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DIVIDENDS WITH TAX AND CAPITAL INJECTION IN A SPECTRALLY NEGATIVE LÉVY RISK MODEL

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Abstract. We consider a risk model driven by a spectrally negative Lévy process. From the surplus dividends are paid and capital injections have to be made in order to keep the surplus positive. In addition, tax has to be paid for dividends, but injections lead to an exemption from tax. We generalize the results from [**12, 13**] and show that the optimal dividend strategy is a two-barrier strategy. The barrier depends on whether an immediate dividend would be taxed or not. For a risk process perturbed by diffusion with exponentially distributed claim sizes, we show how the value function and the barriers can be determined.

1. Introduction

Consider a spectrally negative Lévy process $\{X_t^0\}$ and $X_0^0 = x$ with $\mathsf{E}[|X_t^0|] < \infty$. This process models the surplus of an insurance portfolio. A classical measure for the risk is the ruin probability $\mathsf{P}[\inf_t X_t < 0]$. For literature on the ruin problem, see, for example, [1, 10] or [8, 9], where also the problem of differentiability is considered. An alternative measure was introduced by de Finetti [5]. Dividends may be paid from the surplus, and the value of the process is the expected discounted value of the dividend payments until ruin. This problem was for example also considered in [4, 6, 11]. Kulenko and Schmidli [7] considered in addition capital injections. Each time the surplus becomes negative, capital injections have to be made in order to cover the deficit. That means, the surplus process is reflected in zero. Choosing a dividend strategy $\{D_t\}$, the capital injection process $\{L_t\}$ is the minimal process such that

$$X_t^D = X_t^0 - D_t + L_t$$

remains positive. Here, L and D are increasing processes with $D_{0-} = L_{0-} = 0$. The first who considered this model were Shreve et al. [14] in a diffusion setup. For Lévy processes, this problem has been considered in [3].

Recently, Schmidli [12, 13] has introduced tax payments on dividends. More specifically, from dividends a part $1-\gamma$ has to be paid as tax and only $\gamma\, dD_t$ counts to the value. But if a capital injection dL_t is made, the same amount may be paid as dividend without tax in the future. That means that no tax has to be paid for the next dL_t dividend payments. After that, tax is applied to dividends again. Note that if tax would be applied to all of the dividends, the optimisation problem would just be the problem considered in [7] with a modified penalizing factor η (defined below). It turns out that the value function is strongly connected to the problem considered by [7]. The optimal strategy is very simple. There are two dividend barriers $b_>$ and b_0. The first barrier is

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applied if an immediate dividend would not be taxed. The second barrier is applied if an immediate dividend is taxed. In this paper, we will show that the same holds true if the underlying risk process is a Lévy process.

Note that Albrecher and Ivanovs [2] have shown recently how to calculate the value function. However, they only consider two-barrier strategies. We will actually show below that this kind of strategy is indeed optimal. Moreover, in general, it is hard to find the scale functions used in [2]. Our approach via the HJB equation gives an alternative way to calculate the solution.

The paper is organised as follows. We first recall the problem without tax. In Section 3, we introduce taxes and show a verification theorem. The solution to the problem is given in Section 4, and it is verified that the proposed solution satisfies the Hamilton–Jacobi–Bellman equation (3). The proof is simpler than the corresponding proof in [12]. Then we discuss the case $\eta = 1$, where dividends and capital injections are measured equally. We discuss when one or both barriers are kept at zero. In particular, we find that the barrier can only be at zero if the Brownian part is absent, or if $\eta = 1$. Finally, in Section 7, we consider the example of a risk process perturbed by Brownian motion with exponentially distributed claim sizes.

2. Dividends and capital injections

We first review the problem without tax. This model is considered in [2, 3], where the approach is via scale functions. Choosing a dividend strategy D, the corresponding value is

$$V^D(x) = \mathsf{E}\left[\int_0^\infty e^{-\delta t}\, dD_t - \eta \int_0^\infty e^{-\delta t}\, dL_t\right],$$

where $\delta > 0$ is a preference parameter and $\eta \ge 1$ is a penalising factor for the capital injections. In the problem without tax, $\eta > 1$ is necessary in order that it is not optimal to keep the surplus at zero. The value function is then $V^n(x) = \sup_D V^D(x)$.

By the argument given in Lemma 1 of [7], the value function is concave. This implies that there exists an optimal strategy and this strategy is of barrier type. Indeed, since we have a linear upper bound, we can apply an argument given in [6] and show that dividends will be paid for capital large enough. Because of the concavity, the derivative of the value function must be one above any point where dividends are paid. That means that all surplus above a dividend barrier $b_>$ is paid as dividend, whereas no dividend is paid whenever the surplus is below $b_>$.

Because the Lévy process is spectrally negative, the moment generating function of X_t^0 exists and is of the form

$$\mathsf{E}\left[\exp\left\{r\left(X_t^0 - x\right)\right\}\right] = \exp\{t\psi(r)\},$$

where

$$\psi(r) = cr + \frac{1}{2}\sigma^2 r^2 - \int_0^\infty (1 - e^{-rz})\, dM(z).$$

Here M is the so called Lévy measure and $\psi(r)$ is finite for all $r \ge 0$ and strictly convex. Since the process is integrable, we have $\int_0^\infty z\, dM(z) < \infty$ and the usual term $z\mathbb{1}_{|z|\le 1}$ in the integral can be dropped. The diffusion approximation is obtained with $M(\mathbb{R}) = 0$. In the classical model, $\sigma = 0$ and $dM(z) = \lambda\, dF(z)$, where λ is the claim intensity and $F(z)$ is the claim size distribution. If, in addition, $\sigma > 0$, we obtain the risk model perturbed by Brownian motion. If $\sigma = 0$ and $dM(z) = \gamma e^{-\beta z}/z$, we obtain the Gamma process.

