F-DIVIDED SHEAVES TRIVIALIZED BY DOMINANT MAPS ARE ESSENTIALLY FINITE

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Abstract. By a result of Biswas and Dos Santos on a smooth and projective variety over an algebraically closed field, a vector bundle trivialized by a proper and surjective map is essentially finite; that is, it corresponds to a representation of the Nori fundamental group scheme. In this paper we obtain similar results for nonproper nonsmooth algebraic stacks over arbitrary fields of characteristic $p > 0$. As a byproduct we have the following partial generalization of the Biswas–Dos Santos result in positive characteristic: on a pseudo-proper and inflexible stack of finite type over $k$, a vector bundle which is trivialized by a proper and flat map is essentially finite.

Introduction

Let $X$ be a smooth and projective variety over an algebraically closed field $k$ with a rational point $x \in X(k)$. Then the étale universal cover $\tilde{X}_x$ of $X$ is an fpqc torsor under the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ (thought of as a profinite constant group scheme over $k$). Thus any representation of $\pi_1^{\text{ét}}(X, x)$ on a finite-dimensional $k$-vector space $V$ will give rise to a vector bundle $\tilde{X}_x \times_{\pi_1^{\text{ét}}(X, x)} V$ on $X$ by fpqc descent. This defines a functor from the category of finite-dimensional $\pi_1^{\text{ét}}(X, x)$-representations to the category of vector bundles on $X$. It is Lange and Stuhler who first observed (in [LS, 1.2]) that a vector bundle on $X$ is in the essential image of this functor if and only if the vector bundle is trivialized by a finite étale cover of $X$.

A similar result holds also for the Nori fundamental group scheme, introduced in [Nori]. Let $X$ be a proper, geometrically connected and geometrically reduced scheme over a field $k$ equipped with a rational point $x \in X(k)$. Nori defined the notion of essentially finite vector bundles on $X$ and proved that they form a Tannakian category over $k$, whose associated group scheme is called the Nori fundamental group scheme of $(X, x)$. Directly from Nori’s definition, one has that a vector bundle on $X$ is essentially finite if and only if it is trivialized by a torsor over $X$ under some finite $k$-group scheme. However, torsors under finite group schemes only correspond to Galois covers in the étale case. Therefore, one should expect more; for instance, that vector bundles trivialized by flat and finite maps are essentially finite. In [BDS] and [AM] this question has been answered for normal, projective varieties over an algebraically closed field: in this case a vector bundle is essentially finite if and only if it is trivialized by a proper and surjective morphism.

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Meanwhile Nori’s construction and the notion of essentially finite vector bundles has been largely extended. In [BV] N. Borne and A. Vistoli introduced the notion of the Nori fundamental gerbe for a fibered category \( \mathcal{X} \) over a field \( k \) via a universal property and also an abstract notion of an essentially finite object in an additive and monoidal category. The fibered categories over \( k \) which are inflexible over \( k \) (see [L.]) are the ones admitting a Nori gerbe. For instance, if \( \mathcal{X} \) is a reduced stack of finite type, then it is inflexible if and only if \( k \) is integrally closed in the ring \( H^0(\mathcal{O}_\mathcal{X}) \) (see Remark [L.]). A fibered category \( \mathcal{X} \) over \( k \) is pseudo-proper over \( k \) if for all \( E \in \text{Vect}(\mathcal{X}) \) the \( k \)-vector space \( H^0(E) \) is finite dimensional. Examples of pseudo-proper stacks are of course proper algebraic stacks over \( k \) but also arbitrary affine gerbes. The previous result of Nori is generalized by showing that for a pseudo-proper and inflexible fibered category \( \mathcal{X} \) over \( k \), the category of essentially finite vector bundles on \( \mathcal{X} \) is a \( k \)-Tannakian category, and its associated gerbe is the Nori fundamental gerbe (see [BV, Theorem 7.9]). Applying this to an affine gerbe, they also concluded that, in a \( k \)-Tannakian category, the full subcategory of essentially finite objects is the \( k \)-Tannakian subcategory of objects having a finite monodromy gerbe, which is the gerbe corresponding to the \( k \)-Tannakian subcategory generated by this object (see [BV, Corollary 7.10]).

Based on [BV] and [EH], we also get in [TZ] a Tannakian description of the Nori fundamental gerbe for not necessarily pseudo-properly fibered categories using the language of stratified sheaves and \( F \)-divided sheaves. The theory applies to all reduced and inflexible algebraic stacks of finite type over \( k \). In particular in positive characteristic we studied Tannakian categories of \( F \)-divided sheaves \( \text{Fdiv}(\mathcal{X}/k) \) and \( \text{Fdiv}_\infty(\mathcal{X}/k) \) and prove that their essentially finite objects are the representations of the Nori étale fundamental gerbe and Nori fundamental gerbe of \( \mathcal{X}/k \), respectively.

In this paper we consider the Lange–Stuhler or Biswas–Dos Santos style theorem in non-pseudo-proper settings. In what follows we assume that the ground field \( k \) has positive characteristic. An object in an additive monoidal category is called trivial or free if it is isomorphic to a finite direct sum of copies of the unit object. Given a map of algebraic stacks \( f : \mathcal{U} \to \mathcal{X} \), we denote by \( \text{Fdiv}(\mathcal{X}/k)_f \) the full subcategory of \( \text{Fdiv}(\mathcal{X}/k) \) of objects trivialized by \( f \), that is whose pullback along \( f \) is free in \( \text{Fdiv}(\mathcal{U}/k) \). When \( \mathcal{X} \) is reduced and inflexible, we denote by \( \text{Fdiv}_\infty(\mathcal{X}/k)_f \) the sub-Tannakian category of \( \text{Fdiv}_\infty(\mathcal{X}/k) \) generated by the objects trivialized by \( f \). Here are our main results:

**Theorem 1.** Let \( \mathcal{X} \) be a geometrically connected (resp., reduced and inflexible) algebraic stack of finite type over \( k \). If \( E \in \text{Fdiv}(\mathcal{X}/k) \) (resp., \( E \in \text{Fdiv}_\infty(\mathcal{X}/k) \)) is an essentially finite object, then there exists a surjective finite and étale (resp., finite and flat) map \( f : \mathcal{U} \to \mathcal{X} \) trivializing \( E \). Conversely, let \( f : \mathcal{U} \to \mathcal{X} \) be a flat and surjective map of algebraic stacks of finite type over \( k \). Then \( \text{Fdiv}(\mathcal{X}/k)_f \) (resp., \( \text{Fdiv}_\infty(\mathcal{X}/k)_f \)) is a Tannakian subcategory and, if one of the conditions below is satisfied, its associated gerbe is finite and étale (resp., profinite):

1. the map \( f \) is proper;
2. the map \( f \) is geometrically connected, in which case \( \text{Fdiv}(\mathcal{X}/k)_f = \text{Vect}(k) \);
3. the stack \( \mathcal{U} \) is geometrically irreducible.

In this generality we are unable to prove the result for arbitrary flat and surjective maps \( f \), although we believe this should be true. On the other hand, by adding some more regularity, we can even drop the flatness hypothesis.
**Theorem II.** Let $\mathcal{X}$ be a geometrically unibranch (e.g., normal, see Appendix A) algebraic stack of finite type over $k$. If $\mathcal{X}$ is geometrically connected (resp., reduced and inflexible), then an object in $F\text{div}(\mathcal{X}/k)$ (resp., $F\text{div}_\infty(\mathcal{X}/k)$) is essentially finite if and only if it is trivialized by a dominant morphism of finite type from an algebraic stack.

By lifting vector bundles to objects of $F\text{div}_\infty(\mathcal{X}/k)$, we are able to deduce from Theorem II this partial generalization of [BDS] and [AM]:

**Corollary I.** Let $\mathcal{X}$ be a pseudo-proper and inflexible algebraic stack $\mathcal{X}$ of finite type over $k$. Then a vector bundle on $\mathcal{X}$ is essentially finite if and only if it is trivialized by a proper and flat map $f: \mathcal{U} \to \mathcal{X}$ from an algebraic stack such that $H^0(\mathcal{O}_{\mathcal{U} \times_{\mathcal{X}} \mathcal{U}})$ is a finite $k$-algebra.

Essentially, we drop the smoothness or the normality assumption, but we require the trivialization to be flat. The finiteness condition is needed in this generality because $\mathcal{U}$ or $\mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ may not be pseudo-proper. For instance, asking that the space of global sections of all coherent sheaves on $\mathcal{X}$ be finite dimensional solves this issue.

Another generalization of [BDS] and [AM] is contained in [TZ3], where we introduce a different approach to this problem which allows us to extend their results to normal, connected and strongly pseudo-proper algebraic stacks of finite type over an arbitrary field $k$. In spirit the main theorems in [TZ3] are quite different from the main theorems in this paper, but Corollary I provides a bridge.

Another application of the theory and, more precisely, of Corollary I is the following. In [BV2] Borne and Vistoli introduce the virtually unipotent fundamental gerbe of a fibered category $\mathcal{X}$ over $k$, denoted by $\mathcal{X} \to \Pi_{\mathcal{X}/k}^\text{vu}$: it is defined as a provirtually unipotent affine gerbe such that any map $\mathcal{X} \to \Gamma$ to a virtually unipotent gerbe factors uniquely through $\Pi_{\mathcal{X}/k}^\text{vu}$ (see Definitions 1.15 and 1.16). If $k$ has positive characteristic, and if $\mathcal{X}$ is pseudo-proper, geometrically reduced, and geometrically connected over $k$ and has an atlas from a reduced Noetherian scheme, then the representations of $\Pi_{\mathcal{X}/k}^\text{vu}$ have a Tannakian interpretation: they correspond, up to Frobenius, to subsequent extensions of essentially finite vector bundles (see [BV2], Section 10, Theorem 10.7). We deduce the following:

**Corollary II.** Let $\mathcal{X}$ be a pseudo-proper, geometrically reduced, and geometrically connected algebraic stack of finite type over a field of positive characteristic $k$ such that $\dim_k H^1(\mathcal{X}, E) < \infty$ for all vector bundles $E$ on $\mathcal{X}$. Then the virtually unipotent gerbe $\Pi_{\mathcal{X}/k}^\text{vu}$ and the Nori fundamental gerbe $\Pi_{\mathcal{X}/k}^\text{n}$ coincide.

The above result holds only in positive characteristic. Corollary II fails for a stack like $\mathcal{B}_k \mathbb{G}_a$, so that the condition on $H^1$ is also necessary: it ensures that all $\mathbb{G}_a$-torsors of $\mathcal{X}$ and of its covers come from a finite subgroup of $\mathbb{G}_a$ (see Lemma 2.4).

Finally as application of Theorem II we prove the following:

**Corollary III.** Let $\Gamma$ be a gerbe of finite type over a field $k$, that is, a gerbe which is an algebraic stack of finite type over $k$. Then all objects of $F\text{div}(\Gamma/k)$ and $F\text{div}_\infty(\Gamma/k)$ are essentially finite. In particular

$F\text{div}(\Gamma/k) \simeq \text{Vect}(\Gamma_{\text{ét}})$ and $F\text{div}_\infty(\Gamma/k) \simeq \text{Vect}(\hat{\Gamma})$,

where $\Gamma_{\text{ét}}$ and $\hat{\Gamma}$ are the proétale and profinite quotients, respectively.
The paper is divided as follows. In the first section we recall various properties and definitions, such as $F$-divided sheaves and Nori gerbes, and state the results we will use about them. The second section is dedicated to the proof of Theorem I and Corollaries I and II while the third one to the proof of Theorem II and Corollary III. Finally, in the appendices we discuss the notion of geometrically unibranch algebraic stacks and the behavior of $F\text{div}(-/k)$ under finite field extensions.

