ON FLAT SUBMAPS
OF MAPS OF NONPOSITIVE CURVATURE

A. YU. OLSHANSKII AND M. V. SAPIR

Abstract. We prove that for every \( r > 0 \) if a nonpositively curved \((p, q)\)-map \( M \) contains no flat submaps of radius \( r \), then the area of \( M \) does not exceed \( C r n \) for some constant \( C \). This strengthens a theorem of Ivanov and Schupp. We show that an infinite \((p, q)\)-map which tessellates the plane is quasi-isometric to the Euclidean plane if and only if the map contains only finitely many nonflat vertices and faces. We also generalize Ivanov and Schupp’s result to a much larger class of maps, namely to maps with angle functions.

Contents

1. Introduction 1
2. Large flat submaps of \((p, q)\)-maps 7
3. \((p, q)\)-maps that are quasi-isometric to \( \mathbb{R}^2 \) 17
4. Maps with angle functions 24
Acknowledgments 26
References 26

1. Introduction

Recall that a map is a finite, connected, and simply connected 2-complex embedded in the Euclidean plane. So its 1-skeleton is a finite, connected plane graph. The cells of dimensions 0, 1, and 2 are called vertices, edges, and faces, respectively. Every edge \( e \) has an orientation; so it starts at the vertex \( e_- \) and ends at \( e_+ \), and \((e^{-1})_- = e_+, (e^{-1})_+ = e_- \) for the inverse edge \( e^{-1} \), which has the same support as \( e \). The degree \( d(o) \) of a vertex \( o \) is the number of oriented edges \( e \) with \( e_- = o \). In particular, every loop \( e \) (an edge which connects a vertex \( o \) with itself) together with \( e^{-1} \) contributes 2 to the degree of \( o \).

If a closed path \( q = e_1 \cdots e_k \) is the boundary of a face \( \Pi \), then the degree \( d(\Pi) \) of \( \Pi \) is the length \( |q| = k \). In particular, if both \( e \) and \( e^{-1} \) occur in the boundary path of a face, they contribute 2 to the degree of that face. Similarly, one defines the perimeter \( |\partial M| \) of a map \( M \) as the length of a closed boundary path of \( M \).

A submap \( N \) of a map \( M \) is the subcomplex bounded by a closed curve which can be made simple by an arbitrary small transformation. So either \( N \) is a map or it can be turned into a map after such small transformation.
In group theory, maps appear most often as van Kampen diagrams. Many algebraic and geometric results about groups (e.g., small cancellation theory and construction of various groups with “extreme properties” such as Tarski monsters (7,8)) are obtained by establishing combinatorial properties of corresponding maps and their submaps. Following is a typical example of such a statement: The area (i.e., the number of faces) of every reduced van Kampen diagram over a finite group presentation is at most linear in terms of its perimeter if and only if the group is hyperbolic. In other words, hyperbolic groups are precisely the finitely presented groups with linear Dehn functions. One recurrent feature of van Kampen diagrams is the existence of “special” submaps in every van Kampen diagram of a large area. For example, in the proof of the upper bound of the Dehn function of a group (7,8) are obtained by establishing combinatorial properties of corresponding maps constructed in (11) using an S-machine, it is proved that if the area of a reduced diagram is large enough, then up to a homotopy that does not change the area very much, the area is concentrated in a few special subdiagrams called discs (these are the subdiagrams simulating the work of the S-machine).

A remarkable result of this kind was proved by Ivanov and Schupp in (6). Recall that an edge e of a map M is called exterior if it belongs to a boundary path of M. A face Π of M is called exterior if its boundary ∂Π has a common edge with ∂(M). An exterior vertex is one of the vertices of the boundary path. Nonexterior faces, vertices, and edges are called interior.

A map M is called a (p,q)-map if every interior face Π in M has degree at least p and the degree of every interior vertex is at least q. Note that if a group presentation \( \mathcal{P} = \langle X, R \rangle \) satisfies the small cancellation condition \( C(p) \) and \( T(q) \), then every reduced van Kampen diagram over \( \mathcal{P} \) is a (p,q)-map if we ignore all interior vertices of degree 2 (as in (6,7)). It is well known (see (7)) that if \( \frac{1}{p} + \frac{1}{q} \) is smaller than \( \frac{1}{2} \) (i.e., the curvature of the presentation is negative), then the group is hyperbolic and its Dehn function is linear. The case when \( \frac{1}{p} + \frac{1}{q} > \frac{1}{2} \) is not interesting. Indeed, by a result of Gol’berg (4,8), every group can be given by a presentation satisfying \( C(5) \) and \( T(3) \) and by a presentation satisfying \( C(3) \) and \( T(5) \) and, hence, by a presentation satisfying \( C(p) \) and \( T(q) \) for every \( p \geq 3, q \geq 3 \) with \( \frac{1}{p} + \frac{1}{q} > \frac{1}{2} \) (see (7)). If \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \) (i.e., the curvature is nonpositive) and so \( (p,q) \) is either \((3,6),(4,4),(6,3)\)), then the group has at most quadratic Dehn function; see (7, Theorem V.6.2).

A submap of a (p,q)-map is called flat if each of its faces is flat, i.e., has degree \( p \), and each interior vertex is flat, i.e., has degree \( q \). The radius of a map is the maximal distance from a vertex to the boundary of the map.

Ivanov and Schupp (6) proved that if a (p,q)-map \( M, \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \), has no flat submaps of radius \( r \) (they call flat submaps regular), then the area of the map is linear in terms of its perimeter with the multiplicative constant depending on \( r \). More precisely, Ivanov and Schupp proved the following:

**Theorem 1.1** (Ivanov and Schupp, (6)). Let \( M \) be a finite (p,q)-map with perimeter \( n \) such that the maximal distance from a vertex in \( M \) to a boundary vertex or to a nonflat vertex or face is \( r \). Then the area of \( M \) does not exceed \( L(r)n \), where \( L(r) \) is some exponential function in \( r \).

Theorem 1.1 implies that if a group presentation \( \mathcal{P} = \langle X, R \rangle \) satisfies conditions \( C(p) \) and \( T(q) \), \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \), and the radius of every flat van Kampen diagram over \( \mathcal{P} \) does not exceed a certain constant, then the Dehn function of the group given by...
\( \mathcal{P} \) is linear, hence the group is hyperbolic. Using this, Ivanov and Schupp proved hyperbolicity of many one-relator groups.

In this paper, we strengthen Theorem 1.1 in two ways. First, we replace the exponential upper bound for \( L(r) \) by a linear upper bound. Second, we extend Theorem 1.1 to a much larger class of maps called maps with angle functions.

Let us call a submap of a \((p, q)\)-map simple if it is bounded by a simple closed curve. Note that the closure of the union of faces from a nonsimple submap may not be simply connected (see Figure 1) while the closure of the union of faces from a simple submap is always simply connected.

![Figure 1](image-url)

**Figure 1.** The thick path \( ABCDEFGAHIJA \) is simple up to an arbitrary small deformation. It bounds a nonsimple submap.

Let \( p = 4, 3, \) or 6. Let \( S^p \) be the usual tessellation of the plane by \( p \)-gons. Then for every \( n \geq 0 \), the standard map \( S^n_p \) is constructed as follows. By definition, \( S^n_p \) is a vertex \( o \) in \( T \). If the submap \( S^n_p \) of \( T \) is constructed, then \( S^{n+1}_p \) is the (closure of the) union of all faces having a common vertex with \( S^n_p \). Then \( S^n_p \) is a simple \((p, q)\)-map. For example, then \( S^4_3 \) is the \((2n \times 2n)\)-square tessellated by unit squares, \( S^3_6 \) is a regular hexagon with side length \( n \) tessellated by equilateral triangles with side length 1, and \( S^n_p \) can be viewed as the weak dual\(^1\) to the \((3, 6)\)-map constructed just as \( S^{3}_{n+1} \), only starting with a triangle face instead of a vertex.

**Remark 1.2.** Every simple \((p, q)\)-map, \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \), of radius \( r \) contains a simple submap \( M' \) which is isomorphic to \( S^n_p \) for \( n = O(r) \). The submap \( M' \) can be obtained in a manner similar to \( S^n_p \). Pick a vertex \( o \) in \( M \) at distance \( r \) from the boundary of \( M \). This is the submap \( M_0 \). If \( M_i \) is already constructed, then \( M_{i+1} \) is obtained from \( M_i \) by adding all faces having a vertex in common with \( M_i \). The process continues until one of the vertices in \( M_i \) is exterior. In that case we set \( M' = M_i \). Of course it should be explained why \( M_{i+1} \) is indeed a simple standard.