Because ψ is convex, $\psi(0) = 0$ and $\lim_{r\to\infty} \psi(r) = \infty$, there is a unique solution $\rho > 0$ to $\psi(\rho) = \delta$. We will need the coefficient ρ below.

Standard arguments show that below the barrier $x \le b_>$, the value function is a viscosity solution for $x \ge 0$ to the integro-differential equation

$$\frac{1}{2}\sigma^2 V^n_{xx}(x) + cV^n_x(x) - \delta V^n(x) - \int_0^\infty \big(V^n(x) - V^n(x-z)\big)\,dM(z) = 0 \tag{1}$$

with $V^n(x) = V^n(0) + \eta x$ for $x < 0$. Above the barrier, $V^n(x) = V^n(b_>) + x - b_>$ for $x > b_>$. The function $V^n(x)$ solves then the Hamilton–Jacobi–Bellman equation

$$\begin{aligned} 0 = \max\Big\{ &\frac{1}{2}\sigma^2 V^n_{xx}(x) + cV^n_x(x) - \delta V^n(x) - \int_0^\infty \big(V^n(x) - V^n(x-z)\big)\,dM(z), \\ &1 - V^n_x(x), V^n_x(x) - \eta\Big\}. \end{aligned} \tag{2}$$

Since any solution f to (1) on $(-\infty, b]$ for a value b such that $f_x(b) = 1$ is the value of the barrier strategy with the barrier at b, we have to choose the solution such that $\inf_x f_x(x) = 1$. The barrier is then chosen such that $V^n_x(b_>) = 1$. In particular, we get $V^n_{xx}(b_>) = 0$. If $\sigma > 0$, then we have, in addition, the boundary condition $V^n_x(0) = \eta$. If $\sigma = 0$, the condition $V^n(x) = V^n(0) + \eta x$ for $x < 0$ implies that $V^n_x(0+) < \eta$; see also [7].

For further use, we denote by $V^0(x)$ the (viscosity) solution to (1) that coincides with the value function on $[0, b_>]$. That is,

$$V^n(x) = \begin{cases} V^0(x), & \text{if } x \le b_>, \\ V^0(b_>) + x - b_>, & \text{if } x > b_>. \end{cases}$$

As in Lemma 5 of [12], it follows that $\lim_{x\to\infty} V^0(x) e^{-\rho x} > 0$.

3. The model with tax

We suppose now that tax at rate $1 - \gamma$ has to be paid. If a capital injection is made, the same amount is exempt from tax. Denote by Y_t the amount of dividends that could immediately be paid without tax. Then for $Y_0 = y$,

$$Y_t = y + L_t - \int_0^t \mathbb{1}_{Y_s > 0}\, dD^c_s - \sum_{s \le t} \min\{\Delta D_s, Y_{s-}\},$$

where $D^c_t = D_t - \sum_{s\le t} \Delta D_s$ is the continuous part of the dividend payments. The value of a dividend strategy D is

$$\begin{aligned} V^D(x,y) = &\int_0^\infty e^{-\delta t}(\mathbb{1}_{Y_t > 0} + \gamma \mathbb{1}_{Y_t = 0})\, dD^c_t \\ &+ \sum_{t \ge 0} e^{-\delta t}\big[\min\{\Delta D_t, Y_{t-}\} + \gamma(\Delta D_t - Y_{t-})^+\big] - \eta \int_0^\infty e^{-\delta t}\, dL_t, \end{aligned}$$

where $x^+ = \max\{x, 0\}$ denotes the positive part of x. Our value function is then $V(x,y) = \sup_D V^D(x,y)$.

Denote by

$$\mathcal{T}_t = \int_0^t \mathbb{1}_{Y_s = 0}\, dD^c_s + \sum_{s \le t} (\Delta D_t - Y_{t-})^+$$

the amount of dividends for which tax has been paid. Then it is easy to see that

$$D_t + Y_t = y + L_t + \mathcal{T}_t.$$

See also [12].

Lemma 1. *The value function $V(x,y)$ is concave in the first coordinate. Moreover, the function $z \mapsto V(x+z, y+z)$ is concave.*

Proof. The proof of Lemma 2 of [12] is also valid for the Lévy risk model. An analogous proof is applied to $z \mapsto V(x+z, y+z)$. □

It can only be optimal to pay dividends if $V_x(x,0) = \gamma$ or $V_x(x,y) + V_y(x,y) = 1$ for $y > 0$. Concavity, therefore, implies that the optimal strategy will be of barrier type. That is, there is a function $b(y)$. Dividends are paid if $X_t \ge b(Y_t)$. More specifically, the process is reflected at the barrier $b(y)$. We will verify below that the function $b(y)$ is of the very simple form $b(y) = b_0 \mathbb{1}_{y=0} + b_> \mathbb{1}_{y>0}$.

The problem is connected to a Hamilton–Jacobi–Bellman equation.

Theorem 1. *Suppose that $f(x,y)$ is a continuous function such that*

$$f(x,y) = f(0, y-x) + \eta x \quad \text{for } x < 0.$$

Suppose further that for $x \ge 0$, $f(x,y)$ is a viscosity solution to

$$\begin{aligned}0 = \max\Big\{ &\frac{1}{2}\sigma^2 f_{xx}(x,y) + c f_x(x,y) - \delta f(x,y) \\ &\quad - \int_0^\infty \big(f(x,y) - f(x-z,y)\big)\, dM(z), \\ &\quad 1 - f_x(x,y) - f_y(x,y),\ f_x(x,y) + f_y(x,y) - \eta \Big\}\end{aligned} \tag{3a}$$

if $y > 0$, and

$$\begin{aligned}0 = \max\Big\{ &\frac{1}{2}\sigma^2 f_{xx}(x,0) + c f_x(x,0) - \delta f(x,0) \\ &\quad - \int_0^\infty \big(f(x,0) - f(x-z,0)\big)\, dM(z), \\ &\quad \gamma - f_x(x,0) \Big\}.\end{aligned} \tag{3b}$$

If $\sigma > 0$, assume that $f_x(0,y) + f_y(0,y) = \eta$. We further assume that there is a function $b(y)$, such that $f_x(x,y) + \mathbb{1}_{y>0} f_y(x,y) > \mathbb{1}_{y>0} + \gamma \mathbb{1}_{y=0}$ for $x < b(y)$ and

$$f_x(x,y) + \mathbb{1}_{y>0} f_y(x,y) = \mathbb{1}_{y>0} + \gamma \mathbb{1}_{y=0} \quad \text{for } x \ge b(y).$$

Then $f(x,y) = V(x,y)$.

