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PROPERTIES OF HIGHLY RELIABLE SYSTEMS WITH PROTECTION IN THE CASE OF POISSON RENEWAL PROCESS

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Abstract. Some upper bounds for characteristics of reliability of a highly reliable system with protection are given for the case where the renewal process in the system is Poissonian.

1. Introduction

Uniform estimates of the deviation between the exponential distribution function and solutions of the integral equation

$$\Phi = \theta L + (1-\theta)\Phi \star K \tag{1}$$

are obtained in the paper [1], where L and K are some distribution functions of non-negative random variables, $\theta \in (0,1)$, and the symbol $\star$ stands for the convolution of distribution functions,

$$\Phi \star K(t) = \int_0^t \Phi(t-x)\, dK(x).$$

These estimates can be used when studying characteristics of the reliability of a system with protection [2, p. 75]. The papers [3–5] are devoted to such studies for the general case.

The aim of this paper is to improve estimates obtained in [1] and to consider different estimates for the case where the renewal process is Poissonian. In particular, we estimate the trouble-free time with a given reliability as well as estimate the distribution function of random intervals between refusals and that of the number of refusals.

The paper is organized as follows. The main results of the paper are formulated in Section 2, while Section 3 is devoted to their proofs.

2. Main conditions and results

A system with protection (see [2, p. 75]) consists of an alternating process and an independent renewal process. The system fails at every renewal moment if the alternating process is under repair at this moment.

We assume throughout this paper that the renewal process in the system is Poissonian with intensity v.

Let $m(F)$ and $m_2(F)$ be the expectation and second moment, respectively, of a random variable with the distribution function $F(t)$.

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The distribution functions of the trouble-free and repairing intervals of the alternating process are denoted by G and B, respectively. Let

$$\hat{\beta}(s) = \int_0^{+\infty} e^{-sx}\, dB(x)$$

be the Stieltjes–Laplace transform of the distribution function B and let $E_a(t) = 1 - e^{-at}$, $t \geq 0$, be the exponential distribution function with parameter a.

If the probability of the event that the trouble-free time of a system is at least t_0 exceeds γ, then t_0 is called the guaranteed trouble-free time with reliability γ.

Theorem 2.1. *The guaranteed trouble-free time t_0 with reliability γ in a system with protection is estimated from below as*

$$t_0 \geq T \ln \frac{1-\theta}{\gamma + 2\theta\varkappa},$$

where $T = \hat{\beta}(v) m(G)/(1-\hat{\beta}(v))$, $\theta = 1 - \hat{\beta}(v)$, and

$$A(t) = \frac{1}{\hat{\beta}(v)} \int_0^t e^{-vx}\, dB(x), \qquad K(t) = G \star A(t),$$

$$m(A) = \frac{1}{\hat{\beta}(v)} \int_0^{+\infty} x e^{-vx}\, dB(x) \leq m(B),$$

$$\varkappa = \frac{m_2(K)}{2m^2(K)} = \frac{m_2(G) + 2m(G)m(A) + m_2(A)}{2\big(m(G) + m(A)\big)^2}.$$

Moreover $m_2(A) \leq m_2(B)$.

The distribution function of the trouble-free time in a system with protection is denoted by $\Phi_1(x)$ if the repairing period of the alternating process started at the initial moment or by $\Psi_1(x)$ if the trouble-free period of the alternating process started at the initial moment.

Theorem 2.2. *The distribution function of the trouble-free time in a system with protection admits the following bounds:*

$$E_q(x) - qm(L) - 2\varkappa \int_0^x \big(1 - B(y)\big) v e^{-vy}\, dy \leq \Phi_1(x),$$

$$\Phi_1(x) \leq (1-\theta) E_q(x) + (1 + 2\varkappa) \int_0^x \big(1 - B(y)\big) v e^{-vy}\, dy,$$

$$\Psi_1(x) \leq (1-\theta) E_q(x) + \theta(1 + 2\varkappa) G(x),$$

$$\Psi_1(x) \geq E_q(x) - q\left(m(L) + \int_0^x \big(1 - G(y)\big)\, dy\right) - 2\varkappa\theta G(x),$$

where

$$r = \frac{\theta}{1-\theta}, \qquad q = \frac{r}{m(G) + m(A)} \leq \frac{r}{m(G)} = \frac{1}{T},$$

$$m(L) = \frac{1}{\theta} \int_0^{+\infty} x\big(1 - B(x)\big)\, dE_v(x) \leq \frac{1}{v}.$$

