^2 W_T^2(s)}}\,dW_T(s)$$

weakly converges to the process $\beta^{(2)}(t) = W^*\left(\beta^{(1)}(t)\right)$ as $T \to \infty$, where

$$\beta^{(1)}(t) = \pi\left[\int_{x_0}^{\zeta(t)} \operatorname{sign} x\,dx - \int_0^t \operatorname{sign}\zeta(s)\,dW(s)\right],$$

$\zeta(t) = x_0 + W(t)$, and the processes $W^*(t)$ and $W(t)$ are independent.

Thus $\beta^{(2)}(t) = W^*\big(\pi L_W(t,\, x_0)\big)$, where

$$L_W(t,\, x_0) = |x_0 + W(t)| - |x_0| - \int_0^t \operatorname{sign}\big(x_0 + W(s)\big)\, dW(s)$$

is the local time of the Wiener process $W(t)$ at the point x_0 in the interval $[0, t]$.

Example 6.2. Consider equation (1), where $a_T(x) = b_T\big[1 + (b_T x - 1)^2\big]^{-1}$. We are going to show that equation (1) belongs to the case $K\,(G_T)$ if

$$G_T(x) = f_T(x) = \int_0^x \exp\left\{-2\int_0^u a_T(v)\, dv\right\} du.$$

Indeed,

$$f'_T(x) = \exp\left\{-2\int_0^x a_T(v)\, dv\right\} = \exp\Big\{-2\operatorname{arctg}(b_T v - 1)\big|_0^x\Big\}$$
$$= \exp\big\{-2\left[\operatorname{arctg}(b_T x - 1) + \operatorname{arctg} 1\right]\big\} \to \sigma_0(x) = \begin{cases} e^{-\frac{3}{2}\pi}, & x > 0,\\ e^{\frac{\pi}{2}}, & x < 0,\end{cases} \quad \text{as } T \to \infty.$$

Since $f'_T(x) a_T(x) + \frac{1}{2}\, f''_T(x) = 0$, we have

$$\left[G'_T(x) a_T(x) + \frac{1}{2}\, G''_T(x)\right]^2 + \left[G'_T(x)\right]^2 = \left[f'_T(x)\right]^2 \le C \le C\left[1 + |G_T(x)|^2\right].$$

We derive $|G_T(x)| \ge C|x|^\alpha$ for all $x \in \mathbb{R}$ with $C = \delta_0$ and $\alpha = 1$ from inequalities $0 < \delta_0 \le G'_T(x) = f'_T(x) \le C_0$. In addition,

$$\left|\int_0^x f'_T(u)\left(\int_0^u \frac{\chi_B\big(G_T(v)\big)}{f'_T(v)}\, dv\right) du\right| \le \frac{C_0}{\delta_0}\left|\int_0^x\int_0^u \chi_B\big(G_T(v)\big)\, dv\, du\right| \le C_1\lambda(B)|x|.$$

Thus condition (A_2) holds for the case of $\psi\,(|x|) = C_1|x|$ and $m = 1$.

Under assumptions (A_4),

$$q_T^{(1)}(x) = G'_T(x) a_T(x) + \frac{1}{2}\, G''_T(x) \equiv 0,$$
$$q_T^{(2)}(x) = \left[G'_T(x)\right]^2 - \sigma_0^2\big(G_T(x)\big) = \begin{cases} \left[G'_T(x)\right]^2 - e^{-3\pi} \to 0, & x > 0,\\ \left[G'_T(x)\right]^2 - e^{\pi} \to 0, & x < 0\end{cases} \quad \text{as } T \to \infty,$$

and

$$\sup_{|x| \le N} f'_T(x)\left|\int_0^x \frac{q_T^{(2)}(v)}{f'_T(v)}\, dv\right| \le \frac{C_0}{\delta_0}\int_{-N}^N \left|q_T^{(2)}(v)\right| dv \to 0 \quad \text{as } T \to \infty.$$

Therefore conditions (A_4) hold for

$$a_0(x) \equiv 0, \qquad \sigma_0(x) = \begin{cases} e^{-\frac{3}{2}\pi}, & \text{if } x > 0,\\ e^{\frac{\pi}{2}}, & \text{if } x \le 0,\end{cases} \qquad y_0 = x_0\sigma_0(x_0).$$

Thus the process $\zeta_T(t) = G_T(\xi_T(t))$ weakly converges to a solution $\zeta(t)$ of the Itô equation

$$\zeta(t) = x_0\sigma_0(x_0) + \int_0^t \sigma_0\big(\zeta(s)\big)\, d\widehat{W}(s)$$

as $T \to \infty$.