Proof. Let D be an arbitrary dividend strategy. Let $n > 0$. We stop the process when n is reached, $\tau_n^D = \inf\{t > 0 \colon X_t^D > n\}$. Then

$$\begin{aligned}\Big\{ f\left(X^D_{\tau_n^D \wedge t}, Y^D_{\tau_n^D \wedge t}\right) e^{-\delta(\tau_n^D \wedge t)} &- \int_0^{\tau_n^D \wedge t} \left[\mathcal{A} f\left(X_s^D, Y_s^D\right) - \delta f\left(X_s^D, Y_s^D\right)\right] e^{-\delta s}\, ds \\ &- \int_0^{\tau_n^D \wedge t} e^{-\delta s} \left[f_x\left(X_s^D, Y_s^D\right) + f_y\left(X_s^D, Y_s^D\right)\right] dL_s^c \\ &- \sum_{\substack{s \le \tau_n^D \wedge t \\ \Delta L_s > 0}} \left(f\left(X_s^D, Y_s^D\right) - f\left(X_{s-}^D, Y_{s-}^D\right)\right) e^{-\delta s} \\ &+ \int_0^{\tau_n^D \wedge t} e^{-\delta s} \left(f_x\left(X_s^D, Y_s^D\right) + \mathbb{1}_{Y_s^D > 0} f_y\left(X_s^D, Y_s^D\right)\right) dD_s^c \\ &- \sum_{\substack{s \le \tau_n^D \wedge t \\ \Delta D_s > 0}} \left(f\left(X_s^D, Y_s^D\right) - f\left(X_{s-}^D, Y_{s-}^D\right)\right) e^{-\delta s} \Big\}\end{aligned}$$

is a (local) martingale, where

$$\mathcal{A}g(x,y) = \frac{1}{2}\sigma^2 g_{xx}(x,y) + c g_x(x,y) - \int_0^\infty \big(g(x,y) - g(x-z,y)\big)\, dM(z).$$

We can choose a localisation sequence $\tilde{\tau}^D_{n,m} \le \tau^D_n$ in order to obtain a martingale. We further assume that $\Delta D_t \Delta L_t = 0$, since it does not make sense to pay dividends and make capital injections at the same time because $\eta \ge 1$. By (3), we have $\mathcal{A}f(x,y) - \delta f(x,y) \le 0$, $f_x(x,y) + f_y(x,y) \le \eta$, $f_x(x,y) + f_y(x,y) \ge 1$ for $y > 0$ and $f_x(x,0) \ge \gamma$. Suppose $\Delta L_s > 0$. Then

$$\begin{aligned}
f\left(X^D_s, Y^D_s\right) - f\left(X^D_{s-}, Y^D_{s-}\right) &= f\left(X^D_s, Y^D_s\right) - f\left(X^D_s - \Delta L_s, Y^D_s - \Delta L_s\right) \\
&= \int_0^{\Delta L_s} \left[f_x\left(X^D_s - z, Y^D_s - z\right) + f_y\left(X^D_s - z, Y^D_s - z\right)\right] dz \\
&\le \eta \Delta L_s.
\end{aligned}$$

Suppose $\Delta D_s > 0$. Then

$$\begin{aligned}
f\left(X^D_s, Y^D_s\right) - f\left(X^D_{s-}, Y^D_{s-}\right) &= f\left(X^D_{s-} - \Delta D_s, \left(Y^D_{s-} - \Delta D_s\right)^+\right) - f\left(X^D_{s-}, Y^D_{s-}\right) \\
&= f\left(X^D_{s-} - \Delta D_s, \left(Y^D_{s-} - \Delta D_s\right)^+\right) \\
&\quad - f\left(X^D_{s-} - \min\left\{\Delta D_s, Y^D_{s-}\right\}, \left(Y^D_{s-} - \Delta D_s\right)^+\right) \\
&\quad + f\left(X^D_{s-} - \min\left\{\Delta D_s, Y^D_{s-}\right\}, \left(Y^D_{s-} - \Delta D_s\right)^+\right) \\
&\quad - f\left(X^D_{s-}, Y^D_{s-}\right).
\end{aligned}$$

We get

$$\begin{aligned}
&f\left(X^D_{s-} - \min\left\{\Delta D_s, Y^D_{s-}\right\}, \left(Y^D_{s-} - \Delta D_s\right)^+\right) - f\left(X^D_{s-}, Y^D_{s-}\right) \\
&\quad = -\int_0^{\min\left\{\Delta D_s, Y^D_{s-}\right\}} \left[f_x\left(X^D_{s-} - z, Y^D_{s-} - z\right) + f_y\left(X^D_{s-} - z, Y^D_{s-} - z\right)\right] dz \\
&\quad \le -\min\left\{\Delta D_s, Y^D_{s-}\right\}.
\end{aligned}$$

Note that $X^D_{s-} - \min\{\Delta D_s, Y^D_{s-}\} \neq X^D_{s-} - \Delta D_s$ if and only if $Y^D_{s-} < \Delta D_s$. Thus, in this case

$$f\left(X^D_{s-} - \Delta D_s, 0\right) - f\left(X^D_{s-} - Y^D_{s-}, 0\right) = -\int_{Y^D_{s-}}^{\Delta D_s} f_x\left(X^D_{s-} - z, 0\right) dz \le -\gamma\left(\Delta D_s - Y^D_{s-}\right).$$