Theorem 2.3. *If the functioning of a system started at the initial moment with the trouble-free period of the alternating process, then the distribution of the number of refusals ξ is such that*

$$\mathsf{P}(\xi \geq n) \leq \theta + (1-\theta)\bar{p}_n(t) + 2\varkappa\theta \min\{n, 1 + qt\},$$

where

$$\bar{p}_{n+1}(t) = \left(1 - \sum_{k=1}^{n} \tau_k\right) E_q(t) + \sum_{k=1}^{n} \tau_k \left(1 - \sum_{i=1}^{n-k} \tau_i\right) E_q^{\star\,2}(t)$$
$$+ \sum_{k=1}^{n-1} \tau_k \sum_{i=1}^{n-k} \tau_i \left(1 - \sum_{j=1}^{n-k-i} \tau_j\right) E_q^{\star\,3}(t) + \dots$$
$$+ \big(\tau_1^{n-1}(1-\tau_1) + (n-1)\tau_1^{n-2}\tau_2\big) E_q^{\star\,n}(t) + \tau_1^n E_q^{\star(n+1)}(t), \qquad n \ge 0;$$
$$\theta_n = \int_0^{+\infty} \frac{(vx)^{n-1}}{(n-1)!} e^{-vx}\, dB(x) = \int_0^{+\infty} \left(E_v^{\star(n-1)}(x) - E_v^{\star\,n}(x)\right) dB(x), \qquad n \ge 1;$$
$$\tau_k = \frac{\theta_{k+1}}{\theta}, \qquad k \ge 1.$$

In particular,

$$\mathsf{P}(\xi \ge 1) = \Psi_1(t) \le \theta + (1-\theta) E_q(t) + 2\varkappa\theta,$$
$$\mathsf{P}(\xi \ge 2) \le \theta + (1-\theta)\left(\frac{\theta_2}{\theta} E_q^{\star\,2}(t) + \frac{\theta - \theta_2}{\theta} E_q(t)\right) + 2\varkappa\theta\big(1 + E_q(t)\big).$$

Remark 2.1. If $t > 0$ is fixed, then the family of numbers $p_0(t) = 1 - \bar{p}_1(t)$ and $p_n(t) = \bar{p}_n(t) - \bar{p}_{n+1}(t)$, $n \ge 1$, is the distribution of the number of events occurring in a memoryless stationary input flow during the interval $(0,t)$. Its moment generating function is given by

$$\sum_{n=0}^{\infty} p_n(t) x^n = e^{qt(\omega(x)-1)},$$

where

$$\omega(x) = \sum_{n=1}^{\infty} \frac{\theta_{n+1}}{\theta}\, x^n$$

(see [7, p. 42]).

Remark 2.2. The mean of the trouble-free time in a system with protection is equal to $T + \frac{1}{v}$ if the system started functioning with a repairing period of the alternating process or to $T + \frac{1}{v} + \frac{1}{m(G)}$ otherwise.

3. Proofs

Proof of Theorem 2.1. If a system started functioning with the repairing period of the alternating process, then the distribution function of the trouble-free time $\Phi_1(t)$ is a solution of equation (1) with $K(t) = G \star A(t)$, where

$$A(t) = \frac{1}{\hat{\beta}(v)} \int_0^t e^{-vx}\, dB(x), \qquad L(t) = \frac{1}{\theta} \int_0^t \big(1 - B(x)\big)\, dE_v(x).$$