Notation

A category fibered in groupoid $\mathcal{X}$ over a ring $R$ is a category fibered in a groupoid over the category of affine $R$-schemes $\text{Aff}/R$. By an fpqc covering $U \rightarrow \mathcal{X}$, we mean a map of fibered categories which is represented by fpqc coverings of algebraic spaces. If $\mathcal{X}$ is an algebraic stack, we denote by $\mathcal{X}_{\text{red}}$ its reduction.

If $\mathcal{X}$ is a fibered category over $\mathbb{F}_p$, one can always define a Frobenius functor $F_\mathcal{X} : \mathcal{X} \longrightarrow \mathcal{X}$ by sending an object $T \rightarrow \mathcal{X}$ to the composition $T \xrightarrow{F_T} T \rightarrow \mathcal{X}$, where $F_T$ is the absolute Frobenius of $T$, and, if $\mathcal{X}$ is defined also over a field extension $k/\mathbb{F}_p$, a relative Frobenius functor $\mathcal{X} \rightarrow \mathcal{X}^{(i,k)}$ for $i \in \mathbb{N}$. Here and in the rest of the paper by $\mathcal{X}^{(i,k)}$ we mean the base change of $\mathcal{X}$ over the $F_k^i : k \rightarrow k$. When $k$ is clear from the context, we will simply write $\mathcal{X}^{(i)}$. Please refer to [TZ, Notations and Conventions] for details.

In this paper we will freely talk about affine gerbes over a field (often improperly called just gerbes) and Tannakian categories and use their properties. Please refer to [TZ, Appendix B] for details. If $\mathcal{C}$ is a $k$-Tannakian category, then its associated affine gerbe over $k$ is denoted by $\Pi \mathcal{C}$: if $A$ is a $k$-algebra, then $\Pi \mathcal{C}(A)$ is the category of $k$-linear, monoidal, and exact functors $\mathcal{C} \rightarrow \text{Vect}(A)$.

If $\mathcal{C}$ is a monoidal category, we will often denote by $1 \mathcal{C}$ or simply $1$ the unit object of $\mathcal{C}$. An object in $\mathcal{C}$ is called trivial or free if it is isomorphic to $1 \oplus m \mathcal{C}$ for some $m \in \mathbb{N}$.

1. Preliminaries

Everything in this section is either contained in or is an easy consequence of results in [BV] and [TZ]. We collect some important theorems and definitions here for the convenience of the reader.

Definition 1.1. Let $k$ be a field of characteristic $p > 0$, and let $\mathcal{X}$ be a category fibered in groupoid over $k$. We define $\text{Fdiv}(\mathcal{X}/k)$ to be the category with the following.

**Objects:** tuples $(E_i, \sigma_i)_{i \in \mathbb{N}}$, where $E_i$ is a vector bundle on $\mathcal{X}^{(i)}$, and an isomorphism $\sigma_i : \phi_{i+1}^* E_{i+1} \rightarrow E_i$, where $\phi_{i+1} : \mathcal{X}^{(i)} \rightarrow \mathcal{X}^{(i+1)}$ is the relative Frobenius map of $\mathcal{X}^{(i)}/k$.

**Morphisms:** a morphism from $(E_i, \sigma_i)_{i \in \mathbb{N}}$ to $(E'_i, \sigma'_i)_{i \in \mathbb{N}}$ is just a collection of morphisms $a_i : E_i \rightarrow E'_i$ making the diagram

\[
\begin{array}{ccc}
E_i & \xrightarrow{\sigma_i} & E_i \\
\phi_{i+1}^* E_{i+1} & \downarrow & \phi_{i+1}^* E_{i+1} \\
\phi_{i+1}^*(a_{i+1}) & \xrightarrow{\sigma_i'} & E'_i
\end{array}
\]

commutative.
Given a natural number $i \in \mathbb{N}$, we also define the category $\text{Fdiv}_i(\mathcal{X}/k)$ as follows:

**Objects:** triples $((\mathcal{F}, \mathcal{G}, \lambda))$, where $\mathcal{F} \in \text{Vect}(\mathcal{X})$, $\mathcal{G} = (G_i, \sigma_i)_{i \in \mathbb{N}} \in \text{Fdiv}(\mathcal{X}/k)$, and $\lambda : F^* \mathcal{F}_{/k} \xrightarrow{\cong} \mathcal{G}_0$ is an isomorphism (here $F$ denotes the absolute Frobenius of $\mathcal{X}$);

**Morphisms:** a morphism from $((\mathcal{F}, \mathcal{G}, \lambda))$ to $((\mathcal{F}', \mathcal{G}', \lambda'))$ is just a pair of morphisms $\phi : \mathcal{F} \to \mathcal{F}'$ and $\varphi : \mathcal{G} \to \mathcal{G}'$ making the diagram

\[
\begin{array}{ccc}
F^* \mathcal{F} & \xrightarrow{\lambda} & \mathcal{G}_0 \\
\downarrow \phi^* & & \downarrow \varphi_0 \\
F^* \mathcal{F}' & \xrightarrow{\lambda'} & \mathcal{G}'_0
\end{array}
\]

commutative.

The categories $\text{Fdiv}_i(\mathcal{X}/k)$ are additive and monoidal, and they have a $k$-linear structure coming from the one on $\mathcal{F}$. There are monoidal and $k$-linear functors $\text{Fdiv}_i(\mathcal{X}/k) \to \text{Fdiv}_{i+1}(\mathcal{X}/k)$, $(\mathcal{F}, \mathcal{G}, \lambda) \to (\mathcal{F}, F^* \mathcal{G}, F^* \lambda)$, and we define $\text{Fdiv}_\infty(\mathcal{X}/k) := \text{lim}_{i \in \mathbb{N}} \text{Fdiv}_i(\mathcal{X}/k)$ (see [TZ, Definition 5.10] for details).

**Remark 1.2.** Note that if $k$ is a perfect field, then one can easily check that $\text{Fdiv}_0(\mathcal{X}/k)$ is equivalent to the category defined as follows:

**Objects:** tuples $(E_i, \sigma_i)_{i \in \mathbb{N}}$, where $E_i$ is a vector bundle on $\mathcal{X}^{(i)}$, $\sigma_i : \phi_{1,i}^* E_{i+1} \cong E_i$, for all $i \geq 1$, and $\sigma_0 : \phi_{-i,0}^* E_0 \cong \phi_{-i,1}^* E_1$.

**Morphisms:** a morphism from $(E_i, \sigma_i)_{i \in \mathbb{N}}$ to $(E_i', \sigma_i')_{i \in \mathbb{N}}$ is just a collection of morphisms $a_i : E_i \to E_i'$ which are compatible with $\sigma_i$ and $\sigma_i'$.

Using this equivalence, one can deduce that if $k$ is perfect and if $\mathcal{X}$ admits a representable fpqc covering from a reduced scheme $\mathcal{X}$, then the natural transition functors $\text{Fdiv}_i(\mathcal{X}/k) \to \text{Fdiv}_{i+1}(\mathcal{X}/k)$ are fully faithful.

**Remark 1.3.** An important property of $\text{Fdiv}$ that will be used throughout the paper is that it is insensitive to nilpotent thickenings, that is if $\mathcal{X}' \to \mathcal{X}$ is a nilpotent closed immersion, then $\text{Fdiv}(\mathcal{X}/k) \to \text{Fdiv}(\mathcal{X}'/k)$ is an equivalence (see [TZ] Lemma 6.24). In particular if $\mathcal{X}$ is a Noetherian algebraic stack and $\mathcal{X}_{\text{red}}$ its reduction, then $\text{Fdiv}(\mathcal{X}/k) \simeq \text{Fdiv}(\mathcal{X}_{\text{red}}/k)$.

**Definition 1.4.** Given a $k$-algebra $A$, we set

\[ A_{\text{et},k} = \{ a \in A \mid \exists \text{ a separable polynomial } f \in k[x] \text{ s.t. } f(a) = 0 \}. \]

Alternatively, $A_{\text{et},k}$ is the union of all $k$-subalgebras of $A$ which are finite and étale over $k$. When the base field is clear from the context, we will simply write $A_{\text{et}}$.

**Remark 1.5.** ([TZ, Lemma 2.6]). Let $\mathcal{X}$ be a quasi-compact and quasi-separated algebraic stack over $k$. Then $\mathcal{X} \to \text{Spec } \mathcal{H}^0(\mathcal{O}_{\mathcal{X}})_{\text{ét}}$ is geometrically connected and, if $\mathcal{X}$ is connected, $\mathcal{H}^0(\mathcal{O}_{\mathcal{X}})_{\text{ét}}$ is a field. Moreover, $\mathcal{X}$ is geometrically connected over $k$ if and only if $\mathcal{H}^0(\mathcal{O}_{\mathcal{X}})_{\text{ét}} = k$.

**Definition 1.6.** ([BV, Section 5]). If $\mathcal{X}$ is a category fibered in a groupoid over $k$, the Nori fundamental gerbe (resp., étale Nori fundamental gerbe) of $\mathcal{X}/k$ is a profinite (resp., pro-étale) gerbe $\Pi$ over $k$ together with a map $\mathcal{X} \to \Pi$ such that for all finite (resp., finite and étale) stacks $\Gamma$ over $k$ the pullback functor

\[ \text{Hom}_k(\Pi, \Gamma) \to \text{Hom}_k(\mathcal{X}, \Gamma) \]
is an equivalence. If this gerbe exists, it is unique up to a unique isomorphism, and it will be denoted by $\Pi_{X/k}^N$ (resp., $\Pi_{X/k}^{N,\text{ét}}$) or by dropping the /k if it is clear from the context.

We call $X$ inflexible if it is nonempty and all maps to a finite stack over $k$ factor through a finite gerbe over $k$.

**Remark 1.7.** By [BV] Definition 5.7, pp. 13 $X$ admits a Nori fundamental gerbe if and only if it is inflexible. By [TZ] Theorem 4.4 if $X$ is reduced, quasi-compact and quasi-separated, then $X$ is inflexible if and only if $k$ is algebraically closed in $H^0(\mathcal{O}_X)$. In particular if $X$ is geometrically connected and geometrically reduced, then it is inflexible.

By [TZ] Proposition 4.3 if $X$ is quasi-compact and quasi-separated, then $X$ admits a Nori étale fundamental gerbe if and only if $X$ is geometrically connected over $k$.

**Definition 1.8 ([BV] Definition 7.7, pp. 21).** Let $C$ be an additive and monoidal category. An object $E \in C$ is called finite if there exist $f \neq g \in \mathbb{N}[X]$ polynomials with natural coefficients and an isomorphism $f(E) \simeq g(E)$, it is called essentially finite if it is a kernel of a map of finite objects of $C$. We denote by $\text{EFin}(C)$ the full subcategory of $C$ consisting of essentially finite objects.

**Definition 1.9 ([BV] Definition 7.1, pp. 20).** A category $X$ fibered in a groupoid over a field $k$ is pseudoproper if it satisfies the following two conditions:

1. there exists a quasi-compact scheme $U$ and a morphism $U \to X$ which is representable, faithfully flat, quasi-compact, and quasi-separated;
2. for all vector bundles $E$ on $X$ the $k$-vector space $H^0(X,E)$ is finite dimensional.

**Example 1.10 ([BV] Example 7.2, pp. 20).** Examples of pseudo-proper fiber categories are proper schemes, finite stacks, and affine gerbes.