Assumptions of Remark 3.2 with $\beta^{(1)}(t) = t/2$ hold for the functions $g_T(x) = \cos(b_T x)$ if $\hat{g}_T(x) = g_T\left(G_T^{-1}(x)\right)$, where $G_T^{-1}(x)$ are the inverse functions to $G_T(x)$ and $g_0(x) \equiv \frac{1}{2}$. Hence

$$\beta_T^{(2)}(t) = \int_0^t \cos\big(b_T\xi_T(s)\big)\, dW_T(s)$$

weakly converges to $\frac{1}{\sqrt{2}}W^*(t)$ as $T \to \infty$, where $W^*(t)$ is a Wiener process.

Example 6.3. Let

$$a_T(x) = -\frac{1}{4}\frac{b_T^2 x}{1+b_T^2x^2}$$

in equation (1). In this case, equation (1) belongs to the class $K(G_T)$ for $G_T(x) = x^2$, and conditions (A$_4$) hold for $a_0(x) = \frac{1}{2}$, $\sigma_0(x) = 2\sqrt{|x|}$, and $y_0 = x_0^2$. According to Remark 2.4, the process $\zeta_T(t) = \xi_T^2(t)$ weakly converges to a solution $\zeta(t)$ of equation

$$\zeta(t) = x_0^2 + \frac{1}{2}\,t + 2\int_0^t \sqrt{\zeta(s)}\,d\widehat{W}(s) \tag{16}$$

as $T \to \infty$. Assumptions of Theorem 3.2 hold for the functions

$$g_T(x) = \frac{\sqrt[4]{b_T}}{\sqrt{\ln b_T}}\,\frac{\cos(b_T x)}{\sqrt[8]{1+b_T^2x^2}}$$

if

$$g_0(x) = \frac{1}{4}\,\frac{1}{\sqrt[4]{|x|}}, \qquad \hat{g}_T(x) = \frac{\sqrt[4]{b_T}}{\sqrt{\ln b_T}}\,\frac{\cos\left(b_T\sqrt{|x|}\right)}{\sqrt[8]{1+b_T^2x^2}}.$$

In this case, $\hat{g}_T\left(x^2\right) = g_T(x)$, and the process

$$\beta_T^{(2)}(t) = \frac{\sqrt[4]{b_T}}{\sqrt{\ln b_T}}\int_0^t \frac{\cos\bigl(b_T\xi_T(s)\bigr)}{\sqrt[8]{1+b_T^2\xi_T^2(s)}}\,dW_T(s)$$

weakly converges to the process $W^*\left(\beta^{(1)}(t)\right)$ as $T \to \infty$ by Theorem 3.2, where

$$\beta^{(1)}(t) = \frac{2}{3}\left[\zeta^{\frac{3}{4}}(t) - |x_0|^{\frac{3}{2}}\right] - \int_0^t \sqrt[4]{\zeta(s)}\,d\widehat{W}(s),$$

$\bigl(\zeta(t), \widehat{W}(t)\bigr)$ is a solution of equation (16), and $W^*(t)$ is a Wiener process such that $W^*(t)$ and $\beta^{(1)}(t)$ are independent.

Example 6.4. Let $a_T(x) = b_T\chi_{[0,\lambda/b_T]}(x)$ and $\lambda > 0$ in equation (1). If

$$G_T(x) = f_T(x) = \int_0^x \exp\left\{-2\int_0^u a_T(v)\,dv\right\}du,$$

then equation (1) belongs to the class $K(G_T)$, and conditions (A$_4$) hold for $a_0(x) = 0$,

$$\sigma_0(x) = \begin{cases} e^{-2\lambda}, & x > 0,\\ 1, & x \le 0,\end{cases}$$

and $y_0 = x_0\sigma_0(x_0)$. Thus, according to Remark 2.4, the process $\zeta_T(t) = G_T\bigl(\xi_T(t)\bigr)$ weakly converges to a solution $\zeta(t)$ of the Itô equation