Denote by $\Phi(t)$ a solution of equation (1) with $L(t) = K(t)$. According to Theorem 2 of [1], this solution admits the uniform bound

$$|\Phi(t) - E_q(t)| \le 2r\varkappa. \tag{2}$$

The function $\Phi_1(t)$ can be written in terms of $\Phi(t)$ as

$$\Phi_1(t) = L \star \big(\theta + (1-\theta)\Phi(t)\big). \tag{3}$$

Now relations (2) and (3) imply that

$$\Psi_1(t) \le \Phi_1(t) \le (1-\theta) E_q(t) + \theta(1 + 2\varkappa). \tag{4}$$

Equating the right hand side of this inequality to $1-\gamma$ we obtain an expression involving the guaranteed trouble-free time; then we use the inequality $\frac{1}{q} \geq T$.

The inequalities $m(A) \leq m(B)$ and $m_2(A) \leq m_2(B)$ follow from $A(t) \geq B(t)$, $t > 0$. In turn, the latter inequality follows from Lemma 3.1 below.

Lemma 3.1. *Let $V(t)$ be a non-negative monotone function, and let $F(t)$ be a distribution function of a non-negative random variable,*

$$\theta = \int_0^\infty V(x)\,dF(x), \qquad L(t) = \frac{1}{\theta}\int_0^t V(x)\,dF(x).$$

Then $L(t) \leq F(t)$ for all $t > 0$ if $V(t)$ is a non-decreasing function. Otherwise, $F(t) \leq L(t)$ for all $t > 0$.

Proof of Lemma 3.1. If $V(t)$ is a non-decreasing function, then $F - L$ equals 0 at the origin and increases in the domain where $V(x) < \theta$; then $F - L$ decreases to 0 at infinity. Thus $F - L$ is non-negative. The case of a non-increasing function $V(t)$ is considered analogously.

Lemma 3.1 is proved. □

Theorem 2.1 is proved. □

Proof of Theorem 2.2. The upper bounds are obtained in (4). We derive the lower bounds from the following results.

Lemma 3.2. *For all $t > 0$,*

$$E_q \star L(t) \geq E_q(t) - q\int_0^t \big(1 - L(x)\big)\,dx.$$

Proof. It is clear that

$$\begin{aligned} E_q \star \big(1 - L(t)\big) &= q\int_0^t \big(1 - L(t-x)\big)e^{-qx}\,dx \leq q\int_0^t \big(1 - L(t-x)\big)\,dx \\ &= q\int_0^t \big(1 - L(x)\big)\,dx. \end{aligned}$$

Lemma 3.2 is proved. □

Theorem 2.2 is proved. □

Proof of Theorem 2.3. The functions $\mathsf{P}(\xi \geq n) = \Psi_n(t)$ are minimal solutions of the renewal type equation

$$\Psi_n(t) = \sum_{k=2}^n \theta_k A_k \star G \star \Psi_{n+1-k}(t) + \left(\theta - \sum_{k=2}^n \theta_k\right) L_n \star G(t) + (1-\theta) A \star G \star \Psi_n(t).$$

Since $\frac{(vx)^{n-1}}{(n-1)!}$ is an increasing function, Lemma 3.1 implies that $A_k(t) \leq A(t)$ for all $t > 0$. Thus

$$\begin{aligned} \Psi_n(t) &\leq \sum_{k=2}^n \frac{\theta_k}{\theta} \Phi \star \Psi_{n+1-k}(t) + \left(\theta - \sum_{k=2}^n \theta_k\right)\left(1 + \frac{1-\theta}{\theta}\Phi(t)\right) \\ &= \sum_{k=1}^{n-1} \tau_k \Phi \star \Psi_{n-k}(t) + \left(1 - \sum_{k=1}^{n-1} \tau_k\right)\big(\theta + (1-\theta)\Phi(t)\big), \end{aligned} \tag{5}$$

where $\Phi(t)$ is a solution of equation (1) with $L(t) = K(t)$.

