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FOURIER SERIES AND FOURIER–HAAR SERIES FOR STOCHASTIC MEASURES

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Abstract. The Fourier series and Fourier–Haar series are introduced for general stochastic measures. The convergence of partial sums of these series and the absolute continuity of a stochastic measure are studied. An application is given for the convergence of solutions of the stochastic heat equation.

1. Introduction

Representations of stochastic processes by random series is an important tool for the approximation of random functions. Stochastic processes and random series generated by a general stochastic measure μ defined on Borel subsets of $(0, 1]$ are considered in this paper. The only restriction imposed on the measure μ is the σ-additivity in probability. By analogy with the classical case, we introduce the Fourier series and Fourier–Haar series for general stochastic measures. We prove that the values of μ as well as the values of stochastic integrals with respect to μ can be approximated with the help of these series.

Representations of stochastic processes in the form of random series are studied starting from the well-known Paley–Wiener decomposition for the Wiener process. An analogous decomposition for the fractional Brownian motion is obtained in [1]. Wavelet decompositions of stochastic processes have been studied recently; see, for example, [2].

There is an extensive literature devoted to Fourier series with random coefficients and their sums. Properties of Fourier series with independent coefficients are studied in detail in [3, 4].

The necessary results for Fourier series and Fourier–Haar series used in the current paper can be found, for example, in [5, 6].

The paper is constructed as follows. Section 2 contains necessary results and facts. The proofs of theorems concerning the convergence of Fourier series and the absolute continuity of the stochastic measure are given in Section 3. An application for the convergence of solutions of stochastic heat equation is discussed in Section 4. Some properties of Fourier–Haar series for stochastic measures are presented in Section 5.

2. Preliminaries

The set of all random variables defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ is denoted by $L_0 = L_0(\Omega, \mathcal{F}, \mathsf{P})$. More precisely, L_0 denotes the classes of P-equivalent random variables. The convergence in L_0 is understood as the convergence in probability. Let S be an arbitrary set and let $\mathcal{B}$ be the σ-algebra of subsets of S.

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Definition 2.1. An σ-additive mapping $\mu\colon \mathcal{B} \to L_0$ is called a *stochastic measure.*

We do not assume that μ is nonnegative or is adapted. In this sense, our definition is general. Such a function of sets is called a general stochastic measure in [7].

One can consider $\mu(A) = \int_0^T \mathbf{1}_A(s)\,dX(s)$ as an example of a stochastic measure, where $X(s)$ is a square integrable martingale or a fractional Brownian motion with Hurst index $H > 1/2$. Another example of a stochastic measure is represented by an α-stable measure defined on the Borel σ-algebra (see [8, Chapter 3]). Other examples can be found in [7, Chapters 7 and 8]. Conditions for the differences of a stochastic process with independent increments to generate a stochastic measure can also be found in [7, Chapters 7 and 8].

The theory of integration of real-valued functions with respect to stochastic measures is given in [7, 9], for example. In particular, it is shown in [7, 9] that every measurable function is integrable with respect to any stochastic measure μ. Moreover, an analogue of the Lebesgue dominated convergence theorem holds; see [7, Corollary 1.2] or [9, Proposition 7.1.1].

We say that f_n converges to f (and write $f_n \to f$) μ-almost everywhere if $\mu(A) = 0$ almost surely for each set $A \subset \{f_n \not\to f\}$, $A \in \mathcal{B}$.

The integral of random functions with respect to the real measure dx is understood in the Riemann sense. This type of integral is studied in [10]. Here we briefly discuss the definition and an important property of such an integral.

Definition 2.2. Let $B \subset \mathbb{R}^d$ be a Jordan measurable set and let $\xi\colon B \times \Omega \to \mathbb{R}$ be a measurable random function. We say that ξ is integrable in the set B if, for an arbitrary sequence of partitions

$$B = \bigcup_{1\le k\le k_n} B_{kn}, \quad n \ge 1, \qquad \max_k \operatorname{diam} B_{kn} \to 0, \quad n \to \infty,$$

and all $x_{kn} \in B_{kn}$, the limit in probability of integral sums

$$\int_B \xi(x)\,dx := \mathsf{P}\lim_{n\to\infty} \sum_{1\le k\le k_n} \xi(x_{kn})\mathsf{m}\,(B_{kn})$$

exists.

Here m denotes the Jordan measure, and the sets B_{kn}, $1 \le k \le k_n$, are Jordan measurable in every partition and such that their intersections coincide with the intersections of their boundaries.