**Theorem 1.11 ([BV] Theorem 7.9, Corollary 7.10, pp. 22).** Let $X$ be an inflexible pseudo-proper fibered category over a field $k$. Then the pullback of $X \to \Pi_{X/k}^N$ induces an equivalence $\text{Vect}(\Pi_{X/k}^N) \to \text{EFin}(\text{Vect}(X))$.

Let $C$ be a Tannakian category. Then $\text{EFin}(C)$ is the Tannakian subcategory of $C$ of objects whose monodromy gerbe is finite.

**Definition 1.12.** A field extension $L/k$ is called separably generated (resp., separable) up to a finite extension if there exists an intermediate extension $k \subseteq F \subseteq L$ such that $L/F$ is finite and $F/k$ is separable (resp., separably generated) (see [SP 030I]).

Putting together several results [TZ] Lemma 2.6, Theorem 4.4, Theorem 5.8, Theorem 5.13, Theorem 6.23], we deduce the following:

**Theorem 1.13.** Let $X$ be a connected algebraic stack over $k$ with an fpqc covering $U \to X$ from a Noetherian scheme $U$ such that, for all $u \in U$, $k(u)/k$ is separable up to a finite extension, and set $L = \text{End}_{\text{Fdiv}(X/k)}(1)$. $\text{Fdiv}(X/k) \to \text{Vect}(X)$ is faithful, $L \subseteq H^0(\mathcal{O}_X)$ is a field, $\text{Fdiv}(X/k)$ is an $L$-Tannakian category, and there is an equivalence of $L$-Tannakian categories $\text{Vect}(\Pi_{X/k}^{N,\text{ét}}) \simeq \text{EFin}(\text{Fdiv}(X/k))$. If there exists a map $\text{Spec} F \to X$ where $F$ is a field separably generated up to a finite extension over $k$, then $L = H^0(\mathcal{O}_X)_{\text{ét}}$, so that $L = k$ if and only if $X$ is geometrically connected over $k$. 
Assume $\mathcal{X}$ reduced and set $L_i = \{ x \in H^0(\mathcal{O}_\mathcal{X}) \mid x^p \in L \}$ for $i \in \mathbb{N}$ and $L_\infty = \bigcup_i L_i$. Then $\text{Fdiv}_i(\mathcal{X}/k)$ is an $L_i$-Tannakian category for $i \in \mathbb{N} \cup \{ \infty \}$ and there is an equivalence of $L_\infty$-Tannakian categories $\text{Vect}(\Pi^\infty_{\mathcal{X}/L_\infty}) \simeq \text{EFin}(\text{Fdiv}_\infty(\mathcal{X}/k))$.

Moreover,
$$\text{EFin}(\text{Fdiv}_\infty(\mathcal{X}/k)) \simeq \lim_{\longrightarrow i} \text{EFin}(\text{Fdiv}_i(\mathcal{X}/k))$$

and $(\mathcal{F}, \mathcal{G}, \lambda) \in \text{Fdiv}_i(\mathcal{X}/k)$ is essentially finite if and only if $\mathcal{G}$ is essentially finite in $\text{Fdiv}(\mathcal{X}/k)$. If there exists a map $\text{Spec } F \to \mathcal{X}$ where $F$ is a field separably generated up to a finite extension over $k$ then $L_\infty = k$ if and only if $\mathcal{X}$ is inflexible over $k$.

**Remark 1.14.** A separably generated field extension is separable and all extensions of a perfect field are separable (see [SP, 05DT]). For instance finitely generated field extensions are separably generated up to a finite extension. Thus, Theorem 1.15 applies when $\mathcal{X}$ is a stack of finite type over $k$: if $\mathcal{X}$ is geometrically connected over $k$, then $\text{Fdiv}(\mathcal{X}/k)$ is a $k$-Tannakian category over $k$, and if $\mathcal{X}$ is reduced and inflexible over $k$, then $\text{Fdiv}_\infty(\mathcal{X}/k)$ is a $k$-Tannakian category too.

**Definition 1.15 ([BV2, Definition 6.16]).** A virtually unipotent group scheme over $k$ is an affine group scheme $G$ of finite type over $k$ such that the reduction of the connected component of $G \times_k \overline{k}$ is unipotent, where $\overline{k}$ is an algebraic closure of $k$.

An affine gerbe $\Gamma$ over $k$ is virtually unipotent if $\Gamma \times_k \overline{k} \simeq BG$, where $G$ is a virtually unipotent group scheme over $\overline{k}$. A provirtually unipotent gerbe is an affine gerbe projective limit of virtually unipotent gerbes.

**Definition 1.16 ([BV2, Definition 5.6]).** If $\mathcal{X}$ is a category fibered in a groupoid over $k$, the virtually unipotent fundamental gerbe of $\mathcal{X}/k$ is a provirtually unipotent gerbe $\Pi$ over $k$ together with a map $\mathcal{X} \to \Pi$ such that for all virtually unipotent gerbes $\Gamma$ over $k$ the pullback functor
$$\text{Hom}_k(\Pi, \Gamma) \to \text{Hom}_k(\mathcal{X}, \Gamma)$$

is an equivalence. If this gerbe exists it is unique up to a unique isomorphism, and it will be denoted by $\Pi^\text{VU}_{\mathcal{X}/k}$ or by dropping the $/k$ if it is clear from the context.

**Theorem 1.17 ([BV2, Section 6.3, Theorem 7.1]).** If $\mathcal{X}$ is a quasi-compact, quasi-separated, and geometrically reduced fibered category over $k$ such that $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}) = k$, then $\mathcal{X}$ admits a virtually unipotent fundamental gerbe.

**Definition 1.18 ([BV2, Definition 10.2, Definition 10.6]).** Let $\mathcal{X}$ be a fibered category. A vector bundle $E$ on $\mathcal{X}$ is called an extended essentially finite sheaf if there is a filtration
$$0 = E_{r+1} \subseteq E_r \subseteq \cdots \subseteq E_1 \subseteq E_0 = E$$
in which all quotients $E_i/E_{i+1}$ are essentially finite vector bundles on $\mathcal{X}$.

Assume that $k$ has positive characteristic, and let $F: \mathcal{X} \to \mathcal{X}$ be the absolute Frobenius of $\mathcal{X}$. A vector bundle $E$ on $\mathcal{X}$ is called virtually unipotent if there exists $m \in \mathbb{N}$ such that $F^mE$ is an extended essentially finite vector bundle on $\mathcal{X}$.

**Theorem 1.19 ([BV2, Theorem 10.7]).** Let $\mathcal{X}$ be a pseudo-proper, geometrically reduced, and geometrically connected fibered category admitting an fpqc cover from a Noetherian reduced scheme. Then the pullback $\text{Vect}(\Pi^\text{VU}_{\mathcal{X}/k}) \to \text{Vect}(\mathcal{X})$ is an equivalence onto the full subcategory of $\text{Vect}(\mathcal{X})$ of virtually unipotent sheaves.
2. Trivializations by flat and surjective maps

We fix a field $k$ of positive characteristic. The aim of this section is to prove Theorem [1] and Corollaries [2] and [3]. We start by showing how to trivialize essentially finite $F$-divided sheaves.

**Proposition 2.1.** Let $\Gamma$ be a finite stack over $k$. Then

$$F\text{div}(\Gamma/k) \simeq \text{Vect}(\Gamma_{\text{\acute{e}t}}) \quad \text{and} \quad F\text{div}_\infty(\Gamma/k) \simeq \text{Vect}(\Gamma).$$

**Proof.** By [TZ, Lemma 3.6] we can find an index $i > 0$ and 2-commutative diagrams

$$\begin{array}{ccc}
\Gamma & \xrightarrow{c} & \Gamma^{(i)} \\
\downarrow^{\alpha} & & \downarrow \\
\Gamma_{\text{\acute{e}t}} & \xrightarrow{\sim} & \Gamma_{\text{\acute{e}t}}^{(i)}.
\end{array}$$

The arrow $c$ provides an equivalence of categories between $\Gamma^{(\infty)} := \varinjlim_{i \in \mathbb{N}} \Gamma^{(i)}$ and $\Gamma_{\text{\acute{e}t}}^{(\infty)} := \varinjlim_{i \in \mathbb{N}} \Gamma_{\text{\acute{e}t}}^{(i)}$. Thus we have $F\text{div}(\Gamma) = \text{Vect}(\Gamma^{(\infty)}) = \text{Vect}(\Gamma_{\text{\acute{e}t}}^{(\infty)}) = \text{Vect}(\Gamma_{\text{\acute{e}t}})$ (see [TZ, §6.2, Definition 6.20 and Proposition 6.21]).

Considering the factorization $F^i: \Gamma \xrightarrow{\alpha} \Gamma_{\text{\acute{e}t}} \xrightarrow{\beta} \Gamma$, where $F$ is the absolute Frobenius of $\Gamma$, we get a functor $\Psi: \text{Vect}(\Gamma) \rightarrow F\text{div}(\Gamma)$ mapping $V \in \text{Vect}(\Gamma)$ to $(V, \beta^*V, v) \in F\text{div}(\Gamma)$, where $v: \alpha^*(\beta^*V) \rightarrow F^iV$ is the canonical morphism. Here we identify the forgetful functor $F\text{div}(\Gamma) \rightarrow \text{Vect}(\Gamma)$ with $\text{Vect}(\Gamma_{\text{\acute{e}t}}) \xrightarrow{\alpha^*} \text{Vect}(\Gamma)$. Since $\Psi$ is a section of the forgetful functor $\Phi: F\text{div}_\infty(\Gamma) \rightarrow \text{Vect}(\Gamma)$, we conclude that $\Phi$ is essentially surjective.

Let us prove that it is fully faithful, let $\chi = (V, W, u), \chi' = (V', W', u') \in F\text{div}(\Gamma)$, and let $\lambda: V \rightarrow V'$ be a map. Via $u, u'$ we obtain a map $\delta: \alpha^*W \rightarrow \alpha^*W'$: an arrow $\chi \rightarrow \chi'$ mapping to $\lambda$ is a pair $(\lambda, W \xrightarrow{\delta} W')$ with $\alpha^*\delta = \delta$. Since $\alpha^*$ is faithful, it follows that $\Phi$ is faithful. To prove that it is full, we must prove that there exists $l > 0$ such that $F^l\delta: \alpha^*F^l_{\text{\acute{e}t}}W \rightarrow \alpha^*F^l_{\text{\acute{e}t}}W'$, where $F^l_{\text{\acute{e}t}}$ is the absolute Frobenius of $\Gamma_{\text{\acute{e}t}}$, comes from a map $\gamma: F^l_{\text{\acute{e}t}}W \rightarrow F^l_{\text{\acute{e}t}}W'$. Since the composition $\Gamma_{\text{\acute{e}t}} \xrightarrow{\beta} \Gamma \xrightarrow{\alpha} \Gamma_{\text{\acute{e}t}}$ coincides with $F^l_{\text{\acute{e}t}}$, and since $\beta \circ \alpha = F^i$, it is enough to set $l = i$ and $\gamma = \beta^*\delta$. \hfill \Box