\(^1\)Recall that if \( M \) is a map, then the weak dual map \( \overline{M} \) is obtained by putting a vertex in every (bounded) face, and for every edge shared by two faces of \( M \), connect the two vertices from these faces by an edge crossing that edge. Thus the vertices of \( M \) correspond to faces of \( M \), edges of \( M \) correspond to interior edges of \( M \), and faces of \( M \) correspond to interior vertices of \( M \).
submap provided $M_i$ is already a simple standard submap. It is not as obvious as it seems. The explanation uses Lemma 2.2 below, and it is given in Remark 2.3.

Our main result is the following:

**Theorem 1.3** (see Theorem 2.1 below). If a $(p,q)$-map $M$ does not contain flat simple submaps of radius $\geq r$, then the area of $M$ is at most $cr^n$ for some constant $c$.

Note that the statement of Theorem 2.1 is nontrivial even for the van Kampen diagrams over the standard presentation of $\mathbb{Z}^2$, although there is a significantly easier proof in this case. Theorem 2.1 is applicable to van Kampen diagrams over any $(C(p) - T(q))$-presentations with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, say, the standard presentations of two-dimensional right-angled Artin groups or the fundamental groups of alternating knots.

Theorem 2.1 is proved in section 2. The plan of the proof is the following. First for every map $M$ and every two (real) numbers $p, q$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, we define curvature of faces and vertices of $M$ as numbers proportional to the excessive degrees, and we show that the sum of all curvatures is equal to $p$. Then we assume that $p, q$ are positive integers (so $(p, q)$ is $(3, 6), (4, 4)$, or $(6, 3)$) and note that by a simple transformation of the map, we can assume that all faces of $M$ have degrees between $p$ and $2p - 1$. The perimeter of the map after this transformation increases by a factor of at most $\leq p - 1$ and the set of (simple) flat submaps can only get smaller. The key Contraction Lemma 2.6 says that the perimeter of the interior $M^o$ which is the union of all faces of $M$ having no boundary vertices of $M$, is “substantially smaller” than the perimeter of $M$. From this, we deduce, first, that the area of $M$ is $O(R^n)$, where $R$ is the radius of $M$. Second, we deduce that one can cut $M$ along paths of linear in $n$ total length so that in the resulting map $\tilde{M}$ all nonflat vertices and faces are on the boundary. Then the radius $\tilde{R}$ of $\tilde{M}$ is smaller than $r + p$. Hence $\text{Area}(M) = \text{Area}(\tilde{M}) = O(\tilde{R}^n) = O(r^n)$.

In section 3 we consider infinite maps on the plane, i.e., tessellations of $\mathbb{R}^2$. An infinite map is called **proper** if its support (i.e., the union of all faces, edges, and vertices) is the whole plane $\mathbb{R}^2$ and every disc in $\mathbb{R}^2$ intersects only a finite number of faces, edges, and vertices of the map. Our main result is the following:

**Theorem 1.4** (see Theorem 3.1 below). Let $M$ be a proper $(p, q)$-map with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Then the 1-skeleton of $M$ with its path metric is quasi-isometric to the Euclidean plane if and only if $M$ has only finitely many nonflat vertices and faces.

Our proof of Theorem 1.4 proceeds as follows. Suppose that a proper map $M$ has finite number of nonflat vertices and faces. Then we modify it in a finite sequence of steps. At each step we reduce the number of nonflat faces and vertices while keeping the map quasi-isometric to $M$. As a result we get a proper map $M'$ with at most one nonflat vertex and no nonflat faces. Such a map is naturally subdivided by infinite paths emanating from the nonflat vertex into a finite number of convex infinite submaps, each of which is quasi-isometric to a quadrant of the Euclidean plane. Combining the corresponding quasi-isometries, we get a quasi-isometry between $M'$ and $\mathbb{R}^2$. The main tool in the proof is the notion of infinite **corridor**, that is an infinite sequence of faces in $M$ where each consecutive face shares an edge. This gives the “if” part of the theorem.

For the “only if” part, we prove that if a proper map $M$ has infinitely many nonflat vertices or faces, then for every constant $c$ it contains an infinite $c$-separated
set $S$ of vertices which has a superquadratic growth function (that is the function that for every $n$ gives the number of vertices from $S$ at distance $\leq n$ from a given vertex is superquadratic). This cannot happen if $M$ was quasi-isometric to $\mathbb{R}^2$. The key tool in proving this part of the theorem is the Contraction Lemma 2.6 from the proof of Theorem 2.1. We construct a sequence of submaps $N(r)$ such that the boundaries of $N(r)$ contain large $c$-separated sets of vertices. In order to prove this property of $N(r)$, we use winding numbers of piecewise geodesic paths in $M$ passing through vertices of $\partial(N(r))$ around a vertex which is deep inside $N(r)$.

Bruce Kleiner and Michah Sageev explained to us that the “only if” part of Theorem 1.4 can be deduced from Theorem 4.1 of their paper [1] (joint with Mladen Bestvina). If we view every face of a proper $(p, q)$-map $M$ as a regular Euclidean $n$-gon, then the map $M$ turns into a CAT(0) 2-complex $M'$ which is quasi-isometric to the original map. Then Part 1 of [1, Theorem 4.1] implies that $M$ has a locally finite second homology class whose support $S$ is locally isometric to the Euclidean plane outside some finite ball. It remains to notice that the only such homology class is (up to a scalar multiple) the fundamental class of $M$. Hence $M$ is locally flat outside a finite ball. Bruce Kleiner also explained how to deduce the “if” part of Theorem 1.4 using Riemannian geometry. First we “smooth out” the CAT(0) 2-complex $M$ which is locally flat outside a finite ball to obtain (using the Cartan–Hadamard theorem) a two-dimensional Riemannian manifold $M'$ with the same property and which is quasi-isometric to the map $M$. Then we use the Rauch comparison theorem to establish a bi-Lipschitz equivalence between $M'$ and the Euclidean plane.

Note that our proof of Theorem 1.4 is completely self-contained, short, and uses only basic graph theory.

In section 4 we consider the class of maps with angle functions. Let $o$ be a vertex on the boundary of a face $\Pi$ of a map $M$. A corner of $\Pi$ at $o$ is the pair of two consecutive oriented edges $e$ and $f$ of $\partial(\Pi)$ with $e_+ = f_- = o$. ($f^{-1}$ and $e^{-1}$ define the same corner.) An angle function assigns a nonnegative number (angle) to each corner of each cell. Then the curvature of an interior vertex $o$ is $2\pi$ minus the sum of all angles at $o$. The curvature of an exterior vertex is defined in a similar way. The curvature of a face of perimeter $d$ is the sum of angles of corners of this face minus the sum of angles of a Euclidean $d$-gon (that is $\pi(d - 2)$). A map with an angle function is called flat if all of its faces and interior vertices are of curvature 0. A map with an angle function is called a $(\delta, b)$-map, $\delta > 0$, $b > 0$, if the curvature of every nonflat vertex and face does not exceed $-\delta$ and the degree of every vertex and face does not exceed $b$.

The class of $(\delta, b)$-maps is very large. By Fáry’s theorem (see [2])-11) every finite planar graph $M$ without double edges and loops can be drawn on the Euclidean plane using only straight line segments for edges. The proof from [2] shows that if $M$ is a plane map, then one can assume that the graph with straight edges is isomorphic to $M$ as a 2-complex. For a map with straight edges, we can assign to each corner its Euclidean angle, making the map flat.

Note that many authors considered van Kampen diagrams as maps with angle functions. Some of the earliest implementations of this idea are in the papers [3] by Steve Gersten, [4] by Steve Pride, and [5] by Jim Howie.

It is easy to see that a (finitely presented) group $G$ has a (finite) presentation $P = \langle X \mid R \rangle$ such that every van Kampen diagram over $P$ can be assigned an
angle function which makes the diagram a flat map if and only if one can find a 
finite generating set of the group which does not contain involutions. Such a finite 
generating set exists if and only if the group is not an extension of an Abelian group 
$A$ by the automorphism of order 2 which takes every element of $A$ to its inverse.

We will show in Remark 4.4 that every $(p,q)$-map can be transferred into a $(\delta,b)$-
map with angle function without decreasing the area, or increasing the perimeter 
or the set of flat vertices and faces. Thus the following theorem is a generalization 
of Ivanov and Schupp’s Theorem 1.1 to a much wider class of maps.

**Theorem 1.5.** Suppose that $M$ is a $(\delta,b)$-map of perimeter $n$, and suppose the 
distance of every vertex of $M$ to a boundary vertex or to a nonflat vertex or face of 
$M$ is at most $r$. Then $\text{Area}(M) \leq Ln$, where $L$ is exponential in $r$.