If the usual Riemann integral exists for every trajectory of ξ and for every fixed ω, then ξ is integrable in the sense of Definition 2.2, and moreover both integrals coincide.

Theorem 2.1 (Theorem 4.1 [10])**.** *Let μ be a stochastic measure in $(S, \mathcal{B})$ and let $B \subset \mathbb{R}^d$ be a Jordan measurable set. Assume that $h(x, s)\colon B \times S \to \mathbb{R}$ is a measurable nonrandom function being Riemann integrable on B with respect to dx for every fixed s. We further assume that*

$$|h(x, s)| \le q(s), \qquad \int_B |h(x, s)|\,dx \le q_1(s),$$

where $q, q_1\colon S \to \mathbb{R}$ are integrable on S functions with respect to $d\mu(s)$. Then the random function $\xi(x) = \int_S h(x, s)\,d\mu(s)$ is integrable on B, and

$$\int_B dx \int_S h(x, s)\,d\mu(s) = \int_S d\mu(s) \int_B h(x, s)\,dx.$$

3. Fourier series for general stochastic measures

Let $\mathcal{B}$ be the Borel σ-algebra of subsets of $(0,1]$. For a given stochastic measure μ, consider the following Fourier series. Put

$$
\begin{aligned}
\xi_k &= \int_{(0,1]} \exp\{-2\pi ikt\}\, d\mu(t) \\
&:= \int_{(0,1]} \cos(2\pi kt)\, d\mu(t) - i\int_{(0,1]} \sin(2\pi kt)\, d\mu(t), \qquad k\in\mathbb{Z}.
\end{aligned} \tag{1}
$$

Definition 3.1. The series

$$\sum_{k\in\mathbb{Z}} \xi_k \exp\{2\pi ikt\} \tag{2}$$

is called the *Fourier series for the stochastic measure* μ. The random variables ξ_k are called the *Fourier coefficients of series* (2). *Partial sums of series* (2) are given by

$$s_n(t) = \sum_{|k|\le n} \xi_k \exp\{2\pi ikt\}.$$

Stochastic integrals on the right hand side of (1) are defined for any measure μ, since the integrands are bounded. Thus the Fourier series is well defined for every stochastic measure in $\mathcal{B}$.

Theorem 3.1 below claims that μ is uniquely defined if ξ_k, $k\in\mathbb{Z}$, are given.

Theorem 3.1. *If $\xi_k = 0$ almost surely for $k\in\mathbb{Z}$, then $\mu(A)=0$ almost surely for all sets $A\in\mathcal{B}$.*

Proof. It is known that, for a given stochastic measure μ, there exists a real control measure λ such that $\mu(A)=0$ for all sets A with $\lambda(A)=0$ (see [7, Theorem B.2.2]). Consider the space of functions

$$\widetilde{\mathbb{C}} = \{f\in\mathbb{C}([0,1])\colon f(0)=f(1)\}.$$

The Stone–Weierstrass theorem applied to functions defined on a circle of the unit circumference claims that the trigonometric polynomials are dense in $\widetilde{\mathbb{C}}$ equipped with the uniform metric. It follows from assumptions of the theorem that $\int_{(0,1]} P\, d\mu = 0$ for every trigonometric polynomial P. By analogy with the Lebesgue theorem [7, Proposition 7.1.1], we conclude that $\int_{(0,1]} f\, d\mu = 0$ for all $f\in\widetilde{\mathbb{C}}$. The class $\widetilde{\mathbb{C}}$ is dense in $L_1(\lambda)$, and thus, given an arbitrary $A\in\mathcal{B}$, there are some functions f_n in $\widetilde{\mathbb{C}}$ that λ-almost everywhere converge to $\mathbf{1}_A$. Since λ is a control measure, f_n converge to $\mathbf{1}_A$ almost everywhere with respect to the measure μ. It is clear that the functions

$$g_n = f_n\mathbf{1}_{\{|f_n|<2\}} + 2\mathbf{1}_{\{f_n\ge 2\}} - 2\mathbf{1}_{\{f_n\le -2\}}$$

are uniformly bounded, belong to $\widetilde{\mathbb{C}}$, and μ-almost everywhere converge to $\mathbf{1}_A$. An analogue of the Lebesgue theorem implies that

$$\int_{(0,1]} g_n\, d\mu \xrightarrow{\mathsf{P}} \int_{(0,1]} \mathbf{1}_A\, d\mu = \mu(A).$$

Since $\int_{(0,1]} g_n\, d\mu = 0$, we obtain $\mu(A)=0$ almost surely. ☐

The weak convergence of partial sums s_n is proved in Theorem 3.2.