Putting the above considerations together shows that

$$\begin{aligned}
\Bigg\{ &f\left(X^D_{\tau^D_{n,m}\wedge t}, Y^D_{\tau^D_{n,m}\wedge t}\right) e^{-\delta\left(\tau^D_{n,m}\wedge t\right)} - \eta \int_0^{\tau^D_{n,m}\wedge t} e^{-\delta s}\, dL_s \\
&+ \int_0^{\tau^D_{n,m}\wedge t} e^{-\delta s}\left(\mathbb{1}_{Y^D_s > 0} + \gamma \mathbb{1}_{Y^D_s = 0}\right) dD^c_s \\
&+ \sum_{s \le \tau^D_{n,m}\wedge t} \left[\min\left\{\Delta D_s, Y^D_{s-}\right\} + \gamma\left(\Delta D_s - Y^D_{s-}\right)^+\right] e^{-\delta s} \Bigg\}
\end{aligned}$$

is a supermartingale. Taking expected values shows that

$$f(x,y) \geq \mathsf{E}\Bigg[f\left(X^D_{\tau^D_{n,m}\wedge t}, Y^D_{\tau^D_{n,m}\wedge t}\right) e^{-\delta(\tau^D_{n,m}\wedge t)} - \eta \int_0^{\tau^D_{n,m}\wedge t} e^{-\delta s}\, dL_s$$
$$+ \int_0^{\tau^D_{n,m}\wedge t} e^{-\delta s}\left(\mathbb{1}_{Y^D_s>0} + \gamma \mathbb{1}_{Y^D_s=0}\right) dD^c_s$$
$$+ \sum_{s\leq \tau^D_{n,m}\wedge t}\left[\min\left\{\Delta D_s, Y^D_{s-}\right\} + \gamma\left(\Delta D_s - Y^D_{s-}\right)^+\right] e^{-\delta s}\Bigg].$$

Letting $m\to\infty$, by bounded convergence and monotone convergence, respectively, we get

$$f(x,y) \geq \mathsf{E}\Bigg[f\left(X^D_{\tau^D_n\wedge t}, Y^D_{\tau^D_n\wedge t}\right) e^{-\delta(\tau^D_n\wedge t)} - \eta \int_0^{\tau^D_n\wedge t} e^{-\delta s}\, dL_s$$
$$+ \int_0^{\tau^D_n\wedge t} e^{-\delta s}\left(\mathbb{1}_{Y^D_s>0} + \gamma \mathbb{1}_{Y^D_s=0}\right) dD^c_s$$
$$+ \sum_{s\leq \tau^D_n\wedge t}\left[\min\left\{\Delta D_s, Y^D_{s-}\right\} + \gamma\left(\Delta D_s - Y^D_{s-}\right)^+\right] e^{-\delta s}\Bigg].$$

As a concave function, $f(x,y)$ is bounded by a linear function. As in the case without tax, we conclude that $\{f(X^D_t, Y^D_t)e^{-\delta t}\}$ is uniformly integrable. Thus, we can let $n\to\infty$, and using, in addition, monotone convergence, we get

$$f(x,y) \geq \mathsf{E}\Bigg[f\left(X^D_t, Y^D_t\right) e^{-\delta t} - \eta \int_0^t e^{-\delta s}\, dL_s + \int_0^t e^{-\delta s}\left(\mathbb{1}_{Y^D_s>0} + \gamma \mathbb{1}_{Y^D_s=0}\right) dD^c_s$$
$$+ \sum_{s\leq t}\left[\min\left\{\Delta D_s, Y^D_{s-}\right\} + \gamma\left(\Delta D_s - Y^D_{s-}\right)^+\right] e^{-\delta s}\Bigg].$$

Finally, letting $t\to\infty$, by the same argument, we obtain

$$f(x,y) \geq \mathsf{E}\Bigg[\int_0^\infty e^{-\delta s}\left(\mathbb{1}_{Y^D_s>0} + \gamma \mathbb{1}_{Y^D_s=0}\right) dD^c_s$$
$$+ \sum_{s<\infty}\left[\min\left\{\Delta D_s, Y^D_{s-}\right\} + \gamma\left(\Delta D_s - Y^D_{s-}\right)^+\right] e^{-\delta s} - \eta \int_0^\infty e^{-\delta s}\, dL_s\Bigg]$$
$$= V^D(x,y).$$

We have used that $f(x,y)$ is bounded by a linear function and thus $\mathsf{E}[f(X^D_t, Y^D_t)]$ is bounded by a linear function in t. Because the strategy D is arbitrary, we have shown that $f(x,y) \geq V^D(x,y)$.

Choose the strategy D^* such that no dividends are paid if $X^*_t < b(Y^*_t)$ and all capital above $b(y)$ is paid as dividend. That is, the process X is reflected at zero and at the barrier $b(y)$. Repeating the arguments above, all inequalities become equalities. Thus, $f(x,y) = V^*(x,y) \leq V(x,y)$. □

4. The solution to the problem

We have defined before the function $V^0(x)$ as the (viscosity) solution to (1) that coincides on $[0,b_>]$ with the value function of the problem without tax. Recall that ρ is the solution to $\psi(\rho)=\delta$. Consider the function

$$g(x,y) = V^0(x) - Ce^{\rho(x-y)}.$$

We choose C such that $\inf_{x\geq 0} g_x(x,0) = \gamma$. Note that such a C exists because we have $\liminf_{x\to\infty} V^0(x)e^{-\rho x} > 0$. Denote the value for which the infimum is attained by b_0. Note that

$$\left[\frac{1}{2}\sigma^2\rho^2 + c\rho - \delta - \int_0^\infty (1-e^{-\rho z})\,dM(z)\right] e^{\rho(x-y)} = 0$$

by the definition of ρ. Thus $g(x,y)$ solves (1). Further, $g_x(x.y) + g_y(x,y) = V_x^0(x)$, and in particular, $g_x(b_>,y) + g_y(b_>,y) = V_x^0(b_>) = 1$. This implies $b_0 > b_>$, unless $b_> = 0$. Further, if $\sigma > 0$, then $g_x(0,y) + g_y(0,y) = V_x^0(0) = \eta$.