The key part of the (very short) proof of Theorem 1.5 (see section 4) is Lemma 
4.2 which shows that in every nonpositively curved map with an angle function, 
the sum of curvatures of all faces and interior vertices exceeds $\pi(2 - n)$ where $n$ is 
the perimeter of the map.

It is easy to see that if we weaken the conditions of Theorem 1.5 by assuming 
only that the curvature of every face or vertex is not positive or if we do not assume 
that the degrees of all vertices and faces are uniformly bounded, the statement of 
the theorem will no longer hold.

Note that unlike Theorem 1.1 the exponential (in $r$) estimate in Theorem 1.5 
cannot be improved even in the case of van Kampen diagrams over group presenta-
tions. For example let $BS(1,2)$ be the Baumslag–Solitar group $\langle a, b \mid b^{-1}ab = a^2 \rangle$. 
Then the van Kampen diagram for the relation $[b^{-n}ab^n, a] = 1$ (see Figure 2 for the 
case $n = 5$) can be viewed as a flat $(\delta,b)$-map $M_n$ with $b = 5$ and any $\delta > 0$ if we 
assign angles in each face as shown on Figure 2. The distance from any vertex of 
$M_n$ to $\partial(M_n)$ is $O(n)$; the perimeter of $M_n$ is also $O(n)$ but the area is exponential 
in $n$.

![Figure 2. The van Kampen diagram for the relation $[b^{-n}ab^n, a] = 1$ in $BS(1,2)$](https://www.ams.org/journal-terms-of-use)
2. Large flat submaps of \((p, q)\)-maps

Recall that we only consider simply connected maps.

Although every edge in a map has an orientation, when counting the numbers of edges (or faces) in a map, we take usually any pair \((e, e^{-1})\) as one nonoriented edge (e.g., \(E\) is the number of nonoriented edges, when we apply Euler’s formula). The boundary path \(p\) of a map or a face is considered up to cyclic permutations and taking inverse paths \(p^{-1}\).

The number of faces in \(M\) is called the area of \(M\), denoted \(\text{Area}(M)\).

Here is a precise formulation of our main result.

**Theorem 2.1.** Let \(p, q\) be positive integers with \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\), \(C = \frac{3}{2}(p - 1)(q + 1)\). Then the area of \(M\) does not exceed \(C(r + p)n\), provided \(M\) contains no simple flat submaps of radius greater than \(r\).

2.1. A lemma about curvatures. Given a pair \((p, q)\) of arbitrary (possibly negative) real numbers with \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\), the curvature of a face \(\Pi\) in a map \(M\) is defined as \(\text{curv}_{p,q}(\Pi) = p - d(\Pi)\). Let \(o\) be a vertex in \(M\). Then let \(\mu(o)\) be the number of times the boundary path \(\gamma\) goes through the vertex \(o\). For example, the multiplicity \(\mu(o)\) is 1 if the closed path \(\gamma\) passes through \(o\) only once, and \(\mu(o) = 0\) if \(o\) is an interior vertex. The curvature \(\text{curv}_{p,q}(o)\) of a vertex \(o\) is defined as \(\frac{2}{q}(q - d(o)) - \mu(o)\).

Let \(I_v = I_v(M)\) be the sum of curvatures of all vertices of \(M\), and let \(I_f = I_f(M)\) be the sum of curvatures of all faces of \(M\).

The following lemma follows from Theorem 3.1 in [7, Chapter V], we include the proof for completeness and because it is significantly easier than in [7].

**Lemma 2.2.** For an arbitrary map \(M\) and arbitrary real \(p, q\) with \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\), we have \(I_v + I_f = p\).

**Proof.** An easy calculation shows that the statement holds for a map consisting of one vertex. Assume that \(M\) has more than one vertex, hence it has no vertices of degree 0. Let us assign weight 1 to every nonoriented edge of the map \(M\). Then the sum of all weights is the number \(E\) of edges in \(M\). Now let us make each edge give \(\frac{1}{q}\) of its weight to each of its vertices (if it has only one vertex, the edge gives it \(\frac{2}{q}\)) and \(\frac{1}{p}\) to each of the (at most two) faces containing that edge. Thus the sum of weights of all vertices of \(M\) is equal to \(\sum_o \frac{1}{q}d(o)\), where \(o\) runs over all vertices of \(M\). The sum of weights of all faces is \(\sum_{\Pi} \frac{1}{p}d(\Pi)\), where the sum runs over all faces of \(M\).

By the assumption \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\), every edge separating two faces becomes completely weightless (it gives \(\frac{1}{p}\) to each of the faces and \(\frac{1}{q}\) to each of its vertices). For the same reason, an edge \(e\) of the boundary path \(\gamma = \partial(M)\) becomes of weight \(\frac{1}{p}\) if it lies on the boundary of a face, or \(\frac{2}{q}\) otherwise. In the later case, the nonoriented edge \(e\) occurs in the path \(\gamma\) twice (with different orientations). Therefore after the redistribution of weights, the sum of weights of all edges in \(M\) is equal to \(\frac{1}{p}n\) where \(n\) is the perimeter of the map.

Thus, the total weight is equal to

\[
E = \sum_o \frac{1}{q}d(o) + \sum_{\Pi} \frac{1}{p}d(\Pi) + \frac{1}{p}n.
\]
Since $E - V - F = -1$ by Euler’s formula, where $V$ and $F$ are numbers of vertices and faces in $M$, respectively, we have

$$-1 = \sum_o \frac{1}{q} d(o) + \sum_i \frac{1}{p} d(\Pi) + \frac{1}{p} n - V - F$$

$$= \sum_o \left( \frac{1}{q} d(o) - 1 \right) + \sum_i \left( \frac{1}{p} d(\Pi) - 1 \right) + \frac{1}{p} n.\quad (2.1)$$

Notice also that $n = \sum_v \mu(o)$ where $v$ runs over all vertices of $M$; indeed, $\mu(o) = 0$ for all interior vertices, and $\mu(o)$ is the number of times the boundary path passes through $o$. Therefore, we can rewrite (2.1) as

$$-1 = \sum_o \left( \frac{1}{q} d(o) - 1 + \frac{1}{p} \mu(o) \right) + \sum_i \left( \frac{1}{p} d(\Pi) - 1 \right).$$

Since the first of these sums is $-\frac{1}{p} I_v$ and the second sum is $-\frac{1}{p} I_f$, we deduce that $I_v + I_f = p$. □

**Remark 2.3.** A result similar to Lemma 2.2 is true for maps on arbitrary surfaces $S$ with boundary. It is easy to see in that case that the right-hand side is equal to $p\chi(S)$ where $\chi(S)$ is the Euler characteristic of the smallest subsurface of $S$ containing the map.

**Remark 2.4.** Let us use Lemma 2.2 to complete the proof from Remark 1.2 about standard submaps of simple flat maps. Here we consider the case $p = q = 4$ only, leaving the other two cases to the reader as an exercise. As in Remark 1.2, let $M$ be a simple flat map of radius $r$. We set $M_0$ to be a vertex $o$ at distance $r$ from the boundary of $M$, and we assume that for $i \geq 0$, the submap $M_i$ is constructed and this submap is simple and isomorphic to the standard map $S^4_r$ which is the $(n \times n)$-square (where $n = 2i$) tessellated by unit squares. Counting the difference between the degrees of the vertices from $\partial M_i$ in $M$ and in $M_i$, we obtain exactly $4n + 4$ oriented edges $e_1, \ldots, e_{4n+4}$ with $(e_j) - \in \partial(M_i)$ and $(e_j) + \notin M_i$. We claim that no two of the edges $e_j, e_k, j \neq k$, are mutually inverse and no two of the vertices $(e_j)_+, (e_k)_+$ coincide.

Indeed, suppose that $e_j = e_k^{-1}$ (see Figure 3). Then this edge and a subpath of $\partial M_i$ bound a submap $N$ having no faces from $M_i$. All exterior vertices of $N$, except for $(e_j)_{\pm}$, have degree at least 3 in $N$ because the degrees of these vertices in $M_i$ are at most 3 while there degrees in $M$ are 4. Therefore, the $(4,4)$-curvature of every face and vertex in the flat map $N$, except for $(e_j)_{\pm}$, are nonpositive, while the curvature of each of $(e_j)_{\pm}$ is at most 1. Hence the sum $I_f + I_v$ of all these curvatures for $N$ is at most $2 < 4$, which contradicts Lemma 2.2.

In case when $(e_j)_+ = (e_k)_+$, we consider the submap $N$, without faces from $M_i$, bounded by $e_j, e_k^{-1}$ and by a subpath of $\partial M_i$. It has at most three vertices of positive curvature (equal to 1), namely, $(e_j)_-, (e_j)_+ = (e_k)_+$, and $(e_k)_-$. This again contradicts Lemma 2.2 since $3 < 4$.