Theorem 3.2. *Let a function $g \in \widetilde{\mathbb{C}}$ be such that its Fourier series converges to g uniformly in the interval $[0,1]$. Then*

$$\int_{(0,1]} g(x)s_n(x)\,dx \xrightarrow{\mathsf{P}} \int_{(0,1]} g(x)\,d\mu(x), \qquad n \to \infty. \tag{3}$$

The integral above of the random function is understood in the sense of Definition 2.2.

Proof. We have

$$\begin{aligned}
\int_{(0,1]} g(x)s_n(x)\,dx &= \int_{(0,1]} g(x)\left(\sum_{|k|\le n} \xi_k \exp\{2\pi ikx\}\right)dx \\
&= \int_{(0,1]} g(x)\,dx \sum_{|k|\le n} \exp\{2\pi ikx\} \int_{(0,1]} \exp\{-2\pi ikt\}\,d\mu(t) \\
&= \sum_{|k|\le n} \int_{(0,1]} g(x)\,dx \exp\{2\pi ikx\} \int_{(0,1]} \exp\{-2\pi ikt\}\,d\mu(t) \\
&\overset{(*)}{=} \sum_{|k|\le n} \int_{(0,1]} \exp\{-2\pi ikt\}\,d\mu(t) \int_{(0,1]} \exp\{2\pi ikx\}g(x)\,dx \\
&= \int_{(0,1]} d\mu(t)\left(\sum_{|k|\le n} \exp\{-2\pi ikt\} \int_{(0,1]} \exp\{2\pi ikx\}g(x)\,dx\right).
\end{aligned}$$

We use Theorem 2.1 to ensure equality $(*)$. When changing the order of integration with respect to dx and $d\mu(t)$ we have chosen $q(t) = q_1(t) = \sup_x |g(x)|$. By assumption of the theorem,

$$\sum_{|k|\le n} \exp\{-2\pi ikt\} \int_{(0,1]} \exp\{2\pi ikx\}\,g(x)\,dx \to g(t)$$

uniformly in $[0,1]$ as $n \to \infty$. Now the analogue of the Lebesgue theorem completes the proof of (3). □

Remark 3.1. Various sufficient conditions for the uniform convergence of Fourier series to g can be found, for example, in Sections II.8 and II.10 of [6]. In particular, the uniform convergence follows if g satisfies the Hölder condition or if g is a continuous function of bounded variation.

The following result provides a condition for a certain absolute continuity of μ with respect to the Lebesgue measure in terms of Fourier coefficients.

Theorem 3.3. *Let $\sum_{k\in\mathbb{Z}} |\xi_k| < +\infty$ almost surely, $\xi(t) = \sum_{k\in\mathbb{Z}} \xi_k \exp\{2\pi ikt\}$, and a function $g \in \widetilde{\mathbb{C}}$ be such that its Fourier series converges to g uniformly in the interval $[0,1]$. Then*

$$\int_{(0,1]} g(t)\,d\mu = \int_{(0,1]} \xi(t)g(t)\,dt.$$

Proof. Note that $\xi(t)$ as the limit of a uniformly convergent Fourier series has continuous trajectories. Thus $\int_{(0,1]} \xi(t)g(t)\,dt$ is defined as a usual Riemann integral for every fixed $\omega \in \Omega$, and its value coincides with the value provided by Definition 2.2.

By Theorem 3.2,

$$\int_{(0,1]} g(t)s_n(t)\,dt \xrightarrow{\mathsf{P}} \int_{(0,1]} g(t)\,d\mu(t).$$

Hence, it is sufficient to show that

$$\int_{(0,1]} g(t)s_n(t)\,dt \xrightarrow{\mathsf{P}} \int_{(0,1]} \xi(t)g(t)\,dt.$$

The latter relation follows from the uniform convergence of integrands for any fixed ω. □