$$\zeta(t) = x_0\sigma_0(x_0) + \int_0^t \sigma_0\bigl(\zeta(s)\bigr)\,d\widehat{W}(s) \tag{17}$$

as $T \to \infty$. Assumptions of Theorem 3.2 with

$$g_0(x) = \frac{\pi}{2}\,\frac{\operatorname{sign} x}{\sigma_0(x)}, \qquad \hat{g}_T(x) = \left(\frac{b_T}{1+b_T^2\left[G_T^{-1}(x)\right]^2}\right)^{\frac{1}{2}}$$

hold for the functions $g_T(x) = \left(\frac{b_T}{1+b_T^2x^2}\right)^{1/2}$, where $G_T^{-1}(x)$ denotes the inverse functions to $G_T(x)$.

Hence the process

$$\beta_T^{(2)}(t) = \int_0^t \sqrt{\frac{b_T}{1+b_T^2\xi_T^2(s)}}\, dW_T(s) \tag{18}$$

weakly converges to the process $W^*\left(\beta^{(1)}(t)\right)$ as $T \to \infty$ according to Theorem 3.2, where

$$\beta^{(1)}(t) = \pi\left[\int_{x_0\sigma_0(x_0)}^{\zeta(t)} \frac{\operatorname{sign} v}{\sigma_0(v)}\, dv - \int_0^t \operatorname{sign}\zeta(s)\, d\widehat{W}(s)\right],$$

$\left(\zeta(t), \widehat{W}(t)\right)$ is a solution of equation (17), $W^*(t)$ is a Wiener process, and $W^*(t)$ and $\beta^{(1)}(t)$ are independent.

Remark 6.1. The classes $K(G_T)$ related to equation (1) are not defined uniquely. In particular, if $a_T(x)$ in equation (1) are the same as in Example 6.4, then equation (1) belongs to the class $K(G_T)$ with $G_T(x) = x^2$, and conditions (A_4) hold if $a_0(x) = 1$, $\sigma_0(x) = 2\sqrt{|x|}$, and $y_0 = x_0^2$. According to Remark 2.4, the process $\zeta_T(t) = \xi_T^2(t)$ weakly converges as $T \to \infty$ to a solution $\zeta(t)$ of the Itô equation

$$\zeta(t) = x_0^2 + t + 2\int_0^t \sqrt{\zeta(s)}\, d\widehat{W}(s). \tag{19}$$

Here $\zeta(t) \ge 0$ with probability one for all $t \ge 0$. Moreover, assumptions of Theorem 3.2 with

$$\hat{g}_T(x) = \left(\frac{b_T}{1+b_T^2|x|}\right)^{\frac{1}{2}}, \qquad g_0(x) = \frac{\pi}{4\sqrt{|x|}}$$

hold for the functions $g_T(x)$ defined in Example 6.4.

Thus, by Theorem 3.2, the process $\beta_T^{(2)}(t)$ defined by relation (18) weakly converges to $W^*\left(\beta^{(1)}(t)\right)$ as $T \to \infty$, where

$$\beta^{(1)}(t) = \pi\left[\sqrt{\zeta(t)} - |x_0| - \widehat{W}(t)\right],$$

$\left(\zeta(t), \widehat{W}(t)\right)$ is a solution of equation (19), $W^*(t)$ is a Wiener process, and $W^*(t)$ and $\beta^{(1)}(t)$ are independent.

Example 6.5. Let $a_T(x) \equiv 0$ in equation (1). Then $\xi_T(t) = x_0 + W_T(t)$ is a solution of equation (1) of the class $K(G_T)$ for $G_T(x) = x$. In this case, conditions (A_4) hold for $a_0(x) = 0$ and $\sigma_0(x) = 1$. Assumptions of Theorem 3.3 with $c_0 = 1$, $b_0^2 = \frac{1}{2}$ hold for $g_T(x) = b_T \sin(b_T x)$. According to Remark 2.4, the process $\xi_T(t)$ weakly converges to $\zeta(t) = x_0 + \widehat{W}(t)$ as $T \to \infty$, where $\widehat{W}(t)$ is a Wiener process, and

$$\beta_T^{(1)}(t) = \int_0^t b_T \sin\bigl(b_T\xi_T(s)\bigr)\, ds$$

weakly converges to $\sqrt{2}\,\widehat{W}(t)$ as $T \to \infty$ in view of Theorem 3.3.

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