Now if we assume that edges $e_1, \ldots, e_{4n+4}$ are enumerated clockwise, we see that for each $j$, $j + 1$, $j = 1, \ldots, 4n + 4$ (where $j + 1$ is 1 if $j = 4n + 4$), must belong to the same face $\Pi_j$ which shares a vertex with $\partial(M_i)$, and every face sharing a vertex with $\partial(M_i)$ is one of the $\Pi_j$. Each pair of consecutive faces $\Pi_j, \Pi_{j+1}$ share exactly one edge $e_{j+1}$. Hence, $M_{i+1}$ is isomorphic to the $((n + 2) \times (n + 2))$-square tessellated by unit squares. The boundary of $M_{i+1}$ is simple since otherwise a
part of this boundary bounds a flat submap $N$ with at most one exterior vertex of positive curvature contrary to Lemma 2.2 for $N$ since $1 < 4$.

2.2. Weakly exterior faces and the interior of a $(p,q)$-map. In this section, $p$ and $q$ are positive integers satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

A face (edge) in a map $M$ is called strongly interior if it does not share a vertex with $\partial(M)$, otherwise it is called weakly exterior.

Lemma 2.5. Let $M$ be a $(p,q)$-map, and let $o_1, \ldots, o_m$ be its exterior vertices (counted counterclockwise). Then the number of weakly exterior faces does not exceed $\sum d(o_i) - 2m$.

Proof. We use induction on the number of faces in $M$. If $M$ has a cut vertex, then $M$ is a union of two submaps $M_1$ and $M_2$ intersected by a vertex, and it is easy to see that the statement for $M$ follows from the statements for $M_1$ and $M_2$. Thus we can assume that the boundary path of $M$ is simple. Then every exterior vertex $o_j$ belongs to $d(o_j) - 1$ weakly exterior faces, and two vertices $o_j, o_{j+1}$ (addition modulo $m$) belong to one face. So the sum $\sum (d(o_j) - 1) = \sum d(o_j) - m$ overcounts weakly exterior faces by at least $m$. The statement of the lemma then follows. \(\square\)

By the interior $M^0$ of $M$ we mean the union of all strongly interior faces of $M$ and their vertices and edges. Note that $M^0$ may be empty. It may also not be connected, in which case it coincides with the union of its maximal simple submaps $M^0_1, M^0_2, \ldots$. It follows that every edge of $\partial M^0_i$ belongs to a face of $M^0_i$ and to a weakly exterior face of $M$.

Hence the intersection of different submaps is either empty or consists of one vertex. Let us call these submaps the components of $M^0$. Thus the boundary paths $\eta_1^0, \eta_2^0, \ldots$ of the components $M^0_1, M^0_2, \ldots$ are simple (see Figure 3). Below we denote by $\eta$ the union of these boundaries and set $|\eta| = \sum |\eta_i^0|$.

In the next lemma we induct on the type of a map $M$. By definition the type $\tau = \tau(\Pi)$ of a face $\Pi$ in $M$ is the number of interior (nonoriented) edges in $\partial \Pi$. If $m = \text{Area}(M)$ and $(\tau_1, \tau_2, \ldots, \tau_m)$ is the $m$-tuple of the types of all the faces of $M$ with $\tau_1 \geq \tau_2 \geq \cdots$, then $\tau(M)$ is the infinite string $(\tau_1, \ldots, \tau_m, -1, -1, \ldots)$. We order types lexicographically: $\tau(M) \geq \tau(M') = (\tau_1', \tau_2', \ldots)$ if $\tau_1 > \tau_1'$ or $\tau_1 = \tau_1'$.
but \( \tau_2 > \tau'_2 \), and so on. For example, a map with only one face has the type \((0, -1, -1, \ldots)\). It is easy to check that the set of types is well ordered.

**Lemma 2.6** (Contraction lemma). Assume that a \((p, q)\)-map \(M\) has at least one face. Also assume that the degree of every face of \(M\) is at least \(p\). Let \(x\) be the boundary path of \(M\) and \(\eta\) is defined as above. Then

\[
|\mathbf{x}| - |\mathbf{\eta}| \geq -I_f - 2I_v^1 + p,
\]

where \(I_f\) is the sum of the \((p, q)\)-curvatures of all faces and \(I_v^1\) is the sum of \((p, q)\)-curvatures of all interior vertices of \(M\).

**Proof.** Let us denote \(-I_f - 2I_v^1\) by \(J = J(M)\). Thus we need to prove that \(|\mathbf{x}| - |\mathbf{\eta}| \geq J + p\).

**Step 1.** The statement of the lemma is true if \(M\) has only one face \(\Pi\), because we have \(|\mathbf{x}| = d(\Pi) \geq p\), \(|\mathbf{\eta}| = 0\), \(I_f = p - d(\Pi)\), and \(I_v^1 = 0\). Since the smallest type of a \((p, q)\)-map that has faces is the type \((0, -1, -1, \ldots)\) of a map consisting of one face (recall that we assume that \((p, q)\)-maps do not have vertices of degree 1), this gives the base of induction and we can assume that

\((U_1)\) the area of \(M\) is greater than 1.

**Step 2.** Suppose that \(M\) can be cut into two maps \(M_1\) and \(M_2\) with a smaller number of faces by a path of length at most 1. Defining parameters \(x(j), \eta(j), I_f(j), I_v^1(j)\) of the map \(M_j\), \(j = 1, 2\), in the natural way \((x(j)\) is the boundary path of \(M_j\), etc.), we have

- \(|x(1)| + |x(2)| \leq |\mathbf{x}| + 2\),
- \(|\mathbf{x}| = |\mathbf{x}(1)| + |\mathbf{x}(2)|\),
- \(I_f = I_f(1) + I_f(2)\) and \(I_v^1 = I_v^1(1) + I_v^1(2)\) since no interior vertex of \(M\) became exterior after the cutting.
Since $p \geq 2$, the statement of the lemma follows from inequalities $|r(j)| - |y(j)| \geq -I_f(j) - 2I^1_f(j) + p$ (where $j = 1, 2$), which hold, since $\tau(M_j) < \tau(M)$ because $\text{Area}(M_j) < \text{Area}(M)$. Thus, we may assume further that

\[(U_2) \ M \text{ has no cutting paths of length } \leq 1, \text{ hence, in particular, the boundary path } r \text{ is simple.} \]

**Step 3.** Assume there is an exterior vertex $o$ of degree $d > 3$. Let $e_1, e_2, \ldots, e_d$ be all edges ending in $o$, so that $e_1$ and $e_2$ are on the boundary path of a face $\Pi_1$, and $e_2$ and $e_3$ are on the boundary path of a face $\Pi_2$, etc. Suppose $(e_1)_- = o'$. Then let us split the edge $e_2$ into two edges by a new vertex $o''$ in the middle of $e_2$ and replace $e_1$ with a new edge $e'_1$ going from $o'$ to $o''$. Note that this transformation does not change the type of $M$. Indeed, since $e_1$ is an exterior edge, the only faces that are changed by this transformation are $\Pi_1, \Pi_2$, but the number of interior edges on $\partial(\Pi_j), j = 1, 2$, does not change (one of the two edges which are parts of the exterior edge $e_2$ is exterior in the new map and one is interior, and the new edge $e'_1$ is exterior as was $e_1$) (see Figure 5). Hence $\tau(\Pi_j), j = 1, 2$, does not change, and the type of the map does not change either. Also this transformation does not change the set of interior vertices of the map and their degrees. The degree of $\Pi_2$ increases by 1 (because of the new vertex $o'$). Thus both $-I_f$ and $|r|$ increase by 1, and $I^1_i$ does not change. Hence the value of $|r| - J$ does not change. The degree of the vertex $o$ decreases by 1 and the degree of the new vertex $o'$ is 3. Therefore, by doing this transformation, we will eventually get a map with the degrees of all exterior vertices at most 3 with the same path $y$ and the difference $|r| - J$ as for $M$.

![Figure 5. Step 3](image-url)

So we continue the proof under the additional assumption

\[(U_3) \text{ the degree of every exterior vertex is at most 3.} \]

**Remark 2.7.** Note that $(U_3)$ implies that every weakly exterior face of $M$ is exterior, i.e., every face that shares a vertex with $\partial(M)$ also shares an edge with $\partial(M)$.