4. Convergence of solutions of the heat equation

As an application of the results obtained in the preceding section we consider the convergence of solutions of the stochastic heat equation governed by a measure μ:

$$du(t,x) = a^2\Delta u(t,x)\,dt + \sigma(t,x)\,d\mu(t), \qquad u(0,x) = u_0(x), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Here the operator Δ acts with respect to the variable x. The solution of this equation is understood in the mild sense,

$$u(t,x) = \int_{\mathbb{R}^d} p(t,x-y)u_0(y)\,dy + \int_{(0,t]} d\mu(r) \int_{\mathbb{R}^d} p(t-r,x-y)\sigma(r,y)\,dy\,. \tag{4}$$

In equality (4), $p(t,x) = (4a^2\pi t)^{-d/2}e^{-|x|^2/4a^2t}$ denotes the fundamental solution of the heat equation. Changing definitions in an obvious way, we consider Fourier series for functions defined on the interval $[0,T]$ rather than on the interval $[0,1]$. In what follows, the symbol C stands for a constant whose precise value does not matter for our reasoning.

Some applications of the Fourier transform to the convergence of solutions of equation (4) are given in [11].

We impose the following restrictions.

- **A1.** The function $u_0(y) = u_0(y,\omega)\colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ is measurable and bounded for every fixed ω.
- **A2.** The function $\sigma(r,y)\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is measurable and bounded.
- **A3.** $|\sigma(r_1,y_1) - \sigma(r_2,y_2)| \le C\,(|r_1 - r_2| + |y_1 - y_2|)$.

Theorem 4.1. *Assume that conditions* A1–A3 *hold. Let* s_n *be the partial sums of series* (2) *and let*

$$u_n(t,x) = \int_{\mathbb{R}^d} p(t,x-y)u_0(y)\,dy + \int_{(0,t]} s_n(r)\,dr \int_{\mathbb{R}^d} p(t-r,x-y)\sigma(r,y)\,dy\,. \tag{5}$$

Then, for all $(t,x) \in [0,T) \times \mathbb{R}^d$,

$$u_n(t,x) \xrightarrow{\mathsf{P}} u(t,x), \qquad n \to \infty,$$

where $u(t,x)$ *is defined by equality* (4).

Proof. Assumptions A1 and A2 ensure that the integrals in (4) and (5) are well defined. The integral in (5) with respect to dr is understood in the sense of Definition 2.2 (Theorem 2.1 guarantees that this integral is well defined, too). Put

$$g(z,r) = \int_{\mathbb{R}^d} p(t-r,x-y)\sigma(r,y)\,dy, \qquad z = (t,x),\ 0 \le r < t,$$

and extend g for $t \le r \le T$ in a linear way such that $g(z,0) = g(z,T)$. For $r_1, r_2 \in [0,t)$, we have

$$
\begin{aligned}
&|g(z,r_1) - g(z,r_2)| \\
&\quad = C\left|\int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{4a^2(t-r_1)}}}{(t-r_1)^{d/2}}\,\sigma(r_1, x-y)\,dy - \int_{\mathbb{R}^d} \frac{e^{-\frac{|y|^2}{4a^2(t-r_2)}}}{(t-r_2)^{d/2}}\,\sigma(r_2, x-y)\,dy\right| \\
&\quad \overset{(*)}{=} C\left|\int_{\mathbb{R}^d} e^{-|v|^2}\sigma\left(r_1, x - v\sqrt{4a^2(t-r_1)}\right)\,dv\right. \\
&\qquad\qquad\qquad \left. - \int_{\mathbb{R}^d} e^{-|v|^2}\sigma\left(r_2, x - v\sqrt{4a^2(t-r_2)}\right)\,dv\right| \\
&\quad = C\left|\int_{\mathbb{R}^d} e^{-|v|^2}\left(\sigma\left(r_1, x - v\sqrt{4a^2(t-r_1)}\right) - \sigma\left(r_2, x - v\sqrt{4a^2(t-r_2)}\right)\right)dv\right| \\
&\quad \overset{(**)}{\le} C\left(|r_1 - r_2| + \left|\sqrt{4a^2(t-r_1)} - \sqrt{4a^2(t-r_2)}\right|\right)\int_{\mathbb{R}^d}(1+|v|)e^{-|v|^2}\,dv.
\end{aligned}
$$

We used the change of variables $v = y/\sqrt{4a^2(t-r)}$ for equality $(*)$ and assumption A4 for equality $(**)$.