**Proof of Theorem 1** first part. We consider first the case of $F\text{div}(X)$. So let $E \in F\text{div}(X)$, which is essentially finite and denotes by $\Gamma$ the monodromy gerbe of $E$. In particular there are maps $X \rightarrow \Pi_{F\text{div}(X)} \xrightarrow{\alpha} \Gamma$ and $V \in \text{Vect}(\Gamma)$ such that $\alpha^*V \simeq E$. The profinite quotient of $\Pi_{F\text{div}(X)}$ is $\Pi_X^{N, \text{\acute{e}t}}$ by Theorem [1.13] and therefore there is also a factorization $\alpha: \Pi_{F\text{div}(X)} \rightarrow \Pi_X^{N, \text{\acute{e}t}} \rightarrow \Gamma$. In particular $\Gamma$ is a finite and étale gerbe, i.e., a finite gerbe which has an étale and surjective map from a scheme étale over $k$. We have a 2-commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\alpha} & \Pi_{F\text{div}(X)} \\
\downarrow \alpha & & \downarrow b \\
\Gamma & \xrightarrow{\beta} & \Pi_{F\text{div}(\Gamma)} \xrightarrow{\beta^*} \Pi_X^{N, \text{\acute{e}t}}
\end{array}$$

where $b$ is the canonical morphism.
and, by Proposition 2.1, the lower horizontal arrows are equivalences. Using the universal property of \( \Pi_X^{\text{ét}} \), one can conclude that \( \alpha \) fits in the commutative diagram. Thus \( \alpha^* V \simeq b^*(\alpha_\Gamma V) \), that is there exists an \( F \)-divided sheaf on \( \Gamma \) pulling back to \( \mathcal{X} \to \Gamma \). Because of the existence of a finite and étale atlas of \( \Gamma \), there exists a finite separable extension \( L/k \) with a map \( \text{Spec} L \to \Gamma \). As by Proposition 2.1, \( F\text{div}(L/k) \cong \text{Vect}(L) \), so \( \mathcal{E} \) becomes trivial on \( \mathcal{X} \times_{\Gamma} \text{Spec}(L) \). But the map \( \mathcal{X} \times_{\Gamma} \text{Spec}(L) \to \mathcal{X} \), as a pullback of \( \text{Spec}(L) \to \Gamma \), is a finite étale cover.

The case of \( F\text{div}_\infty \) is analogous, just replace \( \Pi_X^{\text{N, ét}} \) with \( \Pi_X^N \).

We now concentrate on the converse problem in Theorem 1.

Proof of Theorem 1 second part. By Theorem 1.13 it is enough to consider the case of \( F\text{div} \). The category \( F\text{div}(\mathcal{X}/k) \) is a sub-Tannakian category of \( F\text{div}(\mathcal{X}/k) \) because in a Tannakian category any subobject or quotient of a trivial object is trivial, \( F\text{div}(\mathcal{U}/k) = \prod_{i \in I} F\text{div}(\mathcal{U}_i/k) \), where each \( \mathcal{U}_i \) is a connected component of \( \mathcal{U} \) and, for all \( i \), \( F\text{div}(\mathcal{U}_i/k) \) is Tannakian over \( H^0(\mathcal{O}_{\mathcal{U}_i})_{\text{ét}} \) by Theorem 1.13.

For all finite extensions of fields \( l/k \) we have the following 2-commutative diagram.

\[
\begin{array}{ccc}
\Pi_{F\text{div}(\mathcal{X} \times_{\mathcal{X}/l})} & \xrightarrow{c} & \Pi_{F\text{div}(\mathcal{X})} \times_k l \\
\downarrow{d} & & \downarrow{b} \\
\Pi_{F\text{div}(\mathcal{X} \times_{\mathcal{X}/l})_{\text{ét}}} & \xrightarrow{a} & \Pi_{F\text{div}(\mathcal{X})_{\text{ét}}} \times_k l
\end{array}
\]

By Proposition 3.3 \( c \) is an isomorphism. So to prove that \( \Pi_{F\text{div}(\mathcal{X}/k)} \) is finite and étale, we may replace \( k \) by a finite field extension. In particular we can assume that all connected components of \( \mathcal{U} \cap \mathcal{X} \), \( \mathcal{U} \cap \mathcal{X} \cap \mathcal{X} \cap \mathcal{X} \mathcal{U} \) are geometrically connected or, using Theorem 1.13 that the rings \( H^0(\mathcal{O}_{\mathcal{U}})_{\text{ét}}, H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}}, \) and \( H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}} \) are products of copies of \( k \). In all the cases considered, we can also assume that \( \mathcal{U} \) is geometrically connected.

Since \( F\text{div} \) is an fpqc stack, \( F\text{div}(\mathcal{X}/k) \) is equivalent to the category of trivial objects \( \mathcal{O}_{\mathcal{U}}^{\text{ét}} \in F\text{div}(\mathcal{U}/k) \) equipped with descent data, i.e., an isomorphism between the two pullbacks of the trivial object along \( \mathcal{U} \times \mathcal{X} \mathcal{U} \to \mathcal{U} \mathcal{U} \mathcal{U} \) satisfying the cocycle condition.

Denote by \( I, J \) the set of connected components of \( \mathcal{U} \times \mathcal{X} \mathcal{U} \) and \( \mathcal{U} \mathcal{U} \mathcal{U} \times \mathcal{X} \mathcal{U} \), respectively. Restricting to each connected components and using Theorem 1.13 we see that the pull-back functors

\[
\text{Vect}(\text{Spec}(H^0(\mathcal{O}_{\mathcal{U}})_{\text{ét}})) \simeq \text{Fdim}(\text{Spec}(H^0(\mathcal{O}_{\mathcal{U}})_{\text{ét}})/k) \to \text{Fdim}(\mathcal{U}/k),
\]

\[
\text{Vect}(\text{Spec}(H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}})) \simeq \text{Fdim}(\text{Spec}(H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}})/k) \to \text{Fdim}(\mathcal{U} \times \mathcal{X} \mathcal{U}/k),
\]

\[
\text{Vect}(\text{Spec}(H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}})) \simeq \text{Fdim}(\text{Spec}(H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}})/k) \to \text{Fdim}(\mathcal{U} \times \mathcal{X} \mathcal{U} \times \mathcal{X} \mathcal{U}/k)
\]

are fully faithful. Thus \( F\text{dim}(\mathcal{X}/k) \) is equivalent to the category of vector bundles \( V \) on \( \text{Spec}(H^0(\mathcal{O}_{\mathcal{U}})_{\text{ét}}) = \text{Spec} k \) with an isomorphism between the two pullbacks of \( V \) to \( \text{Spec}(H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}}) = \text{Spec} k^I \) satisfying the cocycle condition in \( \text{Spec}(H^0(\mathcal{O}_{\mathcal{U} \times \mathcal{X} \mathcal{U} \times \mathcal{X} \mathcal{U}})_{\text{ét}}) = \text{Spec} k^J \).

So \( V \) is a \( k \)-vector space and the data of the isomorphism between the pullbacks is a collection of automorphisms \( \sigma_Z : V \to V \) for all \( Z \in I \). Denote by
pr_{ij}: \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U} \times \mathcal{U} and pr_i: \mathcal{U} \times \mathcal{U} \to \mathcal{U} the projections. Given \( D \in J \), we denote by \( Z_{ij,D} \in I \) the unique connected component containing \( \text{pr}_{ij}(D) \).

It is easy to see that the cocycle conditions on the collection \((\sigma_Z)_{Z \in I}\) translate into the relation

\[
\sigma_{Z_{13,D}} = \sigma_{Z_{23,D}} \circ \sigma_{Z_{12,D}} \quad \text{for all } D \in J.
\]

If \( G \) is the quotient of the free nonabelian group over the symbols \((e_Z)_{Z \in I}\) by the relations \( e_{Z_{13,D}} e_{Z_{23,D}} e_{Z_{12,D}} \) for \( D \in J \), it follows that \( \text{Fdiv}(\mathcal{X}/k) \simeq \text{Rep}_G \). If \( f \) is geometrically connected, then \( |I| = |J| = 1 \), and therefore \( G \) is trivial as required.

So assume that either \( f \) is proper or \( \mathcal{U} \) is geometrically irreducible. We are going to show that \( |G| \leq |I| \).

Consider the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{U} \times \mathcal{U} \times \mathcal{U} & \xrightarrow{\text{pr}_{23}} & \mathcal{U} \times \mathcal{U} \\
\downarrow\text{pr}_{12} & & \downarrow\text{pr}_1 \\
\mathcal{U} \times \mathcal{U} & \xrightarrow{\text{pr}_2} & \mathcal{U}.
\end{array}
\]

Given \( Z, W \in I \), we have \( \text{pr}_1(Z) \cap \text{pr}_1(W) \neq \emptyset \): if \( f \) is proper, then \( \text{pr}_1(Z) \) and \( \text{pr}_1(W) \) are open and closed and thus equal to \( \mathcal{U} \); if \( \mathcal{U} \) is irreducible, it is the intersection of two nonempty open subsets. In particular there exists \( D \in J \) such that \( Z_{12,D} = W, Z_{23,D} = Z \), and therefore \( e_Z e_W = e_{Z_{13,D}} \) in \( G \). Thus it remains to show that for all \( Z \in I \) there exists \( W \) such that \( e_Z^2 = e_W \) in \( G \).

Notice that the diagonal in \( J \) determines the relation \( e_\Delta = e_{\Delta}^2 \), where \( \Delta \in I \) is the diagonal. Thus \( e_\Delta = 1 \) in \( G \). Now consider the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{U} \times \mathcal{U} \times \mathcal{U} & \xrightarrow{\text{pr}_{13}} & \mathcal{U} \times \mathcal{U} \\
\downarrow\text{pr}_{12} & & \downarrow\text{pr}_1 \\
\mathcal{U} \times \mathcal{U} & \xrightarrow{\text{pr}_1} & \mathcal{U}.
\end{array}
\]

Since \( \text{pr}_1(Z) \subseteq \text{pr}_1(\Delta) = \mathcal{U} \), there exists \( D \in J \) such that \( Z_{12,D} = Z \) and \( Z_{13,D} = \Delta \). Thus \( e_{Z_{23,D}} e_Z = e_\Delta = 1 \), as required. \( \square \)

**Remark 2.2.** In the above proof the crucial property of \( \text{Fdiv} \) we used is that it is a stack in the fppf topology. Whether this property is true also for \( \text{Str} \) and \( \text{Crys} \) is unclear.

**Lemma 2.3.** Let \( \mathcal{X} \) be an inflexible and pseudo-proper fiber category over \( k \), let \( V \in \text{Vect}(\mathcal{X}) \), and denote by \( F: \mathcal{X} \to \mathcal{X} \) the absolute Frobenius. If there exists \( m \in \mathbb{N} \) such that \( F^m \ast V \in \text{Vect}(\mathcal{X}) \) is essentially finite, then \( V \) is essentially finite too.

**Proof.** We can consider the case \( m = 1 \) only. Set \( n = \text{rk} V \). The vector bundle \( V \) is given by an \( \mathbb{F}_p \)-map \( v: \mathcal{X} \to \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n,\mathbb{F}_p}) \): the stack \( \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n,\mathbb{F}_p}) \) has a universal vector bundle \( E \) of rank \( n \) such that \( v^* E \simeq V \). By Theorem 1.11 \( V \) is essentially finite if and only if \( v \) factors as \( \mathcal{X} \xrightarrow{\phi} \Gamma \to \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n,\mathbb{F}_p}) \), where \( \Gamma \) is a finite \( k \)-gerbe and \( \phi \) is \( k \)-linear. The vector bundle \( F^* V \) corresponds to the composition \( \mathcal{X} \xrightarrow{\phi} \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n,\mathbb{F}_p}) \xrightarrow{F} \mathcal{B}_{\mathbb{F}_p}(\text{GL}_{n,\mathbb{F}_p}) \), where \( F \) is the absolute Frobenius.
of $B_{G_p}(GL_{n,F_p})$. Thus we have a diagram

$$
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\psi} & B_{G_p}(GL_{n,F_p}) \\
\phi \downarrow & & \downarrow F \\
\Gamma & \rightarrow & B_{G_p}(GL_{n,F_p}),
\end{array}
$$

where $\Gamma$ is a finite $k$-gerbe and the square is 2-Cartesian. We conclude by showing that $\Delta$ is a finite gerbe over $k$.