**Step 4.** Assume there is a vertex $o$ of degree 2 on an exterior face $\Pi$. Let us join two edges incident with $o$ into one edge and remove the vertex $o$. This does not change $\tau(M)$ because only exterior edges and vertices are affected. Then the boundary $y$ of the interior of the map does not change, the contribution of $\Pi$ to $J$ decreases by 1, contributions of all other faces and vertices remain the same, and $|r|$ also decreases by 1, hence $|r| - |y| - J$ will not change. Hence,
(U₄) we can remove vertices of degree 2 of r (joining pairs of edges that share these vertices), provided the property \(d(\Pi) \geq p\) is preserved, and we can split edges of \(r\) by new vertices of degree 2 without changing \(|r| - |\sigma| - J\).

**Step 5.** Suppose an exterior face \(\Pi\) of \(M\) has boundary path of the form \(wv\), where \(u\) is a maximal subpath of the boundary path \(\partial\Pi\) contained as a subpath in the boundary path \(r\) of \(M\). Note that \(|w| \geq 2\), because we excluded cutting paths of length \(\leq 1\) by (U₂), and \(M\) has more than one face by (U₁). Also note that \(|u| > 0\) by Remark 2.7.

Suppose that \(w\) has an exterior vertex \(o\) which is not equal to the end vertices of \(w\). Then we add a vertex \(o'\) of degree 2 on \(u\) (using (U₄)) and connect \(o\) and \(o'\) by a new edge \(g\) cutting up \(\Pi\) into two faces of degrees \(d₁\) and \(d₂\), where \(d₁ + d₂ - 3 = d = d(M)\). For the new map \(M'\) (with parameters \(r', \eta', \eta\), etc.), we have \(|r'| = |r| + 1\), \(\eta' = \eta\). Instead of the face \(\Pi\) of curvature \(p - d\), we have two faces with curvatures \(p - d₁\) and \(p - d₂\). Hence \(I_f - I_f' = 3 - p\). Since the degrees of interior vertices were preserved, we have \(J - J = 3 - p\) and \((|r' - J|) - (|r| - J) = p - 2\). According to (U₄), the same difference \(p - 2\) has a map with additional vertices of degree 2 on \(u\). So we may assume that \(d₁, d₂ \geq p\).

Cutting along \(g\), we obtain new maps \(M₁\) and \(M₂\) with \(\tau(M_j) < \tau(M)\) \((j = 1, 2)\), since \(\Pi\) is subdivided into two faces with \(\tau(\Pi_j) < \tau(\Pi_j)\), \(j = 1, 2\), \(\tau(\Pi_j)\) is computed in \(M_j\). For the parameters \(x_j, \eta_j, \tau_j\), etc., of the maps \(M_j, j = 1, 2\), we have \(|x₁| + |x₂| = |x| + 3\), \(|\eta₁| + |\eta₂| = |\eta|\), and \(J₁ + J₂ = J' = J + 3 - p\). So, by induction on the type, we obtain \(|x| - |\sigma| - J - p = (|x₁| - |\eta₁| - J₁ - p) + (|x₂| - |\eta₂| - J₂ - p) - 3 + 3 \geq 0 + 0 - 3 + 3 = 0\), as desired (see Figure 6). Thus, we may assume that

(U₅) for every exterior face \(\Pi\) as above, the path \(w\) has no exterior vertices except its end vertices.

![Figure 6. Step 5](image_url)

**Step 6.** Properties (U₁)–(U₅) imply the property

(U₆) for every weakly exterior face \(\Pi\) of \(M\), we have \(\partial(\Pi) = eufv\) where \(u = u(\Pi)\) is the subpath of the boundary path of \(M\), \(|u(\Pi)| > 0\), and \(e, f\) are edges with exactly one exterior vertex while \(v = v(\Pi)\) has no exterior vertices.
Using the notation of property \((U_6)\), suppose now that \(|v(\Pi)| > 1\), i.e., \(v(\Pi) = v'v''\) with \(|v'|, |v''| > 0\). Let \(o\) be the last vertex of \(v'\). Then we add a new vertex of degree 2 on an edge of \(u\) (subdividing that edge into two edges) and add a new edge \(t\) connecting \(o\) and \(o'\) (see Figure 7). As a result, the face \(\Pi\) of degree \(d = d(\Pi)\) is subdivided into two faces: a face \(\Pi'\) of degree \(d'\) and a face \(\Pi''\) of degree \(d''\), where \(d' + d'' = d + 3\). Let \(M'\) be the new map with parameters \(\gamma', \gamma', I'_f, J', \) etc. We have \(\tau(M') < \tau(M)\) since \(\tau(\Pi') < \tau(\Pi)\) by \((U_6)\).

By \((U_4)\) we can add new vertices of degree 2 to \(\partial(\Pi'), \partial(\Pi'')\), so we can assume that \(d' = d(\Pi') \geq p, d'' = d(\Pi'') \geq p\). The contributions of \(\Pi'\) and \(\Pi''\) to \(I'_f\) are \(p - d'\) and \(p - d''\), respectively, while the contribution of \(\Pi\) to \(I_f\) was \(p - d\). So \(I_f - I'_f = p - (d' + d'' - 3) + (d' - p) + (d'' - p) = 3 - p\). The contribution of the interior vertex \(o\) to \(2I_o\) is greater than its contribution to \(2I'_o\) by \(\frac{2p}{q}\) since this vertex is incident to the new edge \(t\), i.e., \(2(I'_o) - 2I_o = 2p/q\). Thus \(J - J' = p - 3 - \frac{2p}{q} = -1\) because \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\). However we also have \(|x| - |x'| = 1\) since one edge is subdivided by the vertex \(o'\), and so \(|x'| - J' = x - J\). Thus we can assume

\((U)\) \(M\) satisfies \((U_6)\) and for every exterior face \(\Pi, |v(\Pi)| \leq 1\).

\[\text{Figure 7. Step 6}\]

**Step 7.** Now we consider the cases \(p = 3, 4, 6\) separately. Since \(|v(\Pi)| \leq 1\) by \((U)\), we have \(|u(\Pi)| \geq p - 3\), and by \((U_4)\) one may assume that every exterior face has degree \(p\) if \(p > 3\) and has degree 4 if \(p = 3\).

Since every exterior vertex has degree 2 or 3 (by \((U_5)\)), the difference \(|x| - |y|\) is not smaller than the sum \(S\) of \(|u(\Pi)| - |v(\Pi)|\) for all exterior faces \(\Pi\).

**Case** \(p = 4, q = 4\). In this case the degree of every exterior face is 4. Then a vertex of degree 2 can occur only on the path \(u(\Pi)\) for some exterior face \(\Pi\) with \(|v(\Pi)| = 0\). Let \(N\) be the number of vertices of degree 2 on \(\partial(M)\). Then the contribution of the face containing that vertex to the sum \(S\) is 2 and \(S \geq 2N\). The contribution of faces \(\Pi\) with \(|u(\Pi)| = |v(\Pi)| = 1\) is 0. The sum \(I_o^h\) of curvatures of exterior vertices is \(\left(\frac{4}{4}(4 - 2) - 1\right)N + \left(\frac{4}{4}(4 - 3) - 1\right)(|x| - N) = N\).

By Lemma 22, \(I_o^h + I_o^h + I_f = 4\). Thus \(N + I_o^h + I_f = 4\), hence \(2N + 2(I_o^h + I_f) = 8\), and \(2N \geq J + 8\) because \(J = -2I_o^h - I_f\), and \(I_f \leq 0\) by the assumption of the lemma. Therefore

\(|x| - |y| \geq J + 8 \geq J + p\).
Case $p = 6, q = 3$. Now every exterior face of $M$ has degree 6. Let $N_1$ be the number of weakly exterior faces (which, as we can assume by Remark 2.7, are exterior faces) $\Pi$ with $|\nu(\Pi)| = i$, $i = 0, 1$. Then $|\nu(\Pi)|$ is 4 or 3, respectively, and so $|x| \geq 4N_0 + 3N_1$ and $|y| \leq N_1$ since by property $(U)$ $\nu(\Pi) \leq 1$ for all weakly exterior faces $\Pi$. So $|x| - |y| \geq 4N_0 + 2N_1$.

Every exterior face has either three or two vertices of degree 2. It is easy to compute now that the sum $I_0^b$ of curvatures of the vertices of $x$ is

\[
(3N_0 + 2N_1) \left( \frac{6}{3} (3 - 2) - 1 \right) + (|x| - 3N_0 - 2N_1) \left( \frac{6}{3} (3 - 3) - 1 \right)
= -|x| + 6N_0 + 4N_1 = 2N_0 + N_1.
\]

Since, by Lemma 2.2, $I_0^b + I_1^i + I_f = 6$, we have $2N_0 + N_1 + I_0^i + I_f = 6$. Hence $J + p < 12 + J \leq 2(-I_0^i - I_f + 6) = 4N_0 + 2N_1 \leq |x| - |y|$, as required.