Note that $\bar{p}_n(t)$ satisfies the recurrence equation

$$\bar{p}_n(t) = \left(1 - \sum_{k=1}^{n-1} \tau_k\right) E_q(t) + \sum_{k=1}^{n-1} \tau_k \bar{p}_{n-k} \star E_q(t). \tag{6}$$

Now we prove the inequality

$$\begin{aligned}
\Psi_n(t) &\le \theta + (1-\theta)\bar{p}_n(t) \\
&\quad + 2\varkappa\theta \Biggl(1 + \sum_{k=1}^{n-1} \tau_k E_q(t) + \sum_{k=1}^{n-2} \tau_k \sum_{i=1}^{n-1-k} \tau_i E_q^{\star 2}(t) \\
&\qquad + \sum_{k=1}^{n-3} \tau_k \sum_{i=1}^{n-2-k} \tau_i \sum_{j=1}^{n-1-k-i} \tau_j E_q^{\star 3}(t) + \dots \\
&\qquad + \left(\tau_1^{n-2} + (n-2)\tau_1^{n-3}\tau_2\right) E_q^{\star(n-2)}(t) + \tau_1^{n-1} E_q^{\star(n-1)}(t) \Biggr)
\end{aligned} \tag{7}$$

for $n \ge 1$.

Indeed, inequality (7) for $n = 1$ follows from (4). Now we change n for $n-k$ in (7) and convolute the result with $\Phi(t)$. According to inequality (2),

$$(1-\theta)\bar{p}_{n-k} \star \Phi(t) \le (1-\theta)\bar{p}_{n-k} \star E_q(t) + 2\varkappa\theta\bar{p}_{n-k}(t).$$

For other terms, $\Phi(t) \le 1$. Therefore

$$\begin{aligned}
&\Psi_{n-k} \star \Phi(t) \\
&\quad \le \theta + (1-\theta)\bar{p}_{n-k} \star E_q(t) \\
&\qquad + 2\varkappa\theta \Biggl(1 + \bar{p}_{n-k}(t) + \sum_{i=1}^{n-k-1} \tau_i E_q(t) + \sum_{i=1}^{n-k-2} \tau_i \sum_{j=1}^{n-k-1-i} \tau_j E_q^{\star 2}(t) + \dots \\
&\qquad\quad + \left(\tau_1^{n-k-2} + (n-k-2)\tau_1^{n-k-3}\tau_2\right) E_q^{\star(n-k-2)}(t) \\
&\qquad\qquad + \tau_1^{n-k-1} E_q^{\star(n-k-1)}(t) \Biggr) \\
&\quad = \theta + (1-\theta)\bar{p}_{n-k} \star E_q(t) \\
&\qquad + 2\varkappa\theta \Biggl(1 + E_q(t) + \sum_{i=1}^{n-k-1} \tau_i E_q^{\star 2}(t) + \sum_{i=1}^{n-k-2} \tau_i \sum_{j=1}^{n-k-1-i} \tau_j E_q^{\star 3}(t) + \dots \\
&\qquad\quad + \left(\tau_1^{n-k-2} + (n-k-2)\tau_1^{n-k-3}\tau_2\right) E_q^{\star(n-k-1)}(t) \\
&\qquad\qquad + \tau_1^{n-k-1} E_q^{\star(n-k)}(t) \Biggr).
\end{aligned} \tag{8}$$

Now we substitute (8) into the right hand side of (5) and use the inequality

$$\theta + (1-\theta)\Phi(t) \le \theta(1+2\varkappa) + (1-\theta)E_q(t),$$

which follows from (2). Finally we derive (7) from (6).

Since the sums in (7) do not exceed 1, we get

$$\Psi_n(t) \le \theta + (1-\theta)\bar{p}_n(t) + 2\varkappa\theta \left(1 + E_q(t) + E_q^{\star 2}(t) + \dots + E_q^{\star(n-1)}(t)\right),$$

whence we obtain the statement of Theorem 2.3 as the terms in brackets do not exceed 1 and

$$1 + E_q(t) + E_q^{\star 2}(t) + \dots = 1 + qt. \qquad \square$$

The statement of Remark 2.2 is obtained by integrating (3) and (1).

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