Our candidate for a solution is the function

$$f(x,y) = \begin{cases} g(x,y), & \text{if } y>0 \text{ and } 0\leq x\leq b_>,\\ g(x,0), & \text{if } y=0 \text{ and } 0\leq x\leq b_0,\\ g(b_0,0)+\gamma(x-b_0), & \text{if } y=0 \text{ and } x>b_0,\\ f\left(\max\{x-y,b_>\},(y+b_>-x)^+\right) & \\ \quad +\min\{y,x-b_>\}, & \text{if } y>0 \text{ and } x>b_>,\\ f(0,y-x)+\eta x, & \text{if } x<0. \end{cases} \tag{4}$$

This is indeed the case.

Theorem 2. *Let $f(x,y)$ be the function defined by* (4)*. Then $f(x,y) = V(x,y)$.*

Proof. First note that for $x < 0$,

$$f(x,y) = f(0,y-x) + \eta x = V^0(0) + \eta x - Ce^{\rho(x-y)} = V^{\mathrm{n}}(x) - Ce^{\rho(x-y)}.$$

Assume for the moment that $f(x,y)$ is concave in x. We then need to show that $f(x,y)$ fulfils the HJB equation (3). For $0\leq x\leq b_>$ if $y>0$ and $0\leq x\leq b_0$ if $y=0$, we have

$$\frac{1}{2}\sigma^2 g_{xx}(x,y) + cg_x(x,y) - \delta g(x,y) - \int_0^\infty \big(g(x,y)-g(x-z,y)\big)\,dM(z) = 0$$

by definition. Further, $f_x(x,y) + f_y(x,y) = V_x^0(x) \geq 1$ if $y>0$, and by the definition of $g(x,y)$, $f_x(x,0)\geq\gamma$. Further, $f_x(x,y)+f_y(x,y) = V_x^0(x)\leq\eta$ for $y>0$. This shows that the equation holds below the barrier.

Let $y=0$ and $x>b_0$. Then $f(x,0) = f(b_0,0)+\gamma(x-b_0)$ and, therefore, $f_x(x,0)=\gamma$ and $f_{xx}(x,0)=0$. We have to consider

$$c\gamma - \delta[f(b_0,0)+\gamma(x-b_0)] + \int_0^\infty \big(f(x-z,0)-f(b_0,0)-\gamma(x-b_0)\big)\,dM(z).$$

The integrand is a concave function and therefore also the expression is concave. It vanishes at $x=b_0$ because (1) is fulfilled at that point. The derivative at $x=b_0$ is

$$-\delta\gamma + \int_0^\infty \big(f_x(b_0-z,0)-\gamma\big)\,dM(z),$$

because we can interchange integral and derivative for a concave integrand. We claim that this derivative is non-positive. This will show (3b) by the concavity. Consider for $x>b_0$ small enough,

$$\begin{aligned} &c\frac{g_x(x,0)-\gamma}{x-b_0} - \delta\frac{g(x,0)-g(b_0,0)}{x-b_0}\\ &\qquad+\int_0^\infty\left(\frac{g(x-z,0)-g(b_0-z,0)}{x-b_0} - \frac{g(x,0)-g(b_0,0)}{x-b_0}\right)dM(z)\\ &\quad= -\frac{1}{2}\sigma^2\frac{g_{xx}(x,0)}{x-b_0}\leq 0, \end{aligned}$$

where the last inequality follows because the function changes from concave to convex at b_0. Thus, we can let $x \downarrow b_0$ and our claim is shown. In particular,

$$\frac{1}{2}\sigma^2 f_{xx}(x,y) + cf_x(x,y) - \delta f(x,y) - \int_0^\infty \bigl(f(x,y) - f(x-z,y)\bigr)\,dM(z)$$

is decreasing in x.

Let now $y > 0$ and $b_> < x \le b_> + y$. Then

$$f(x,y) = f(b_>, y + b_> - x) + x - b_> = V^0(b_>) - Ce^{\rho(x-y)} + x - b_>.$$

Hence

$$f_x(x,y) = 1 - C\rho e^{\rho(x-y)} = 1 - f_y(b_>, y + b_> - x)$$

and

$$f_{xx}(x,y) = -C\rho^2 e^{\rho(x-y)}.$$

In particular,

$$f_x(x,y) + f_y(x,y) = 1.$$

Then, using the definition of ρ, we obtain

$$\begin{aligned}
-\frac{1}{2}\sigma^2 C\rho^2 e^{\rho(x-y)} &+ c\Bigl(1 - C\rho e^{\rho(x-y)}\Bigr) + \int_0^{x-b_>} \Bigl[Ce^{\rho(x-y)} - Ce^{\rho(x-y-z)} - z\Bigr]\,dM(z) \\
&- \int_{x-b_>}^\infty \Bigl[V^0(b_>) - Ce^{\rho(x-y)} + x - b_> - V^0(x-z) + Ce^{\rho(x-y-z)}\Bigr]\,dM(z) \\
&- \delta\Bigl(V^0(b_>) - Ce^{\rho(x-y)} + x - b_>\Bigr) \\
&= c - \int_0^{x-b_>} z\,dM(z) - \int_{x-b_>}^\infty \bigl[V^0(b_>) - V^0(x-z) + x - b_>\bigr]\,dM(z) \\
&- \delta\bigl(V^0(b_>) + x - b_>\bigr) \\
&= c - \int_0^\infty [V^n(x) - V^n(x-z)]\,dM(z) - \delta V^n(x) \\
&= \frac{1}{2}\sigma^2 V^n_{xx}(x) + cV^n_x(x) - \int_0^\infty [V^n(x) - V^n(x-z)]\,dM(z) - \delta V^n(x) \le 0
\end{aligned}$$

with equality in $x = b_>$, where we used that $V^n(x)$ solves (2). Thus, we have shown that (3a) is fulfilled for $y > 0$ and $b_> < x \le b_> + y$. Note that the derivative of the right-hand side

$$\int_0^\infty [V^n_x(x-z) - 1]\,dM(z) - \delta$$

is decreasing in x. That implies that

$$\frac{1}{2}\sigma^2 f_{xx}(x,y) + cf_x(x,y) - \delta f(x,y) - \int_0^\infty \bigl(f(x,y) - f(x-z,y)\bigr)\,dM(z)$$

is concave and thus decreasing in x.