Case $p = 3, q = 6$. Let $N_1$ be the number of exterior faces $\Pi$ of degree 4 with $|\nu(\Pi)| = 2$, $|\nu(\Pi)| = 0$, and let $N_0$ be the number of exterior faces $\Pi$ of degree 4 with $|\nu(\Pi)| = |\nu(\Pi)| = 1$. Then we have $|x| \geq 2N_1 + N_0$, $|y| \leq N_0$, and $I_0^b = N_1 + \frac{1}{2}(|x| - N_1) = \frac{1}{2}(|x| + N_1)$ because the curvature of an exterior vertex of degree 2 (resp., 3) is $\frac{1}{3}(6 - 2) - 1 = 1$ (resp., $\frac{1}{2}$). Since $I_0^b + I_0^i + I_f = 3$, by Lemma 2.2 we get $|x| = 2I_0^b - N_1 = 2(-I_0^i - I_f + 3) - N_1$. Note that $-I_f \geq N_0 + N_1$ since each quadrangle contributes $-1$ to the sum $I_f$. Hence

$|x| = 2(-I_0^i - I_f + 3) - N_1 = J - I_f + 6 - N_1 \geq J + N_0 + N_1 + 6 - N_1 = J + N_0 + 6$.

Therefore, $|x| - |y| \geq J + N_0 + 6 - N_0 \geq J + 6 > J + p$, as desired. □

2.3. Adjustment. Note that it is enough to prove Theorem 2.1 for simple maps. Let $M$ be a simple $(p, q)$-map, $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Every face of $M$ of degree $\geq 2p$ can be subdivided by diagonals into faces of degrees $p + 1, \ldots, 2p - 1$. Every vertex $o$ of degree $d \geq 2q$ can be replaced by two (nearby) vertices $o', o''$ connected by an edge such that $d(o') + d(o'') = d(o) + 2$ and $d(o') = d(o'') = q + 1$. It can be done so that if $o$ is an exterior vertex, then exactly one of the vertices $o', o''$ is exterior in the resulting map. Each such transformation increases the total number of vertices and faces of negative $(p, q)$-curvature but it does not increase the number and the curvatures of the vertices and faces having nonnegative curvature. Since the sum $I_f + I_o$ of $(p, q)$-curvatures of all vertices and faces cannot exceed $p$ by Lemma 2.2 any sequence of subdivisions of vertices and faces as above terminates. Clearly, if $M$ is a $(p, q)$-map, then the new map $M'$ is again a $(p, q)$-map, it has nonsmaller area than $M$, the same perimeter, and the set of (simple) flat submaps in $M'$ is a subset of the set of (simple) flat submaps of $M$ because the subdivisions do not introduce new flat vertices or faces. The map $M'$ satisfies the condition that

(B) the degree of every face (every vertex) of the simple $(p, q)$-map $M'$ is less than $2p$ (resp., $2q$).

If we have a $(p, q)$-map $M$ with condition (B), then one can construct a new map $M'$ satisfying condition (B) where all faces (not just interior ones) have degrees at least $p$. Namely, one subsequently cuts out the exterior faces of degree less than $p$ (and also the edges containing the vertices of degree 1 if such edges appear). The perimeter of $M'$ is at most $(p - 1)n$, where $n$ is the perimeter of $M$, $\text{Area}(M') \geq \text{Area}(M) - n$, and the maps $M$ and $M'$ have the same flat submaps.

The map $M'$ is a $(p, q)$-map satisfying the additional condition that
(D) the degree of every face of $M'$ is $\geq p$.

Let us call a $(p,q)$-map with the additional assumptions (B) and (D) a \{p,q\}-map. It is easy to see that Theorem 2.4 follows from

**Theorem 2.8.** The area of an arbitrary simple \{p,q\}-map $M$ does not exceed \((\frac{3q}{2}+1)(r+p)n\), provided $M$ contains no simple flat submaps of radius greater than $r$.

Indeed, let $M'$ be the $(p,q)$-map obtained from a $(p,q)$-map $M$ satisfying the assumption of Theorem 2.4 and condition (B) after removing some exterior faces. By Theorem 2.8 we have $\text{Area}(M') \leq (\frac{3q}{2}+1)(r+p)(p-1)n$. Hence $\text{Area}(M) \leq (\frac{3q}{2}+1)(r+p)(p-1)n + n \leq \frac{q}{2}(q+1)(r+p)(p-1)n$.

2.4. **Connecting nonflat vertices and faces with the boundary.** Note that all nonflat faces and all interior nonflat vertices of a \{p,q\}-map $M$ have negative curvatures.

If $M$ contains nonflat faces or vertices, there exists a subgraph $\Gamma$ of the 1-skeleton of $M$ such that every nonflat face or vertex of $M$ can be connected with $\partial M$ by a path in $\Gamma$. We will assume that $\Gamma$ is chosen with minimal number of edges $D(M)$. Then $\Gamma$ will be called a connecting subgraph of $M$.

**Remark 2.9.** The minimality of $\Gamma$ implies that every vertex of $\Gamma$ can be connected in $\Gamma$ with $\partial M$ by a unique reduced path. It follows that $\Gamma$ is a forest, where every maximal subtree has exactly one vertex on the boundary $\partial M$.

We shall use the notation from section 2.2. Thus $r$ is the boundary path of $M$, $z$ is the union of the boundary paths of components of $M^0$, etc.

**Lemma 2.10.** We have $D(M) \leq (p-1)|x|$.

**Proof.** We shall induct on the area $m$ of $M$. The statement is true if $m = 1$ since in this case $D(M) = 0$.

Let $D_0 = D(M^0)$, and let $\Gamma_0$ be the corresponding connecting subgraph of the interior $M^0$. By the induction hypothesis, $D_0 \leq (p-1)|\eta|$.

Let $A_0$ (resp., $A$) be the number of nonflat faces and interior vertices in $M^0$ (in $M$). By Remark 2.9 $\Gamma_0$ has at most $A_0$ vertices on $\partial M^0$, and so one needs at most $A_0$ paths $\gamma_1, \gamma_2, \ldots$ to connect them with $\partial M$. Besides there are $A - A_0$ nonflat faces and interior vertices in $M$ which are not counted in $A_0$. One can connect them with $\partial M$ adding at most $A - A_0$ connecting paths $\eta_1, \eta_2, \ldots$ to obtain a graph $\Gamma'$ connecting all nonflat faces and interior vertices of $M$ with $\partial(M)$.

The lengths $|\gamma_i|$ and $|\eta_j|$ cannot exceed half of the maximum of the perimeters of faces, and so $|\gamma_i|, |\eta_j| \leq p-1$ by property (B). Therefore, $D(M) \leq D_0 + A(p-1)$.

Since every nonflat face (resp., nonflat interior vertex) has curvature at most $-1$ (resp., at most $-\frac{1}{2}$), we have $A \leq -I_f - 2I_v = J \leq |x| - |\eta| - p$ by Lemma 2.6.

Therefore,

$D(M) \leq D_0 + J(p-1) \leq (p-1)|\eta| + J(p-1) \leq (p-1)(|\eta| + (|x| - |\eta| - p)) < (p-1)|x|$.

\[ \square \]

2.5. **Cutting the map along its connecting subgraph and the proof of Theorem 2.8.** As before, $p,q$ are positive integers with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

**Lemma 2.11.** Let $M$ be a \{p,q\}-map of radius 0 (i.e., all vertices of $M$ are exterior) and perimeter $n > 0$. Then $\text{Area}(M) \leq \frac{q(n-2)}{2p}$.
Proof. Induction on the number of faces in $M$. The statement is easy to check if $M$ contains only one face. If $M$ contains more than one face, then $M$ has a cut vertex or cut edge. In each of the two cases, the cut vertex or the cut edge separates $M$ into two submaps $M_1$ and $M_2$ with perimeters $n_1$, $n_2$ such that $n_1 + n_2 \leq n + 2$. Therefore,

$$\text{Area}(M) = \text{Area}(M_1) + \text{Area}(M_2) \leq \frac{q(n_1 - 2)}{2p} + \frac{q(n_2 - 2)}{2p} \leq \frac{q(n - 2)}{2p}. \quad \square$$

Lemma 2.12. Let $M$ be a $\{p,q\}$-map. Then the sum $I_v^b$ of curvatures of exterior vertices satisfies $I_v^b \geq p$.

Proof. Indeed, we have $I_v^b + I_v^e + I_f = p$ by Lemma 2.2. Since $I_v^b \leq 0$ and $I_f \leq 0$, we have $I_v^b \geq p$. \square

Lemma 2.13. Let $M$ be a $\{p,q\}$-map with perimeter $n > 0$. Then the number $N$ of weakly exterior faces of $M$ is at most $\frac{2}{p} n - q$.