The map $F: B_{G_p}(GL_{n,F_p}) \rightarrow B_{G_p}(GL_{n,F_p})$ is induced by the Frobenius of $GL_{n,F_p}$. Since this last map is a surjective group homomorphism with finite kernel, it follows that $F$ and therefore $\Delta \rightarrow \Gamma$ is a finite relative gerbe. This plus the assumption that $\Gamma$ is a finite gerbe implies that $\Delta$ is a finite gerbe. \hfill $\Box$

**Proof of Corollary 1.** If $V$ is an essentially finite vector bundle on $\mathcal{X}$, then by Theorem 1.11 there exists $\psi: \mathcal{X} \rightarrow \Gamma$, where $\Gamma$ is a finite gerbe, and $W \in \text{Vect}(\Gamma)$ such that $\psi^*W \simeq V$. If Spec $L \rightarrow \Gamma$ is any map from a finite field extension of $k$, then $f: U = \mathcal{X} \times_L \text{Spec} L \rightarrow \mathcal{X}$ is a finite and flat map and the pullback along this map of $V$ is free. Since $U \times_\mathcal{X} U \rightarrow \mathcal{X}$ is finite and flat, the pushforward of the structure sheaf is locally free of finite rank on $\mathcal{X}$, and therefore the set of its global sections forms a finite-dimensional $k$-vector space.

Consider now a proper, flat, and surjective map $f: U \rightarrow \mathcal{X}$ as in the statement, and consider $V \in \text{Vect}(\mathcal{X})$ such that $f^*V$ is free. We are first going to extend $V$ to some object in $\text{Fdiv}_i(\mathcal{X}/k)$ trivialized by $f$ for some $i > 0$. If $Z$ is a stack of finite type over $k$ and we apply Theorem 1.13 on the connected components, we see that $\text{Fdiv}(Z/k) \rightarrow \text{Vect}(Z)$ is faithful, and the set of endomorphisms of the unit object of $\text{Fdiv}(Z/k)$ is identified with $\text{H}^1(\mathcal{O}_Z)_{\text{et}}$. In particular for all $i$ we have

$$
\text{End}_{\text{Fdiv}_i(Z/k)}(1) = \{ x \in \text{H}^0(\mathcal{O}_Z) \mid x^i \in \text{H}^0(\mathcal{O}_Z)_{\text{et}} \} \subseteq \text{H}^0(\mathcal{O}_Z)
$$

via the functor $\text{Fdiv}_i(Z/k) \rightarrow \text{Vect}(Z)$. In particular if $\text{H}^0(\mathcal{O}_Z)$ is a finite $k$-algebra, the above inclusion is an equality for $i$ big enough, which also means that the functor $\text{Fdiv}_i(Z/k) \rightarrow \text{Vect}(Z)$ restricted to the full subcategory of free objects is fully faithful.

By fppf descent along $f: U \rightarrow \mathcal{X}$, the vector bundle $V$ is given by a free object of rank $rk V = r$ on $U$ with an isomorphism of the two pullbacks in $U \times_\mathcal{X} U$ satisfying the cocycle condition of the triple product. By the discussion above for $i$ big enough this also determines a descent data on the free object of rank $r$ in $\text{Fdiv}_i(U/k)$. Since $\text{Fdiv}_i$ and $\text{Vect}$ are stacks in the fppf topology, there exists $E \in \text{Fdiv}_i(\mathcal{X}/k)$ of the form $(V, W, \lambda)$ with $W \in \text{Fdiv}(X/k)$ such that $f^*E \in \text{Fdiv}_i(U/k)$ is trivial. By the definition of $\text{Fdiv}_i(\mathcal{X}/k)$, $W_0 \simeq F^{i*}V$, where $F$ is the absolute Frobenius of $\mathcal{X}$ and $F^*W$ is trivial in $\text{Fdiv}(U/k)$. By Lemma 2.3 and Theorem 1 we can conclude that $V$ is essentially finite. \hfill $\Box$

**Lemma 2.4.** Let $\mathcal{X}$ be an algebraic stack over a field $k$ of positive characteristic such that $\text{H}^1(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a finite-dimensional $k$-vector space. Then any $\mathbb{G}_a^r$-torsor over $\mathcal{X}$ comes from a torsor under a finite subgroup scheme of $\mathbb{G}_a$.

**Proof.** We can assume $r = 1$. We interpret $\text{H}^1(\mathcal{X}, \mathcal{O}_\mathcal{X})$ as the set of isomorphism classes of $\mathbb{G}_a$-torsors over $\mathcal{X}$. If $\phi: \mathbb{G}_a \rightarrow \mathbb{G}_a$ is a group homomorphism, then it
induces a map of sets $\Lambda(\phi): H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$. Thus we obtain a map

$$\Lambda: \text{End}_{\text{groups}}(\mathcal{G}_a) \to \text{End}_{(\text{Sets})}(H^1(\mathcal{O}_X)).$$

Both sides are left $k$-algebras and, by the definition of the $k$-vector space structure on $H^1(\mathcal{O}_X)$, $\Lambda$ is a morphism of $k$-algebras.

Consider the relative Frobenius $F: \mathcal{G}_a \to \mathcal{G}_a$, that is $F(x) = x^p$ functionally and a nonzero polynomial $R(Y) \in k[Y]$. The map $R(F): \mathcal{G}_a \to \mathcal{G}_a$ is a nonzero group homomorphism. Since $\mathcal{G}_a$ is connected and reduced, $R(F)$ is surjective and its kernel is finite. If $R(Y) = \sum_j \lambda_j Y^j$ and $v \in H^1(\mathcal{O}_X)$, then

$$\Lambda(R(F))v = \sum_j \lambda_j \Lambda(F)^j(v).$$

Since $H^1(\mathcal{O}_X)$ is a finite-dimensional $k$-vector space, the vectors $v$, $\Lambda(F)(v)$, $\Lambda(F)^2(v)$, ... must be eventually linearly dependent. Thus there exists a nonzero polynomial $R[Y] \in k[Y]$ such that $\Lambda(R(F))v = 0$. This means that the $\mathcal{G}_a$-torsor $v$ becomes trivial under the map $B_k(\mathcal{G}_a) \to B_k(\mathcal{G}_a)$ induced by $R(F)$. Since $R(F)$ is surjective, it follows that $v$ comes from a torsor under the finite group scheme $\text{Ker}(R(F))$.

**Lemma 2.5.** Let $\mathcal{X}$ be an algebraic stack over a field $k$ of positive characteristic such that $\dim_k H^1(\mathcal{X}, E) < \infty$ for all vector bundles $E$ on $\mathcal{X}$. Let

$$\mathcal{G}_0 \to \mathcal{G}_1 \to \cdots \to \mathcal{G}_{N-1} \to \mathcal{G}_N = 0$$

be a sequence of surjective maps of quasi-coherent sheaves on $\mathcal{X}$ such that $\text{Ker}(\mathcal{G}_{l-1} \to \mathcal{G}_l)$ is free of finite rank for $1 \leq l \leq N$. Then there exists a finite flat surjective map $f: \mathcal{X}' \to \mathcal{X}$ such that $f^*\mathcal{G}_l$ is free of finite rank for all $l$.

**Proof.** Using induction, it is enough to show that if $0 \to O^m_{\mathcal{X}} \to \mathcal{G} \to O^n_{\mathcal{X}} \to 0$ is an exact sequence, there exists a finite flat surjective map $f: \mathcal{X}' \to \mathcal{X}$ such that $f^*\mathcal{G}$ is free. The sheaf $\mathcal{G}$ is given by an element $x \in \text{Ext}^1(O^m_{\mathcal{X}}, O^n_{\mathcal{X}})$. Moreover, there exists an isomorphism $\text{Ext}^1(O^m_{\mathcal{X}}, O^n_{\mathcal{X}}) \cong H^1(\mathcal{O}_X)^{mn}$ functorial in $\mathcal{X}$, so that $x$ corresponds to a sequence $x_i \in H^1(\mathcal{O}_X)$. If $x_i$ corresponds to the $\mathcal{G}_a$-torsor $h_i: \mathcal{P}_i \to \mathcal{X}$, then by construction $h_i^*x_i = 0$. By Lemma 2.4 there exists a finite flat surjective map $f_i: \mathcal{X}_i \to \mathcal{X}$ factoring through $h_i: \mathcal{P}_i \to \mathcal{X}$. In particular $f_i^*x_i = 0$. Thus $f = \prod_i f_i: \prod_i \mathcal{X}_i \to \mathcal{X}$ is a finite flat surjective map such that $f^*x = 0$, which means that $f^*\mathcal{G} \cong f^*O^m_{\mathcal{X}} \oplus f^*O^n_{\mathcal{X}}$.

**Proof of Corollary 1** By Theorem 1.19 we must prove that if $E$ is a virtually unipotent vector bundle on $\mathcal{X}$, then it is essentially finite. Thanks to Lemma 2.4 we can assume that $E$ is an extended essentially finite sheaf. We are going to show that there exists a finite and flat map $f: \mathcal{X}' \to \mathcal{X}$ such that $f^*E$ is free. Since $\mathcal{X}$ is inflexible, the conclusion will follow from Corollary 1.

Since $\mathcal{X}$ is inflexible, there exists a map $\mathcal{X} \to \Gamma$ to a finite gerbe such that all essentially finite quotients $\mathcal{E}_i/\mathcal{E}_{i+1}$ of a filtration of $\mathcal{E}$, as in Definition 1.18, comes from $\Gamma$. If $L/k$ is a finite extension with $\Gamma(L) \neq \emptyset$, the pullback of Spec $L \to \Gamma$ along $\mathcal{X} \to \Gamma$ gives a finite flat surjective map $g: \mathcal{Y} \to \mathcal{X}$ trivializing all essentially finite quotients $\mathcal{E}_i/\mathcal{E}_{i+1}$. Applying Lemma 2.5 on the sheaf $(g^*E)^\vee$ over $\mathcal{Y}$, we find a finite flat surjective map $h: \mathcal{X}' \to \mathcal{Y}$ such that $h^*g^*E^\vee$ is free. The composition $f: \mathcal{X}' \xrightarrow{h} \mathcal{Y} \xrightarrow{g} \mathcal{X}$ gives the desired finite flat surjective map.
3. Trivializations by dominant maps

We fix a field \( k \) of characteristic \( p > 0 \). For the definition and properties of geometrically unibranch stacks, we refer to Appendix A. In this section we are going to prove Theorem \( \ref{thm:3.1} \) and deduce Corollary \( \ref{cor:3.2} \) from it. The crucial point is the following result.

**Theorem 3.1.** Let \( \mathcal{X} \) be a geometrically unibranch algebraic stack locally of finite type over \( k \), and let \( U \subseteq \mathcal{X} \) be a dense open subset. Then the functor \( \text{Fdiv}(\mathcal{X}/k) \rightarrow \text{Fdiv}(U/k) \) is fully faithful and stable under subobjects; that is, if \( F \in \text{Fdiv}(\mathcal{X}/k) \) and \( E_U \subseteq F_U \) in \( \text{Fdiv}(U/k) \), then there exists \( E \subseteq F \) in \( \text{Fdiv}(\mathcal{X}/k) \) inducing the given inclusion.