Thus the function $g(z,r)$ is of bounded variation on the interval $[0,T]$, is continuous, and is uniformly approximated by its Fourier series according to [6, Theorem II.8.1]. Now Theorem 3.2 completes the proof. □

5. Fourier–Haar series for general stochastic measures

We turn back to a stochastic measure μ defined on the Borel σ-algebra in the interval $(0,1]$. Consider the function $\tilde{\mu}(t) = \mu((0,t])$, $0 \le t \le 1$. Our aim is to obtain an approximation of $\tilde{\mu}(t)$ by step functions with the help of the Fourier–Haar series with random coefficients.

In what follows, we use the following notation:

$$
\begin{gathered}
d_k^i = i2^{-k}, \quad k \ge 0,\ 0 \le i \le 2^k, \qquad \Delta_1 = \Delta_0^0 = (0,1), \qquad \overline{\Delta}_1 = [0,1], \\
\Delta_n = \Delta_k^i = \left(d_k^{i-1}, d_k^i\right), \qquad \overline{\Delta}_n = \left[d_k^{i-1}, d_k^i\right], \qquad 2^k + 1 \le n \le 2^{k+1}, \\
\Delta_n^+ = \left(\Delta_k^i\right)^+ = \left(d_k^{i-1}, d_{k+1}^{2i-1}\right) = \Delta_{k+1}^{2i-1}, \qquad \Delta_n^- = \left(\Delta_k^i\right)^- = \left(d_{k+1}^{2i-1}, d_k^i\right) = \Delta_{k+1}^{2i}.
\end{gathered}
$$

The system of functions

$$\chi = \{\chi_n(x),\ n \ge 1\}, \qquad x \in [0,1],$$

where $\chi_1(x) \equiv 1$ and

$$
\chi_n(x) = \begin{cases} 0, & x \notin \overline{\Delta}_n, \\ 2^{k/2}, & x \in \Delta_n^+, \\ -2^{k/2}, & x \in \Delta_n^-, \end{cases}
$$

for $2^k + 1 \le n \le 2^{k+1}$ is the classical orthonormal Haar system. At the points of discontinuity, we put

$$\chi_n(x) = \left(\chi_n(x-) + \chi_n(x+)\right)/2, \qquad \chi_n(0) = \chi_n(0+), \qquad \chi_n(1) = \chi_n(1-).$$

Let trajectories of the function $\tilde{\mu}(t)$ be Riemann integrable for every ω. Then its Fourier–Haar coefficients are defined by the equality

$$\eta_n = \int_{[0,1]} \tilde{\mu}(t)\chi_n(t)\,dt = 2^{k/2}\left(\int_{\Delta_n^+} \tilde{\mu}(t)\,dt - \int_{\Delta_n^-} \tilde{\mu}(t)\,dt\right). \tag{6}$$

Below we use the integration by parts formula provided by Theorem 5.1.

Theorem 5.1. *Let $\tilde{\mu}(t)$ be an integrable function on the interval $[a,b]$ in the sense of Definition 2.2. Then*

$$\int_{[a,b]} \tilde{\mu}(t)\,dt = b\tilde{\mu}(b) - a\tilde{\mu}(a) - \int_{(a,b]} t\,d\mu. \tag{7}$$

Proof. Consider partitions $a = t_0 < t_1 < \cdots < t_j = b$ whose diameters tend to zero. Using an analogue of the Lebesgue theorem in equality $(*)$ below we get

$$\int_{(a,b]} t\,d\mu \overset{(*)}{=} \mathsf{P}\lim_{j\to\infty} \int_{(a,b]} \sum_{i=1}^{j} t_i \mathbf{1}_{(t_{i-1},t_i]}(t)\,d\mu = \mathsf{P}\lim_{j\to\infty} \sum_{i=1}^{j} t_i\big(\tilde{\mu}(t_i) - \tilde{\mu}(t_{i-1})\big)$$
$$= b\tilde{\mu}(b) - a\tilde{\mu}(a) - \mathsf{P}\lim_{j\to\infty} \sum_{i=1}^{j} \tilde{\mu}(t_{i-1})\,(t_i - t_{i-1})\,.$$

The latter sums tend to $\int_{[a,b]} \tilde{\mu}(t)\,dt$ according to Definition 2.2. □

Using (7), we get from equality (6) that

$$\eta_n = 2^{k/2}\Bigg(-d_k^i\tilde{\mu}\big(d_k^i\big) + 2d_{k+1}^{2i-1}\tilde{\mu}\big(d_{k+1}^{2i-1}\big) - d_k^{i-1}\tilde{\mu}\big(d_k^{i-1}\big) - \int_{\left(d_k^{i-1},d_{k+1}^{2i-1}\right]} t\,d\mu + \int_{\left(d_{k+1}^{2i-1},d_k^i\right]} t\,d\mu\Bigg) \tag{8}$$

for $2^k+1 \le n \le 2^{k+1}$. The random variable η_n given by the right hand side of (8) is defined for an arbitrary stochastic measure μ even if its trajectories are not integrable in the Riemann sense. Throughout below, we assume that the random variable η_n is defined by equality (8).