Let now $y > 0$ and $b_> + y < x \le b_0 + y$. Then $f(x,y) = f(x-y,0) + y$. In particular, $f_x(x,y) + f_y(x,y) = 1$. Further, $f_x(x,y) = f_x(x-y,0)$, $f_{xx}(x,y) = f_{xx}(x-y,0)$ and

$$\begin{aligned}
&\frac{1}{2}\sigma^2 f_{xx}(x-y,0) + cf_x(x-y,0) - \int_0^{x-y-b_>} [f(x-y,0) - f(x-y-z,0)]\,dM(z) \\
&\quad - \int_{x-y-b_>}^{x-b_>} [f(x-y,0) + y - f(b_>, y + b_> + z - x) - x + z + b_>]\,dM(z) \\
&\quad - \int_{x-b_>}^{\infty} [f(x-y,0) + y - f(x-z,y)]\,dM(z) - \delta[f(x-y,0) + y] \\
&= \int_{x-y-b_>}^{x-b_>} [f(b_>, y + b_> + z - x) - f(x-y-z,0) + x - y - z - b_>]\,dM(z) \\
&\quad + \int_{x-b_>}^{\infty} [f(x-z,y) - f(x-y-z,0) - y]\,dM(z) - \delta y \\
&= \int_{x-y-b_>}^{x-b_>} \left[V^0(b_>) - V^0(x-y-z) + x - y - z - b_>\right]\,dM(z) \\
&\quad + \int_{x-b_>}^{\infty} \left[V^0(x-z) - V^0(x-y-z) - y\right]\,dM(z) - \delta y \\
&= \int_{x-y-b_>}^{\infty} [V^n(x-z) - V^n(x-y-z) - y]\,dM(z) - \delta y \\
&= \int_0^{\infty} [V^n(x-z) - V^n(x-y-z) - y]\,dM(z) - \delta y \\
&= \frac{1}{2}\sigma^2 V^n_{xx}(x) + cV^n_x(x) - \delta V^{\mathrm{n}}(x) - \int_0^{\infty} [V^n(x) - V^n(x-z)]\,dM(z) \\
&\quad - \left[\frac{1}{2}\sigma^2 V^n_{xx}(x-y) + cV^{\mathrm{n}}_x(x-y) - \delta V^n(x-y)\right. \\
&\quad \left. - \int_0^{\infty} [V^n(x-y) - V^{\mathrm{n}}(x-y-z)]\,dM(z)\right] \\
&\le 0,
\end{aligned}$$

where we used that

$$\begin{aligned}
0 = {}& \frac{1}{2}\sigma^2 f_{xx}(x-y,0) + cf_x(x-y,0) - \delta f(x-y,0) \\
& - \int_0^{\infty} [f(x-y,0) - f(x-y-z,0)]\,dM(z).
\end{aligned}$$

For the last inequality we used that

$$\frac{1}{2}\sigma^2 V^n_{xx}(v) + cV^n_x(v) - \delta V^{\mathrm{n}}(v) - \int_0^{\infty} [V^n(v) - V^n(v-z)]\,dM(z)$$

is a decreasing function in v. See also the argument in the case $y = 0$ and $x > b_0$. Thus, (3a) is fulfilled for $x \le b_0 + y$. Note that the derivative of the expression considered is

$$\int_0^{\infty} [V^n_x(x-z) - V^n_x(x-y-z) - y]\,dM(z) \le 0,$$

because $V^n(v)$ is concave. This implies that

$$\frac{1}{2}\sigma^2 f_{xx}(x,y) + cf_x(x,y) - \delta f(x,y) - \int_0^{\infty} \big(f(x,y) - f(x-z,y)\big)\,dM(z)$$

is decreasing in x.

Let now $y > 0$ and $b_0 + y < x$. Then

$$f(x,y) = f(x-y,0) + y = f(b_0,0) + y + \gamma(x-y-b_0).$$

We have $f_x(x,y) = \gamma$, $f_y(x,y) = 1-\gamma$ and $f_{xx}(x,y) = 0$. Thus, $f_x(x,y) + f_y(x,y) = 1$. We consider

$$c\gamma - \delta f(x,y) - \int_0^\infty [f(x,y) - f(x-z,y)]\,dM(z).$$

As proved above, this expression is negative for $x = b_0 + y$ because $f_x(b_0,0) = \gamma$ and $f_x(b_0,0) = 0$. Taking the derivative yields

$$\int_0^\infty \{f_x(x-z,y) - \gamma\}\,dM(z) - \delta\gamma.$$

We see that this expression is decreasing in x. Thus, the expression considered is concave in x. At $x = b_0 + y$, we obtain

$$\int_0^\infty \{f_x(b_0 + y - z, y) - \gamma\}\,dM(z) - \delta\gamma.$$

We claim that the derivative at $b_0 + y$ is non-positive. Then concavity will show that (3a) is fulfilled.