In particular, if \( \mathcal{X} \) is geometrically irreducible and quasi-compact, then \( \Pi_{\text{Fdiv}(U/k)} \rightarrow \Pi_{\text{Fdiv}(\mathcal{X}/k)} \) is a quotient of \( k \)-gerbes.

We first show how to obtain Theorem \( \ref{thm:3.1} \) from the above result.

**Proof of Theorem \( \ref{thm:3.1} \) using Theorem \( \ref{thm:3.1} \).** We have to prove the “if” part and, thanks to Theorem \( \ref{thm:1.1} \), we just have to consider the case of Fdiv. Let \( f: \mathcal{Y} \rightarrow \mathcal{X} \) be a dominant map of finite type, and let \( E \in \text{Fdiv}(\mathcal{X}/k) \), which is trivialized by \( f \). We want to show that \( E \) is essentially finite. Replacing \( \mathcal{Y} \) by an atlas, we can assume that \( \mathcal{Y} = Y \) is a scheme.

First we assume that \( f \) is also flat. In this case we take an irreducible open subset \( Y_0 \subseteq X \times_k Y \) and let \( Y_0 \subseteq X \times_k k' \) be a model of \( Y_0 \) over a finite field extension \( k'/k \). By Lemma \( \ref{lem:A.3} \) and Propositions \( \ref{prop:B.3} \) and \( \ref{prop:B.5} \) we can assume \( k' = k \). Consider the image \( U := f(Y_0) \subseteq \mathcal{X} \). From Theorem \( \ref{thm:1.1} \) we see that \( E|_U \) is an essentially finite object over \( U/k \), and from Theorem \( \ref{thm:3.1} \) we can conclude that also \( E \) is essentially finite in \( \text{Fdiv}(\mathcal{X}/k) \), as required.

Now we come back to the general case. Let \( X \rightarrow \mathcal{X} \) be a smooth atlas with \( X \) quasi-compact. Using Lemma \( \ref{lem:A.3} \) and Propositions \( \ref{prop:B.3} \) and \( \ref{prop:B.5} \) we may extend \( k \) a little bit and assume that \( X \) has a \( k \)-rational point \( x \). Using the first part of Theorem \( \ref{thm:1.1} \) and the flat case, we may replace \( \mathcal{X} \) by the connected component of \( x \) in \( X \) and assume that \( \mathcal{X} = X \) is a scheme. Using Remark \( \ref{rem:1.3} \) we may assume that \( X \) is a reduced scheme. Since \( X \) is also geometrically unibranch and connected, it is integral. By \( \text{Mat} \) Theorem 24.3) the locus \( V \subseteq U \) of points flat over \( X \) is open and, since \( f \) is dominant, it is nonempty. Replacing \( Y \) by \( V \), we are in the case when \( f \) is flat. \( \square \)

**Example 3.2.** In Theorem \( \ref{thm:3.1} \) the hypothesis that \( \mathcal{X} \) is geometrically unibranch is necessary. Assume that \( k \) is an algebraically closed field of odd characteristic, and consider \( \mathcal{X} = \text{Spec}(A) \), where \( A = k[t]/(x^2 - y^2) \), which is an affine integral scheme of finite type over \( k \), and its open subset \( U = \text{Spec}(A_y) \). We claim that \( \Pi_{\text{Fdiv}(U/k)} \rightarrow \Pi_{\text{Fdiv}(\mathcal{X}/k)} \) is not a quotient. Since the profinite quotient of Fdiv is \( \Pi^{N,\text{et}} \) and, choosing a rational point \( u \in U \), this corresponds to the Grothendieck étale fundamental group \( \pi^{\text{et}} \), it is enough to show that \( \pi^{\text{et}}(U, u) \rightarrow \pi^{\text{et}}(\mathcal{X}, u) \) is not surjective. The scheme \( P = \text{Spec}(A[T]/(T^2 - t)) \) of \( \mu_2 \) torsor over \( X \), and it is nontrivial since \( t \) is not a square in \( A \). On the other hand \( P \) becomes trivial when restricted to \( U \), which implies that there is a quotient \( \pi^{\text{et}}(\mathcal{X}, u) \rightarrow \mu_2 \) such that the composition \( \pi^{\text{et}}(U, u) \rightarrow \pi^{\text{et}}(\mathcal{X}, u) \rightarrow \mu_2 \) is trivial.

Before proving Theorem \( \ref{thm:3.1} \) we collect some preparatory results.
Proposition 3.3. Assume that \( k \) is a field whose absolute Frobenius is finite, and let \( R \) be a complete local \( k \)-algebra whose residue field is finite over \( k \). Then

\[
\text{Fdiv}(R/k) \simeq \text{Vect}(
\) \( R_{\text{ét}} \) \( ).
\]

Proof. By [TZ, Theorem 6.23] it follows that Fdiv\((R/k)\) is an \( R_{\text{ét}} \)-Tannakian category. Thus it is enough to show that any object \( E \in \text{Fdiv}(R/k) \) is trivial. Let \( m \) be the maximal ideal of \( R \), let \( L \) be its residue field, and let \( F \in \text{Fdiv}(R/k) \) be the free object of rank \( \text{rk} E \). Using that Fdiv is insensitive to nilpotent thickenings (Remark 1.3) and that Fdiv\((L/k)\) \( \simeq \text{Vect}(\mathbb{A}^1) \) (Proposition 2.4), there is a compatible system of isomorphisms \( \sigma_n : E \otimes_R R/m^n \rightarrow F \otimes_R R/m^n \) in Fdiv\(( (R/m^n)/k) \). Since the absolute Frobenius of \( k \) is finite, all rings \( R^{(i)} \) are complete with respect to \( m^{(i)} \). As \( E \) is free \( R^{(i)} \)-modules, they are complete with respect to \( m^{(i)} \). Thus these \( \sigma_n \) extend to an isomorphism \( \sigma : E \rightarrow F \), as required. \( \square \)

Lemma 3.4. Let \( A \) be a ring, let \( A \rightarrow A' \) be a faithfully flat map, and let \( A \rightarrow B \) be an injective map. Then \( A = A' \cap B \) inside \( A' \otimes_A B \).

Proof. We provide two proofs:

1. Consider the map \( A \rightarrow A' \cap B \subseteq B \). It is enough to show that \( \phi' := \phi \otimes_A A' \) is an isomorphism. For this it is enough to show that for any element \( x \in A' \cap B \), \( x \otimes 1 \in B \otimes_A A' \) is in the image of \( A' \rightarrow B \otimes_A A' \). But by the definition of \( A' \cap B \), there exists \( a' \in A' \) such that \( 1 \otimes a' = x \otimes 1 \in B \otimes_A A' \). This means that \( \phi' \) is an isomorphism, as claimed.

2. Consider the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f_1} & A' \\
\downarrow{f_2} & & \downarrow{f_2} \\
B & \xrightarrow{g_1} & B \otimes_A A' \\
\downarrow{g_2} & & \downarrow{g_2} \\
B & \xrightarrow{g_1} & B \otimes_A A' \\
\end{array}
\]

with canonical maps. From the diagram it is clear that \( A' \cap B \), as a subset of \( A' \), is contained in the subset of equalizers of \( f_1 \) and \( f_2 \), i.e., \( A' \cap B \subseteq A \). Thus \( A' \cap B = A \). \( \square \)

Lemma 3.5. Let \( A \) be a Noetherian complete local domain over a perfect field \( k \) with fraction field \( K \), and assume that its residue field is finitely generated over \( k \). Then Fdiv\((K/k)\) is a \( K_{\text{ét}} \)-Tannakian category and \( K_{\text{ét}}/k \) is finite.

Proof. By Theorem 1.13 and Remark 1.14 we can conclude that Fdiv\((A/k)\) is an \( A_{\text{ét}} \)-Tannakian category and Fdiv\((K/k)\) is an \( F \)-Tannakian category, where \( F = \text{End}_{\text{Fdiv}(K/k)}(1) \). Let \( B \) be the integral closure of \( A \) in \( K \). Since \( A \) is Nagata [SP, Lemma 10.156.2 and 10.156.8], \( B \) is finite over \( A \), and since \( A \) is Henselian, \( B \) is a local domain. Thus we can assume that \( A \) is normal, and we are going to show that \( F \subseteq A_{\text{ét}} \).

Pick \( x \in F \), that is \( x = (x_n)_{n \in \mathbb{N}} \) with \( x_n \in K \) and \( x_{n+1} = x_n \) (here we use that \( K^{(i)} \simeq K \) because \( k \) is perfect). Given a discrete valuation \( v : K \rightarrow \mathbb{Z} \), one can easily conclude that \( v(x_n) = 0 \) for all \( n \in \mathbb{N} \), which means that \( x_n \in A \) for all \( n \in \mathbb{N} \). In particular \( x \in \text{End}_{\text{Fdiv}(A/k)}(1) = A_{\text{ét}} \), as desired.
We have $K_{\text{et}} \subseteq F \subseteq A_{\text{et}} \subseteq K_{\text{et}}$ and, therefore, $F = K_{\text{et}}$. Finally, if $L$ is the residue field of $A$, we have $K_{\text{et}} = A_{\text{et}} \subseteq L_{\text{et}}$ and $L_{\text{et}}$ is finite over $k$ by [Lang, Chapter VIII, Prop 3.3, pp. 363].

\[\text{Lemma 3.6. Let } X \text{ be a stack of finite type over } k. \text{ Then there exists a finite and purely inseparable extension } L/k \text{ such that } (X \times_k L)_{\text{red}} \text{ is geometrically reduced.}\]

\[\text{Proof. Taking a smooth covering by an affine scheme, one can assume } X = \text{Spec } A. \text{ Consider the exact sequence}\]

\[0 \rightarrow I \rightarrow A \otimes_k k^{1/p} \rightarrow (A \otimes_k k^{1/p})_{\text{red}} \rightarrow 0,
\]

where $k^{1/p}$ is the perfect closure of $k$ and $I \subseteq A \otimes_k k^{1/p}$ is the ideal of nilpotents. Since $I$ is finitely generated, it is defined over a finite intermediate extension $k \subseteq L \subseteq k^{1/p}$. Extending $A$ to $L$, we may assume that $L = k$, i.e., $\exists I_k \subseteq A$ such that $I_k \otimes_k k^{1/p} = I$. Since $I_k \subseteq I$ is nilpotent, $A_{\text{red}} = A/I_k$ is geometrically reduced. \[\square\]

\[\text{Proof of Theorem 3.1. We start by showing the full faithfulness. The restriction functor } F \text{div}(X/k) \rightarrow F \text{div}(U/k) \text{ is faithful even when } X \text{ is not geometrically unibranch. To see this, we may assume that } X \text{ is connected. Let } f : E \rightarrow F \in F \text{div}(X/k) \text{ whose restriction to } U \text{ is 0. Then the image } H \text{ of } f \text{ is an object in } F \text{div}(X/k) \text{ whose restriction to } U \text{ is 0. As the rank of } H \text{ is constant and } U \text{ is nonempty, } H = 0, \text{ i.e., } f|_U = 0.
\]

Since $F \text{div}$ is a stack in the fpqc topology, using the faithfulness, one can easily reduce to the case $X = X = \text{Spec } A$ is affine. Using Lemma 3.5 and that $F \text{div}(X/k) \approx F \text{div}(X^{(i)}/k) \approx F \text{div}((X^{(i)})_{\text{red}}/k)$, we can further assume that $X$ and $U = U$ are also geometrically integral by extending $X$ to a finite extension of $k$, restricting to one connected component, and then taking reduction. We can also assume that $U = \text{Spec } A$, with $a \in A$.