Proof. Let $o_1, \ldots, o_m (m \leq n)$ be the exterior vertices of $M$. Then

$$I_v^b = \sum_j \left( \frac{p}{q} (q - d(o_j)) - \mu(o_j) \right) = -\frac{p}{q} \sum_j d(o_j) + pm - n$$

since $\sum \mu(o_j) = n$. Therefore, we have

$$\sum_j d(o_j) = \frac{q}{p} \left( -I_v^b + pm - n \right) \leq -q + qm - \frac{q}{p} n$$

by Lemma 2.2. By Lemma 2.13 the number $N$ of weakly exterior faces in $M$ is at most $\sum_j d(o_j) - 2m$. Therefore,

$$N \leq -q + (q - 2)m - \frac{q}{p} n \leq -q + \left( q - 2 - \frac{q}{p} \right) n = \frac{q}{p} n - q$$

since $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. \square

Lemma 2.14. If the radius of a $\{p,q\}$-map $M$ is at most $r - 1$ and its perimeter is $n$, then the area of $M$ does not exceed $\frac{2}{p} n r$.

Proof. If $r = 1$, then this follows from Lemma 2.11 so let $r > 1$. Let $M^0$ be the interior of $M$. Its boundary $y$ is the union of the boundaries of the components $M_1^0, M_2^0, \ldots$ having radii $\leq r - 2$. So one may assume by induction on $r$ that $\text{Area}(M^0) < \frac{2}{p} (r - 1) n$, which does not exceed $\frac{2}{p} (r - 1) (n - p)$ by Lemma 2.6. It follows from Lemma 2.13 that $\text{Area}(M) < \frac{2}{p} (r - 1) (n - p) + \frac{2}{p} n - q < \frac{2}{p} n r$. \square

Proof of Theorem 2.8. Let $\Gamma$ be a connecting graph of $M$. Let $e$ be a nonoriented edge of $\Gamma$ with one vertex on $\partial M$. Then cutting $M$ along $e$, one obtains a $\{p,q\}$-map $M_1$ with perimeter $|e| + 2$, where all nonflat faces and vertices are connected with $\partial M_1$ by paths in the graph $\Gamma_1$, where $\Gamma_1$ is obtained from $\Gamma$ by removing the edge $e$. We can continue cutting this way along the edges of $\Gamma$, until we obtain a $\{p,q\}$-map $\bar{M}$ of perimeter $|\Gamma| + 2D(M) \leq (2p - 1)|\Gamma|$ (by Lemma 2.10), where every vertex and every face of curvature $> 0$ is (weakly) exterior. Thus every component of the interior $\bar{M}^0$ of $\bar{M}$ is a simple flat map (see Figure 8).
If \( \bar{M}_0 \) is empty, then the radius \( \bar{r} \) of \( \bar{M} \) is at most \( r - 1 \), since by (B) the degree of every exterior face in \( \bar{M} \) is less than \( 2/p \). Hence, by Lemma 2.14 for \( \bar{M} \) we have
\[
\text{Area}(M) = \text{Area}(\bar{M}) \leq \frac{q}{p} (2p - 1) n \leq \left( \frac{3q}{2} + 1 \right) (r + p) n
\]
since \( \frac{2}{p} (2p - 1) = \frac{3q}{2} + 1 \) by equality \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \), and the theorem is proved. \( \Box \)

If \( \bar{M}_0 \) is not empty, then again by (B), it has a component \( N \) of radius \( \bar{r}_0 \geq \bar{r} - p + 1 \). The map \( N \) is a simple flat submap of \( M \). Hence its radius \( \bar{r}_0 \) does not exceed \( r \). Therefore \( \bar{r} - p + 1 \leq r \) and \( \bar{r} \leq r + p - 1 \). By Lemma 2.14
\[
\text{Area}(M) = \text{Area}(\bar{M}) \leq \frac{q}{p} (2p - 1) (r + p - 1 + 1) n = \left( \frac{3q}{2} + 1 \right) (r + p) n.
\]

3. \((p, q)\)-Maps that are quasi-isometric to \( \mathbb{R}^2 \)

Recall that a metric space \( X \) with distance function \( \text{dist}_X \) is \((L, K)\)-quasi-isometric to a metric space \( Y \) with distance function \( \text{dist}_Y \), where \( L > 1, K > 0 \), if there exists a mapping \( \phi \) from \( X \) to \( Y \) such that \( Y \) coincides with a tubular neighborhood of \( \phi(X) \) and for every two vertices \( o_1, o_2 \) of \( X \) we have
\[
-K + \frac{1}{L} \text{dist}_X(o_1, o_2) < \text{dist}_Y(\phi(o_1), \phi(o_2)) < K + L \text{dist}_X(o_1, o_2).
\]

Two metric spaces \( X \) and \( Y \) are quasi-isometric if there is an \((L, K)\)-quasi-isometry \( X \to Y \) for some \( L \) and \( K \). This relation is reflexive, symmetric, and transitive.

In this section we consider infinite planar maps. Here such a map \( M \) is called proper if the support of \( M \) is the whole plane \( \mathbb{R}^2 \), and every disc on \( \mathbb{R}^2 \) intersects finitely many faces, edges, and vertices of \( M \). The metric on \( M \) is the combinatorial path metric on its 1-skeleton.

**Theorem 3.1.** Let \( M \) be a proper \((p, q)\)-map, where positive integers \( p \) and \( q \) satisfy \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \). Then the 1-skeleton of \( M \) is quasi-isometric to Euclidean plane if and only if \( M \) has a finite number of nonflat vertices and faces.

We will provide a proof for the case \( p = q = 4 \), the other two cases are left for the reader (see Remark 3.10).
3.1. The “if” part of Theorem 3.1 A corridor $B$ of $M$ is a finite sequence of faces, where any two consecutive faces share a gluing boundary edge, and two gluing edges of a face are not adjacent in the boundary path of the face. In detail, a corridor is a sequence

$$(e_0, \Pi_1, e_1, \Pi_2, \ldots, e_{t-1}, \Pi_t, e_t),$$

where $e_{i-1}$ and $e_i^{-1}$ are nonadjacent edges in the boundary of the face $\Pi_i$ for $i = 1, \ldots, t$ (see Figure 9).

The boundary of $B$ has the form $e_0q_0e_1^{-1}(q')^{-1}$, where the sides $q$ and $q'$ consist of nongluing edges of the faces $\Pi_1, \ldots, \Pi_t$ (see Figure 10).

Lemma 3.2. In the above notation, no vertex of a (4,4)-map is passed by a side $q$ or by $q'$ twice.

Proof. Assume that a corridor $B$ is a counterexample. Then we may assume that $q$ is a simple closed path bounding a submap $N$ of minimal possible area. So $N$ contains no faces from $B$. Every vertex of $q$, except for the initial (= terminal) vertex $o$ has degree at least 4 in $M$, and so its degree in $N$ is at least 3. This follows from the definition of a corridor: two gluing edges of a face in a corridor are not adjacent in the boundary path of the face. Thus, the only vertex that can give a positive contribution to the sum $I_f + I_v$ from Lemma 2.2 for the map $N$ is $o$. But this contribution is at most 1, and so we have $I_f + I_v \leq 1$, contrary to Lemma 2.2. 

Figure 9. A corridor

Figure 10. A corridor touching itself
Lemma 3.2 allows us to extend an arbitrary corridor infinitely in both directions:

\[(\ldots, e_{-1}, \Pi_0, e_0, \Pi_1, e_1, \Pi_2, \ldots, e_{t-1}, \Pi_t, e_t, \Pi_{t+1}, \ldots)\]

Here its sides are infinite simple paths subdividing the plane in two parts (because the map $M$ is proper)—these are infinite corridors. One can also consider semi-infinite corridors of the form

\[(e_0, \Pi_1, e_1, \Pi_2, \ldots, e_{t-1}, \Pi_t, e_t, \ldots)\]

Lemma 3.3. Let $M$ be the map from Theorem 3.1 with a finite set of nonflat faces and vertices. Then the 1-skeleton of $M$ is quasi-isometric to the 1-skeleton of a map with finitely many nonflat vertices and without nonflat faces.

Proof. Consider an infinite corridor $B = (\ldots, e_0, \Pi_1, e_1, \Pi_2, \ldots, e_{t-1}, \Pi_t, e_t, \ldots)$ containing a nonflat face $\Pi_i$. Since the number of nonflat faces in $M$ is finite, we can assume that all the faces $\Pi_{i+1}, \ldots$ are flat.