For $b_> + y < x \le b_0 + y$, note that $f(x,y) = f(x-y,0) + y$ and, therefore,

$$f_{xx}(b_0+y,y) = f_{xx}(b_0,0) = 0, \qquad f_x(b_0+y,y) = f_x(b_0,0) = \gamma$$

and $f(b_0,y) = f(b_0,0) + y$. We have seen that

$$\frac{1}{2}\sigma^2 f_{xx}(x,y) + cf_x(x,y) - \delta f(x,y) - \int_0^\infty [f(x,y) - f(x-z-y)]\,dM(z)$$

is a decreasing function in x. Thus, using that $f_{xx}(x,y) \le 0$, we get

$$\begin{aligned}
0 &\ge -\frac{1}{2}\sigma^2 \frac{f_{xx}(x,y)}{b_0+y-x} + c\,\frac{\gamma - f_x(x,y)}{b_0+y-x} - \delta\,\frac{f(b_0,0) - f(x-y,0)}{b_0+y-x} \\
&\quad + \int_0^\infty \left[\frac{f(b_0+y-z,y) - f(x-z,y)}{b_0+y-x} - \frac{f(b_0,0) - f(x-y,0)}{b_0+y-x}\right] dM(z) \\
&\ge c\,\frac{\gamma - f_x(x,y)}{b_0+y-x} - \delta\,\frac{f(b_0,0) - f(x-y,0)}{b_0+y-x} \\
&\quad + \int_0^\infty \left[\frac{f(b_0+y-z,y) - f(x-z,y)}{b_0+y-x} - \frac{f(b_0,0) - f(x-y,0)}{b_0+y-x}\right] dM(z).
\end{aligned}$$

Letting $x \uparrow b_0 + y$, we obtain

$$0 \ge \int_0^\infty [f_x(b_0+y-z,y) - \gamma]\,dM(z) - \delta\gamma,$$

which proves the claim.

Finally, we need to show that indeed $f(x,y)$ is concave in x. Suppose that $f(x,y)$ is not concave. By the construction of the solution, this means that $f(x,0)$ is not concave. Let $b_> < x_0 < b_0$ such that $f(x,0)$ is concave in $(0,x_0)$ and $f_x(x_0,0)$ has a minimum on $(0, x_0+\epsilon)$ for some $\epsilon > 0$. Then $\gamma < f_x(x_0,0) =: \tilde{\gamma} < 1$. The proof above implies that for $x \le x_0$, $f(x,y)$ is the value function of the problem with $\tilde{\gamma}$ instead of γ. Because less tax is paid, we have $f(b_>,0) > V(b_>,0)$. On the other hand, by the proof of Theorem 1, $f(x,y)$ is the value of the strategy with barriers at $b_>$ and b_0. Note that in this proof concavity only was used for the value above the barriers. Thus, $f(b_>,0) \le V(b_>,0)$. This is a contradiction, and hence $f(x,0)$ must be concave on $(0,b_0)$. $\square$

An alternative to solving equation (3) is to use the approach proposed in [2]. For any barriers $b_>$ and b_0, the value function can be expressed in terms of scale functions. One then can find the optimal barriers by maximising the value function.

5. No penalty for injections

Consider now the case $\eta = 1$. Then $b_> = 0$. In order for the value function to make sense, we write

$$\int_0^\infty \delta e^{-\delta t}(D_t^c - L_t^c)\, dt$$

instead of

$$\int_0^\infty e^{-\delta t}\, dD_t^c - \int_0^\infty e^{-\delta t}\, dL_t^c.$$

As in [13, Example 1], the value function without tax is

$$V^n(0) = \delta \int_0^\infty \mathsf{E}\left[X_t^0\right] e^{-\delta t}\, dt = x + \mu/\delta,$$

where $\mu = \mathsf{E}[X_1^0 - x]$. It follows readily that (2) is fulfilled. Thus, we have the initial condition for the solution $V^0(x)$. If $\sigma > 0$, then $V_x^n(0) = 1 = \eta$ by the principle of smooth fit. The solution $V^0(x)$ can also be found in terms of scale function as in [2].

We are now interested in the question when the second barrier is also at zero. The same approach as in [12, Prop. 1] gives that the value function with both barriers at zero becomes

$$\min\{x, y\} + \gamma(x-y)^+ + \delta^{-1}\,\mathsf{E}[X_1^0 - x] - (1-\gamma)\rho^{-1} e^{-\rho(y-x)^+}.$$

By the proof of Theorem 2, we only need to check (3b). That is

$$\begin{aligned}
0 &\ge c\gamma - \delta\bigl(\gamma x + \mu/\delta - (1-\gamma)/\rho\bigr) - \int_0^x \gamma z\, dM(z) \\
&\quad - \int_x^\infty \Bigl[\gamma x - (1-\gamma)/\rho + (1-\gamma)e^{-\rho(z-x)}/\rho + (z-x)\Bigr]\, dM(z) \\
&= -(1-\gamma)\left[\mu + \delta\frac{\gamma}{1-\gamma}\, x - \delta/\rho + \int_x^\infty \left[z - x - \left(1 - e^{-\rho(z-x)}\right)/\rho\right]\, dM(z)\right] \\
&= -(1-\gamma)\left[\mu + \frac{\delta\gamma}{1-\gamma}\, x - \delta/\rho + \int_x^\infty \left(1 - e^{-\rho(v-x)}\right) M\bigl((v,\infty)\bigr)\, dv\right].
\end{aligned}$$

For $x = 0$, the right-hand side is $(1-\gamma)\frac{1}{2}\sigma^2\rho$. This means that if $\sigma > 0$, the barrier cannot be at zero. Therefore, assume $\sigma = 0$. At $x = 0$, the equation is fulfilled. The derivative of the right-hand side is

$$-\delta\gamma + (1-\gamma)\rho \int_x^\infty e^{-\rho(v-x)} M\bigl((v,\infty)\bigr)\, dv = -\delta\gamma + (1-\gamma)\rho \int_0^\infty e^{-\rho v} M\bigl((v+x,\infty)\bigr)\, dv.$$

This is decreasing in x. Hence, the considered expression is concave in x. Thus, the (3b) is fulfilled if the derivative in zero is non-positive. This gives the condition

$$\delta\gamma \ge (1-\gamma)\int_0^\infty \left(1 - e^{-\rho z}\right)\, dM(z) = (1-\gamma)(c\rho - \delta),$$

or equivalently $\delta \ge \rho c(1-\gamma)$.