Let $E, F \in F \text{div}(X/k)$, and consider $\phi_U \in \text{Hom}_{F \text{div}(U)}(E|_U, F|_U)$. Since all $X^{(i)}$ are integral, it is enough to extend each map $E_i|_U \rightarrow F_i|_U$ to $E_i \rightarrow F_i$ and, up to replacing $X$ by $X^{(i)}$, we just have to show this extension for $i = 0$. Shrinking $X$, we can assume that $E_0 = A^{\oplus n}$ and $F_0 = A^{\oplus m}$, so that $\phi_U$ is a matrix with coefficients in $A$, and we must prove those coefficients belong to $A$. Using the functor $F \text{div}(-/k) \rightarrow F \text{div}(\bar{k})$ and Lemma 3.4, we can assume that $k$ is algebraically closed. Denote by $K$ the fraction field of $A$ and by $\phi_K : E \otimes K \rightarrow F \otimes K$ the restriction of $\phi_U$. Consider $p \in X \setminus U$ a closed point and the diagram

\[
\begin{array}{c}
A_p^{\oplus n} \\
\downarrow \\
K^{\oplus n} \xrightarrow{(\phi_K)_u} K^{\oplus m}.
\end{array}
\]

We must show the existence of the dashed arrow or, in other words, that the coefficients of the matrix $\phi_K$ are in $A_p$. Using again Lemma 3.4, it is enough to show that $F \text{div}(A_p/k) \rightarrow F \text{div}(\bar{k}/k)$ is fully faithful, where $A_p$ and $\bar{k}$ are the completion of $A_p$ (which is a domain because $A$ is geometrically unibranch (Remark 3.3)), and its fraction field, respectively. This follows because $F \text{div}(A_p/k) = \text{Vect}(k)$ by Proposition 3.3 and $F \text{div}(\bar{k}/k)$ is a $k$-Tannakian category by Lemma 3.5.

We now show that the full subcategory $F \text{div}(X/k) \subseteq F \text{div}(U/k)$ is stable under taking subobjects. Let $F \in F \text{div}(X/k)$, and let $E|_U \subseteq F|_U$. We will show that there
exists $E \subseteq F$ such that $E|_{U} = E_{U}$. By the full faithfulness we just proved, if the extension $E$ exists it is unique. Since Fdiv is a stack in the fppf topology we can assume $X = X = \text{Spec} \ A$ and also $U = U = \text{Spec} \ A_{a}$ for some $a \in A$. As above we can further assume that $X$ and $U$ are geometrically integral. Denote by $j: U \rightarrow X$ the open embedding. We have an infinite sequence of Cartesian diagrams

$$
\begin{array}{ccc}
U & \rightarrow & U^{(1)} \\
\downarrow{j} & & \downarrow{j^{(1)}} \\
X & \rightarrow & X^{(1)}
\end{array}
\rightarrow \cdots
$$

In particular, since the vertical maps are affine, given $\mathcal{H} \in \text{Fdiv}(U/k)$, there are isomorphisms $\sigma_{n}: (j_{*}^{(n+1)}H_{n+1}|_{X^{(n)}} \rightarrow j_{*}^{(n)}((H_{n+1}|_{U^{(n)}}) \simeq j_{*}^{(n)}H_{n}$, so we can define $j_{*}\mathcal{H}$ as the quasi-coherent F-divided sheaf $(j_{*}^{(n)}H_{n}, \sigma_{n})$. We have $F \subseteq j_{*}j^{*}F$ and $j_{*}(E_{U}) \subseteq j_{*}j^{*}F$, and we define $E_{n} := j_{*}^{(n)}(E_{U})_{n} \bigcap F_{n}$. By construction $j_{*}^{(n)}E_{n} = (E_{U})_{n}$, and there are canonical maps $\tau_{n} : (E_{n+1})_{X^{(n)}} \rightarrow E_{n}$ which are compatible with the transition isomorphisms of $F$. We are going to prove that those $\tau_{n}$ are isomorphisms. Note that this implies that $(E_{n}, \tau_{n}) \in \text{Fdiv}(X/k)$ because $E_{n}$ are automatically locally free by [1Z Theorem I(3), p. 2]. This would conclude the proof, as $(E, \tau_{n})$ gives the desired extension. In order to show that those $\tau_{n}$ are isomorphisms, we can first assume that $k$ is algebraically closed, and by completing at closed points, we may further assume that $A$ is a complete local Noetherian domain with residue field $k$. In particular $\text{Fdiv}(X/k) = \text{Vect}(k)$ by Proposition [5.3] and $\text{Fdiv}(U/k)$ is a $k$-Tannakian category using Theorem [1.13].

Since a subobject of a trivial object in $\text{Fdiv}(U/k)$ is trivial, $E_{U} \subseteq F|_{U}$ is trivial. Thus we can write $F = \bar{E} \oplus G$, where $\bar{E}, G$ are trivial objects in $\text{Fdiv}(X/k)$, and $\bar{E}|_{U} = E_{U}$. Now it is clear that $\bar{E}_{n} = j_{*}^{(n)}(E_{U})_{n} \bigcap F_{n} = E_{n}$ and the transition maps must coincide. Thus $\tau_{n}$ are all isomorphisms. 

\textbf{Proof of Corollary III.} See [BV2 Prop. 3.1] for the notion of gerbes of finite type over $k$. Consider a map $\text{Spec} \ L \rightarrow \Gamma$, where $L/k$ is a finite extension, which exists because $\Gamma$ is of finite type. Since the map $\text{Spec} \ L \rightarrow \Gamma \times_{k} L$ is faithfully flat and affine, and $L$ is smooth over $L$, by [EGAIV Prop 17.7.7] $\Gamma \times_{k} L$ is smooth over $L$. Thus we can conclude that $\Gamma$ is a smooth stack over $k$ and, in particular, geometrically unibranch and geometrically integral. Since by Proposition [2.3] $\text{Fdiv}(L/k) = \text{Vect}(L_{\text{et}})$, applying Theorem II to the flat map $\text{Spec} \ L \rightarrow \Gamma$, we can conclude that all objects of $\text{Fdiv}(\Gamma/k)$ and $\text{Fdiv}_{\infty}(\Gamma/k)$ are essentially finite. The remaining claims follow from Theorem [1.13] taking into account that $\Gamma$ is inflexible over $k$, $\Pi_{\Gamma/k}^{N, \text{et}} = \Gamma_{\text{et}}$, and $\Pi_{\Gamma/k}^{N} = \Gamma$ (see [1Z Definition B11]). 

\textbf{APPENDIX A. GEOMETRICALLY UNIBRANCH ALGEBRAIC STACKS}

\textbf{Definition A.1.} A local ring $R$ is called geometrically unibranch if, denoting by $R_{\text{red}}$ its reduction, $R_{\text{red}}$ is a domain, and its integral closure in its fraction field is a local ring whose residue field is a purely inseparable extension of that of $R$.

A scheme $X$ is called geometrically unibranch at a point $x \in X$ if the ring $\mathcal{O}_{X,x}$ is geometrically unibranch. A scheme is called geometrically unibranch if it is geometrically unibranch at all its points (see [SP]).

\textbf{Example A.2.} All normal schemes are geometrically unibranch.
Remark A.3. Let $R$ be a local ring. By [Ray] pp. 100, Definition 2] $R$ is geometrically unibranch if and only if the strict Henselization $R^{sh}$ has a unique minimal prime, that is $(R^{sh})_{\text{red}}$ is a domain.

By [EGA4] 18.9.1] if $R$ is an excellent local ring and it is geometrically unibranch, then the completion $\hat{R}$ has a unique minimal prime.

If Spec $R$ is geometrically unibranch, then $R$ is geometrically unibranch, but the converse is false because this condition does not pass to localizations [EGA1] Ch 0, 6.5.11, pp. 149].

Remark A.4. In a geometrically unibranch scheme all irreducible components are also connected components. Indeed the localization in a point lying in an intersection of two different irreducible components has at least two minimal primes. In particular a locally connected (e.g., locally Noetherian) geometrically unibranch scheme is a disjoint union of irreducible and geometrically unibranch schemes.

Lemma A.5. Let $k$ be a field, let $X$ be a $k$-scheme, and let $x \in X \times_k \overline{k}$ be a point lying over $x \in X$. Then $X \times_k \overline{k}$ is geometrically unibranch at $x$ if and only if $X$ is geometrically unibranch at $x$.

Proof. We can assume $X = \text{Spec} A$, where $(A, \mathfrak{p})$ is a local ring and $x = \mathfrak{p} \in \text{Spec} A$. Let $k^s$ be the separable closure of $k$ and $P \in X \times_k k^s$ lying over $\mathfrak{p} \in X$. For all $k \subseteq L \subseteq k^s$, set $P_L = P \cap (A \otimes_k L)$. Then

$$\lim_{k \subseteq L \subseteq k^s} (A \otimes_k L)_P = (A \otimes_k k^s)_P,$$

which implies that $A_\mathfrak{p}$ and $(A \otimes_k k^s)_P$ has the same strict Henselization, so that one is geometrically unibranch if and only if the other is. In particular we can assume $k = k^s$ separably closed. In this case $A \otimes_k \overline{k}$ is local so that $\overline{x}$ corresponds to its maximal ideal. Since $A \rightarrow A \otimes_k \overline{k}$ is surjective, purely inseparable ([EGA1] Ch I, Prop. 3.7.1, pp. 246]), and integral, by [SGA4] Exposé VIII, Théorème 1.1] the pullback along Spec $(A \otimes_k \overline{k}) \rightarrow \text{Spec} A$ induces an equivalence of categories between the small étale sites of the two schemes. This easily implies that

$$(A \otimes_k \overline{k})^{sh} \simeq A^{sh} \otimes_A (A \otimes_k \overline{k}) \simeq A^{sh} \otimes_k \overline{k}.$$

Thus Spec $(A \otimes_k \overline{k})^{sh} \rightarrow \text{Spec} A^{sh}$ is a homeomorphism, and the result is clear. □

Proposition A.6. Let $f : X \rightarrow Y$ be a faithfully flat map of schemes, let $x \in X$, and let $y = f(x)$. If $X$ is geometrically unibranch at $x$, then $Y$ is geometrically unibranch at $y$ and the converse holds if all the geometric fibers of $f$ are normal.

Proof. The “if” part follows from the fact that $\mathcal{O}_{Y,y}^{sh} \rightarrow \mathcal{O}_{X,x}^{sh}$ is injective and from [Ray] pp. 100, Definition 2].