Definition 5.1. Let μ be an arbitrary stochastic measure defined on the Borel σ-algebra of the interval $(0,1]$ and let the random variables η_n be defined by equality (8). Then

$$\sum_{n\ge1} \eta_n\chi_n(x) \tag{9}$$

is called the *Fourier–Haar series* for the stochastic measure μ. Accordingly, the sums

$$S_N(x) = \sum_{n=1}^{N} \eta_n\chi_n(x)$$

are called the *partial sums* of series (9).

Similarly to equality (3.8) of [5], we conclude that

$$S_{2^k}(x) = \sum_{i=1}^{2^k} 2^k\left(d_k^i\tilde{\mu}\left(d_k^i\right) - d_k^{i-1}\tilde{\mu}\left(d_k^{i-1}\right) - \int_{\Delta_k^i} t\,d\mu\right)\mathbf{1}_{\Delta_k^i}(x) \tag{10}$$

for $x \ne d_k^i$ and $0 \le i \le 2^k$. Analogously to equality (3.11) of [5], we conclude that

$$S_N(x) = \begin{cases} S_{2^{k+1}}(x), & x \in [0,d_k^i), \\ S_{2^k}(x), & x \in (d_k^i,1], \\ S_{2^k}(x) + \eta_N\chi_N(x), & x = d_k^i, \end{cases} \tag{11}$$

for $\Delta_N = \Delta_k^i$, $2^k+1 \le N < 2^{k+1}$.

Theorem 5.2. *If a point $x \in (0,1]$ is such that $\mu(\{x\}) = 0$ almost surely, then*

$$S_N(x) \xrightarrow{\mathsf{P}} \tilde{\mu}(x), \qquad N \to \infty.$$

Proof. Using equality (10), we conclude that

$$\begin{aligned} S_{2^k}(x) &= \sum_{i=1}^{2^k}\left(\tilde{\mu}\left(d_k^i\right)+(i-1)\mu\left(\Delta_k^i\right)-2^k\int_{\Delta_k^i} t\,d\mu(t)\right)\mathbf{1}_{\Delta_k^i}(x) \\ &= \int_{(0,1]}\sum_{i=1}^{2^k}\Big(\mathbf{1}_{(0,d_k^i]}(t)+\big(i-1-2^k t\big)\,\mathbf{1}_{\Delta_k^i}(t)\Big)\,\mathbf{1}_{\Delta_k^i}(x)\,d\mu(t) \\ &=: \int_{(0,1]} g_k(x,t)\,d\mu(t). \end{aligned}$$

Here $g_k(x,t)\to\mathbf{1}_{(0,x]}(t)$ as $k\to\infty$ for all $t\neq x$. The convergence at the point x does not matter if $\mu(\{x\})=0$ almost surely. An analogue of Lebesgue's theorem implies that $S_{2^k}(x)\xrightarrow{\mathsf{P}}\tilde{\mu}(x)$. In addition,

$$\begin{aligned} \eta_N\chi_N(d_k^i) = -\frac{1}{2}\Bigg(&-d_k^i\tilde{\mu}\big(d_k^i\big)+2d_{k+1}^{2i-1}\tilde{\mu}\big(d_{k+1}^{2i-1}\big)-d_k^{i-1}\tilde{\mu}\big(d_k^{i-1}\big) \\ &-\int_{\left(d_k^{i-1},d_{k+1}^{2i-1}\right]} t\,d\mu+\int_{\left(d_{k+1}^{2i-1},d_k^i\right]} t\,d\mu\Bigg), \end{aligned}$$

and thus $\eta_N\chi_N(x_0)\xrightarrow{\mathsf{P}}0$, $N\to\infty$, for any fixed point $x_0=d_k^i$ if $\mu(\{x_0\})=0$ almost surely (different k result in different i). It remains to apply equality (11). □

It is worth mentioning that the proof above provides an approximation of trajectories of μ in contrast to the proof of Theorem 3.2, where partial Fourier sums are used to approximate integrals with respect to $d\mu$.

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