6. Barriers at zero

A simple strategy is to choose one or both barriers at zero. We can assume that $\eta > 1$. Consider first the problem without tax. Since $V_x^n(x) = \eta \neq 1$ in the case $\sigma > 0$, we conclude that the barrier can only be at zero if $\sigma = 0$. Indeed, in any interval an infinite dividend and capital injection would occur and yield the value $-\infty$. In order that the barrier $b_>$ is at zero, the value of the corresponding strategy $V^n(x) = x + (c - \eta(c-\mu))/\delta$ has to fulfil (2). Thus,

$$\begin{aligned}0 &\geq c - \delta x - \bigl(c - \eta(c-\mu)\bigr) - \int_0^x z\,dM(z) - \int_x^\infty \bigl(x + \eta(z-x)\bigr)\,dM(z)\\ &= (\eta-1)(c-\mu) - \delta x - (\eta-1)\int_x^\infty M\bigl((v,\infty)\bigr)\,dv.\end{aligned}$$

It follows that the right-hand side is concave. Thus, the required equation is fulfilled if the derivative at zero is non-positive, that is

$$(\eta-1)M\bigl((0,\infty)\bigr) - \delta \leq 0.$$

We see that this is only the case if $\delta \geq (\eta-1)M((0,\infty))$. So necessarily $b_> = 0$ is only possible if $M((0,\infty))$ is finite, that means for the classical Cramér–Lundberg risk model. This case is treated in [12].

7. Perturbed risk processes with exponential claims

If $dM(x) = \lambda\alpha e^{-\alpha x}\,dx$ for some $\alpha, \lambda > 0$ and $\sigma^2 > 0$, then we have to solve the equation

$$\begin{aligned}&\frac{1}{2}\sigma^2 V_{xx}^0(x) + cV_x^0(x) - (\lambda+\delta)V^0(x) + \lambda\alpha e^{-\alpha x}\int_0^x V(z)e^{\alpha z}\,dz\\ &\qquad + \lambda e^{-\alpha x}\bigl(V^0(0) - \eta/\alpha\bigr)\\ &\quad = 0.\end{aligned} \tag{5}$$

Taking the derivative yields

$$\begin{aligned}0 &= \frac{1}{2}\sigma^2 V_{xxx}^0(x) + cV_{xx}^0(x) - (\lambda+\delta)V_x^0(x)\\ &\quad - \alpha\left[\lambda\alpha e^{-\alpha x}\int_0^x V(z)e^{\alpha z}\,dz + \lambda e^{-\alpha x}\bigl(V^0(0) - \eta/\alpha\bigr) - \lambda V^0(x)\right]\\ &= \frac{1}{2}\sigma^2 V_{xxx}^0(x) + \left(c + \frac{1}{2}\sigma^2\alpha\right)V_{xx}^0(x) - (\lambda + \delta - \alpha c)V_x^0(x) - \alpha\delta V^0(x),\end{aligned}$$

where we used (5) to replace the integral. We get $V^0(x) = C_1 e^{\rho x} + C_2 e^{-R_1 x} + C_3 e^{-R_2 x}$, where $\rho > 0 > -R_1 > -\alpha > -R_2$. Note that the characteristic polynom can be written as

$$\frac{1}{2}\sigma^2 r^2 + cr - \lambda\frac{r}{\alpha + r} - \delta = 0.$$

Using $V^0(0) = C_1 + C_2 + C_3$, we find from (5) the equation

$$C_1 + C_2 + C_3 - \frac{\eta}{\alpha} - C_1\frac{\alpha}{\alpha+\rho} - C_2\frac{\alpha}{\alpha - R_1} - C_3\frac{\alpha}{\alpha - R_2} = 0.$$

Together with $V_x^0(0) = \eta$ and $V_{xx}^0(b_>) = 0$, that is

$$C_1\rho - C_2R_1 - C_3R_2 = \eta \quad \text{and} \quad C_1\rho^2 e^{\rho b_>} + C_2R_1^2 e^{-R_1 b_>} + C_3R_2^2 e^{-R_2 b_>} = 0,$$

we can determine the constant C_k as functions of $b_>$. Then solving numerically the equation $V_x^0(b_>) = 1$, we find the barrier $b_>$. This yields the constants C_k.

The solution to the problem with tax becomes

$$V(x,y) = e^{\rho x}(C_1 - Ce^{-\rho y}) + C_2 e^{-R_1 x} + C_3 e^{-R_2 x}.$$

From $V_{xx}(b_0, 0) = 0$, that is

$$C = C_1 + \left(C_2 R_1^2 e^{-R_1 b_0} + C_3 R_2^2 e^{-R_2 b_0}\right) \rho^{-2} e^{-\rho b_0},$$

the constant C is determined as a function of b_0. Solving numerically $V_x(b_0, 0) = \gamma$ yields the barrier b_0. Then, C can be determined.

If, for example, $\lambda = \alpha = 1$, $c = 1.2$, $\delta = 0.1$, $\eta = 1.1$, $\sigma^2 = 0.05$ and $\gamma = 0.8$, we find

$$\rho = 0.247264, \qquad R_1 = 0.330707, \qquad R_2 = 48.9166.$$

The barrier is $b_> = 0.133638$ and the solution to (5) becomes

$$V^0(x) = 2.28668 e^{\rho x} - 1.313 e^{-R_1 x} - 0.00205183 e^{-R_2 x}.$$

For the problem with tax we get $b_0 = 0.719681$ and

$$V(x,y) = (2.28668 - 0.737207 e^{-\rho y}) e^{\rho x} - 1.313 e^{-R_1 x} - 0.00205183 e^{-R_2 x}.$$

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