For the converse we can assume $Y = \text{Spec} A$, where $(A, \mathfrak{p})$ is a local and geometrically unibranch domain and $X = \text{Spec} C$. Denote by $\mathfrak{p}$ the prime ideal of $C$ corresponding to $x \in X$. Let $B$ be the integral closure of $A$, which, by assumption, is a local domain whose residue field $l$ is purely inseparable over the residue field $k$. 
of $A$. Given a map $A \to D$, set $C_D = C \otimes_A D$ and consider the coproducts

$$
\begin{array}{ccc}
A & \to & C \\
\downarrow & & \downarrow \\
B & \to & C_B \\
\downarrow & & \downarrow \\
l & \to & C_l
\end{array}
\quad
\begin{array}{ccc}
A & \to & C \\
\downarrow & & \downarrow \\
B & \to & (C_B)_P \\
\downarrow & & \downarrow \\
l & \to & (C_l)_P
\end{array}
$$

Since $B$ is normal and $f$ has normal fibers, it follows that $C_B$ and therefore $(C_B)_P$ are normal rings ([Mat, Theorem 23.9, pp. 184]). Since $A \to C$ is flat and $A \to B$ is injective and integral, it follows that $C_P \to (C_B)_P$ is integral and injective. Since $k = A/p \to B/pB$ is a purely inseparable morphism, $C_P/pC_P \to (C_B)_P/p(C_B)_P$ is purely inseparable too. Thus there is a unique ideal in $(C_B)_P$ which restricts to the maximal ideal of $C_P$. Since a prime ideal in $(C_B)_P$ is maximal if and only if it restricts to a maximal ideal in $C_P$, we conclude that $(C_B)_P$ is a local ring. Thus the connected normal ring $(C_B)_P$ is a domain, and clearly $(C_B)_P$ is the integral closure of $C_P$. The residue field extension of $C_P \to (C_B)_P$ is purely inseparable because $C_P/pC_P \to (C_B)_P/p(C_B)_P$ is a purely inseparable ring morphism.

**Definition B.1.** Given a smooth atlas $\mathcal{X}$, we say that $\mathcal{X}$ is geometrically unibranch at $x$ if there exists a smooth atlas $U \to \mathcal{X}$ and $u \in U$ over $x$ such that $U$ is geometrically unibranch at $u$. The stack $\mathcal{X}$ is called geometrically unibranch if it is geometrically unibranch at all its points.

**Remark A.8.** The notion of being geometrically unibranch for algebraic stacks does not depend on the choice of the atlas thanks to Proposition A.6. Moreover, it is immediate that if an algebraic stack is geometrically unibranch and if only if it has a smooth atlas $U \to \mathcal{X}$ with $U$ geometrically unibranch, and one can verify that statements parallel to Lemma A.5 and Proposition A.6 remain true for algebraic stacks.

**Appendix B. Base change by finite algebraic extensions**

Let $k$ be a field, and let $l/k$ be a finite field extension.

**Definition B.1.** Given a $k$-linear category $\mathcal{C}$, we denote by $\mathcal{C} \otimes_k l$ the category of pairs $(E, \lambda)$, where $E \in \mathcal{C}$ and $\lambda : l \to \text{End}_k(E)$ is a $k$-algebra map.

The category $\mathcal{C} \otimes_k l$ is additive, and it has a natural $l$-linear structure. Moreover if $\mathcal{D}$ is an $l$-linear additive category, then a $k$-linear functor $Q : \mathcal{D} \to \mathcal{C}$ extends uniquely to an $l$-linear functor $Q : \mathcal{D} \to \mathcal{C} \otimes_k l$.

**Remark B.2.** If $\mathcal{X}$ is a category fibered in groupoids with an fpqc atlas over $k$, then the pushforward along $\beta : \mathcal{X} \times_k l \to \mathcal{X}$ induces an equivalence $\beta^* : \text{QCoh}(\mathcal{X} \times_k l) \to \text{QCoh}(\mathcal{X}) \otimes_k l$. Because $l/k$ is finite, it also restricts to an equivalence $\beta^* : \text{Vect}(\mathcal{X} \times_k l) \to \text{Vect}(\mathcal{X}) \otimes_k l$. Under this equivalence the pullback of $\beta$ sends $\mathcal{F} \in \text{QCoh}(\mathcal{X})$ to $(\mathcal{F} \otimes_k l, \rho)$, where $\rho$ is induced by the action on the right component of $\mathcal{F} \otimes_k l$. In particular if $\mathcal{C}$ is a $k$-Tannakian category, then $\mathcal{C} \otimes_k l$ is an $l$-Tannakian category: $\Pi_{\mathcal{C} \times_k l}$ is an $l$-gerbe and $\text{Vect}(\Pi_{\mathcal{C} \times_k l}) \simeq \mathcal{C} \otimes_k l$. 


If \( k \) is of characteristic \( p > 0 \), \( \mathcal{X}/k \) is a category fibered in groupoids with an fpqc atlas, \( \mathcal{X}_l := \mathcal{X} \times_k l \), and \( \beta : \mathcal{X}_l \rightarrow \mathcal{X} \) is the projection, we have the Cartesian diagram

\[
\begin{array}{ccc}
\mathcal{X}_l & \longrightarrow & \mathcal{X}_l^{(1,l)} \longrightarrow \mathcal{X}_l^{(2,l)} \longrightarrow \cdots \longrightarrow \Spec l \\
\downarrow \beta & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{X} & \longrightarrow & \mathcal{X}^{(1)} \longrightarrow \mathcal{X}^{(2)} \longrightarrow \cdots \longrightarrow \Spec k.
\end{array}
\]

This defines a functor \( \beta_* : \text{Fdiv}(\mathcal{X}_l/l) \rightarrow \text{Fdiv}(\mathcal{X}/k) \), which is the right adjoint of the pullback functor \( \beta^* : \text{Fdiv}(\mathcal{X}/k) \rightarrow \text{Fdiv}(\mathcal{X}_l/l) \).

**Proposition B.3.** The unique \( l \)-linear functor \( \beta_\# : \text{Fdiv}(\mathcal{X}_l/l) \rightarrow \text{Fdiv}(\mathcal{X}/k) \otimes_k l \) induced by \( \beta_* \) is a tensor equivalence. If \( \text{Fdiv}(\mathcal{X}/k) \) is a \( k \)-Tannakian category, then \( \text{Fdiv}(\mathcal{X}_l/l) \) is an \( l \)-Tannakian category (with the obvious tensor structure), and the pullback along \( \beta^* : \text{Fdiv}(\mathcal{X}/k) \rightarrow \text{Fdiv}(\mathcal{X}_l/l) \) corresponds to a \( k \)-morphism \( \Pi_{\text{Fdiv}(\mathcal{X}_l/l)} \rightarrow \Pi_{\text{Fdiv}(\mathcal{X}/k)} \) which induces an equivalence

\[
\Pi_{\text{Fdiv}(\mathcal{X}_l/l)} \rightarrow \Pi_{\text{Fdiv}(\mathcal{X}/k)} \times_k l.
\]

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
\text{Fdiv}(\mathcal{X}_l/l) & \overset{\beta_\#}{\longrightarrow} & \text{Fdiv}(\mathcal{X}/k) \otimes_k l \\
\downarrow \alpha & & \downarrow \cong \\
\text{Vect}(\Pi_{\text{Fdiv}(\mathcal{X}/k)} \times_k l) & \overset{\gamma_\#}{\longrightarrow} & \text{Vect}(\Pi_{\text{Fdiv}(\mathcal{X}/k)}) \otimes_k l \\
\downarrow \gamma_* & & \downarrow \cong \\
\text{Fdiv}(\mathcal{X}/k) & \overset{\text{For}_{\mathcal{X}}}{\longrightarrow} & \text{Fdiv}(\mathcal{X}/k) \\
\end{array}
\]

where \( \gamma : \Pi_{\text{Fdiv}(\mathcal{X}/k)} \times_k l \rightarrow \Pi_{\text{Fdiv}(\mathcal{X}/k)} \) is the projection, and \( \text{For}_{\mathcal{X}} \), \( \text{For}_{\Pi} \) are forgetful functors. Applying Remark B.2 to each \( \mathcal{X}^{(i)} \), it is easy to see that \( \beta_\# \) is an equivalence. Assume now that \( \Pi_{\text{Fdiv}(\mathcal{X}/k)} \) is a \( k \)-gerbe. Again by Remark B.2 \( \gamma_\# \) is an equivalence. Thus we get the dashed arrow \( \alpha \), which makes all the rectangles plus \( \gamma_* \), \( \alpha \), commutative. By the uniqueness of the left adjoint, \( \alpha \) also commutes with \( \beta^*, \gamma^* \), and \( \alpha \). Using the fact that \( \alpha, \alpha^*, \gamma^* \) are all tensor functors and the fact that for any \( M \in \text{Fdiv}(\mathcal{X}_l/l) \) there exists \( N \in \text{Fdiv}(\mathcal{X}/k) \) with a surjection \( \beta^* N \rightarrow M \) (e.g., \( \beta^* \beta, M \rightarrow M \)), one can easily deduce that \( \alpha \) is a tensor equivalence. Thus \( \text{Fdiv}(\mathcal{X}_l/l) \) is \( l \)-Tannakian with the Tannakian gerbe \( \Pi_{\text{Fdiv}(\mathcal{X}/k)} \times_k l \).

The last claim follows immediately from the 2-commutative diagram. \( \square \)

**Remark B.4.** Let \( k'/k \) be any field extension, and let \( \mathcal{X} \) be a geometrically connected algebraic stack of finite type over \( k \). Also in this case there is a canonical map of \( k' \)-gerbes

\[
\delta : \Pi_{\text{Fdiv}(\mathcal{X} \times_k k'/k')} \rightarrow \Pi_{\text{Fdiv}(\mathcal{X}/k)} \times_k k',
\]

and using the method employed in [De], one can show that \( \delta \) is a quotient if \( \mathcal{X}/k \) is smooth. If \( \mathcal{X} \) is not smooth, it is unclear to us whether or not the above morphism is surjective. The main problem is that we do not know whether the category of
quasi-coherent $F$-divided sheaves on $X$ is an abelian category. Thus the techniques developed in [De §4] cannot be easily applied.

**Proposition B.5.** Let $\Gamma$ be an affine gerbe over $k$. Let $V \in \text{Vect}(\Gamma)$ be an object. Let $\langle V \rangle$ be the sub-Tannakian category generated by $V$, and denote $\Delta$ the corresponding gerbe. Let $\beta : \Gamma \times_k l \to \Gamma$ be the projection. Then $\langle \beta^*V \rangle \subseteq \text{Vect}(\Gamma \times_k l)$ corresponds to $\Delta \times_k l$. In particular the canonical map

$$(\Gamma \times_k l) \longrightarrow \hat{\Gamma} \times_k l,$$

where $\hat{-}$ denotes the profinite quotient, is an equivalence. In other words if $\mathcal{C}$ is a $k$-Tannakian category, then $\text{EFin}(\mathcal{C}) \otimes_k l \longrightarrow \text{EFin}(\mathcal{C} \otimes_k l)$ is an equivalence.

**Proof.** We have to show that $\langle \beta^*V \rangle = \langle V \rangle \otimes_k l$; that is, an object $W \in \text{Vect}(\Gamma \times_k l)$ belongs to $\langle \beta^*V \rangle$ if and only if $\beta_*W \in \langle V \rangle$. Since $\beta_*\beta^*V \in \langle V \rangle$, $\langle \beta^*V \rangle \subseteq \langle V \rangle \otimes_k l$. Composing with the forgetful functor, we see that $\beta_*W \in \langle V \rangle$ implies that $\beta_*W \in \langle \beta^*V \rangle$. Conversely, if $\beta_*W \in \langle V \rangle$, then $\beta_*\beta_*W \in \langle \beta^*V \rangle$. Thus $W \in \langle \beta^*V \rangle$ as there is a surjection $\beta_*\beta_*W \to W$. This finishes the first claim.

For the second claim we have to show that a vector bundle $V \in \text{Vect}(\Gamma \times_k l)$ is essentially finite if and only if $\beta_*V \in \text{Vect}(\Gamma)$ is essentially finite. By the first claim $\beta_*V$ is essentially finite if and only if $\beta^*\beta_*V$ is essentially finite. Since we have a surjection $\beta^*\beta_*V \to V$ and $\beta^*\beta_*V$ is a finite direct sum of $V$, $V$ is essentially finite if and only if $\beta^*\beta_*V$ is also.

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