Let $p, p'$ be the two sides of $B$ so that each $e_i$ connects a vertex $o_i$ on $p$ with a vertex $o'_i$ on $p'$.

Since $d = d(\Pi_i) \geq 5$, without loss of generality we can assume that the subpath $w = w_0u$, where $u$ is one edge connecting $o''$ with $o_t$ and $|w| > 0$. Then we modify the faces in $B$ as follows: Replace the gluing edge $e_t$ by a new gluing edge $f_t$ connecting $o''$ and $o'_t$, and replace every gluing edge $e_s$ with $s > t$ by the new gluing edge $f_s$ connecting $o_{s-1}$ with $o'_s$ (see Figure 11). Then the degree of $\Pi$ decreases by 1 and the degrees of all other faces are preserved. To complete the proof by induction, it suffices to notice that the 1-skeleton of new map $M'$ is quasi-isometric to the 1-skeleton of $M$, since the distances between the vertices cannot increase/decrease more than two times when we pass from $M$ to $M'$.

Lemma 3.4. Let $B = (\ldots, e_0, \Pi_1, e_1, \Pi_2, \ldots, e_{t-1}, \Pi_t, e_t, \ldots)$ be an infinite corridor in $M$, where every face $\Pi_i$ has degree 4, and so $\Pi$ has the boundary of the form $e_{i-1} f_i(e_i)^{-1} g_i$, where $f_i$ and $g_i$ are edges. Excising the faces of $B$ from $M$ and identifying the edges $f_i$ and $g_i$ (for every $-\infty < i < \infty$), one obtains a new map $M'$ (see Figure 12). We claim that $M'$ is a (4,4)-map whose 1-skeleton is quasi-isometric to the 1-skeleton of $M$.

Proof. Every end vertex of $e_i$ has degree $\geq 4$ in $M$. So the same must be true in $M'$. Hence $M'$ is a (4,4)-map.

If two vertices can be connected in $M'$ by a path $p$ of length $m$, then their preimages in $M$ can be connected in $M$ by a path $q$ of length at most $2m + 1$ since $q$ can be constructed from the edges of $p$ and the gluing edges of $B$. Conversely, the distance between two vertices in $M'$ does not exceed the distance between them in $M$. The quasi-isometry of the 1-skeletons follows.

\[\square\]
Proof of the “if” part of Theorem 3.1 Suppose that $M$ has finitely many nonflat vertices and faces. By Lemma 3.3 we may assume that $M$ has no nonflat faces, and so every face has degree 4. Let two distinct nonflat vertices $o$ and $o'$ be connected by a shortest path $p$. Then we choose an edge $e$ on $p$ and consider an infinite corridor $B$, where $e$ is one of the gluing edges of $B$. Then the transformation $M \to M'$ from Lemma 3.4 decreases the sum of distances between nonflat vertices and replaces $M$ by a quasi-isometric map $M'$. So after a number of such transformations, we shall have a $(4, 4)$-map without nonflat faces and with at most one nonflat vertex $o$. We will use the same notation $M$ for it.

Let us number the edges $e_1, \ldots, e_k$ with initial vertex $o$ in clockwise order; so $o$ lies on the boundaries $e_1, f_1, g_1, e_{i+1}$ (indexes are taken modulo $k$) of $k$ quadrangles $\Pi_1, \ldots, \Pi_k$. Consider the semi-infinite corridors $B_1 = (e_1, \Pi_1, g_1, \Pi_1', \ldots)$ and $C_1 = (e_2, \Pi_1, f_1, \ldots)$ starting with the face $\Pi_1$. They define semi-infinite sides $q_1$ and $q_1'$ starting at $o$. Since $M$ is proper, the paths $q_1, q_2$ bound a submap $Q_1$ of the plane.

There is a semi-infinite corridor $C_1'$ starting with the second edge of $q_1'$ and the face $\Pi_1'$. This corridor has to share the whole side with $C_1$ since it is made of quadrangles and all the vertices, except for $o$, have degree 4. Similarly, the semi-infinite corridor $C_1''$ starting with the third edge of $q_1'$ is glued up to $C_1'$ along the whole side, and so on.

Therefore, $Q_1$ with its path metric is isometric to a standard quadrant of the square grid $\mathbb{Z}^2$ (see Figure 13).

Similarly, we have quadrants $Q_2, \ldots, Q_k$, where each $Q_i$ is bounded by semi-infinite paths $q_i$ and $q_i'$, and as above, we have $q_i' = q_i+1$ (indices modulo $k$). The 1-skeleton of every submap $Q_i$ is quasi-isometric to a quadrant on the Euclidean plane $\mathbb{R}^2$ with the Euclidean metric which, in turn, is quasi-isometric to a part $S_i$ of the plane bounded by two rays with common origin and angle $2\pi/k$ so that the union of all $S_i$ is the whole plane $\mathbb{R}^2$ (use polar coordinates). Combining all these quasi-isometries and using the fact that a quadrant of $\mathbb{R}^2$ is convex, we get a quasi-isometry between the 1-skeleton of $M$ and $\mathbb{R}^2$. □

3.2. The “only if” part of Theorem 3.1 Let $M$ be a proper $(4, 4)$-map having infinitely many nonflat vertices and faces. By contradiction, suppose that $M$ with its path metric $\text{dist}_M$ is $(L, K)$-quasi-isometric ($L > 1, K > 0$) to $\mathbb{R}^2$. If $\text{dist}_M(o_1, o_2) > c$ for some $c \geq KL$, then $\text{dist}_{\mathbb{R}^2}(\phi(o_1), \phi(o_2)) > c_0 = (c - KL)/L$. 


and so the growth of every \( c \)-separated set \( S \) of vertices of \( M \) is at most quadratic, that is the function \( \gamma_{S,o}(r) = |\{o' \in V' \mid \text{dist}_M(o,o') \leq r\}| \) is at most quadratic in \( r \).

The number of vertices in a submap \( M' \) of \( M \) will be denoted by \( \text{area}(M') \) (recall that \( \text{Area}(M') \) denotes the number of faces in \( M' \)).

We start with the following well known lemma:

**Lemma 3.5 ([7, Chapter V, Theorem 6.2]).** If \( N \) is a \((4,4)\)-map with perimeter \( n \), then \( \text{area}(N) \leq kn^2 \) for some constant \( k > 0 \). This also follows immediately from Lemmas 2.13 and 2.6 by induction on \( n \).

Now we are going to modify faces of high degree.

**Lemma 3.6.** There exists a map \( M' \) on the plane

1. with the same set \( V \) of vertices as \( M \);  
2. with an infinite set of nonflat vertices and faces;  
3. for a marked vertex \( o \in V \) and every \( o' \in V \), dist\(_M(o,o') = \text{dist}_{M'}(o,o') \);  
4. for arbitrary vertices \( o',o'' \), we have dist\(_{M'}(o',o'') \leq \text{dist}_{M}(o',o'') \);  
5. the degrees of all faces are at most 6.

**Proof.** Let \( \Pi \) be a face with \( d = d(\Pi) \geq 7 \), and let vertices \( o_1, \ldots, o_d \) be in clockwise order. Consider the difference \( f(i) = \text{dist}(o_i,o) - \text{dist}(o_{i+j},o) \) (indices modulo \( d \)). It is nonnegative if \( o_i \) is the farthest vertex from \( o \) among \( o_1, \ldots, o_d \), and it is nonpositive if \( o_i \) is the closest one. Since \( \text{dist}(o_m,o) - \text{dist}(o_{m+1},o) \leq 1 \), we have \( |f(m) - f(m+1)| \leq 1 \), and so there is \( i \) such that \( f(i) = |\text{dist}(o_i,o) - \text{dist}(o_{i+j},o)| \leq 1 \).

Let us connect vertices \( o_i,o_{i+j} \) by a new, diagonal edge \( e \) inside the cell \( \Pi \), so that \( \Pi \) is divided into two new cells \( \Pi', \Pi'' \) both of degree at least 4 and at least one of degree at least 5. This operation does not introduce any new vertices. Let

---

\[ 2^{\text{A set of points } S \text{ in a metric space } X \text{ is called } c\text{-separated if } \text{dist}_X(o_1,o_2) > c \text{ for every two distinct points } o_1,o_2 \in S.} \]
Let $M'$ be the new map on the plane. It is clear that properties (1) and (4) of Lemma 3.6 hold.

Let us show that the distance from every vertex $o'$ to $o$ in $M'$ is the same as in $M$ (property (3)). Let $g$ be a geodesic in $M'$ connecting $o'$ and $o$. If $g$ does not contain the new edge $e$, then the distance between $o'$ and $o$ did not change. So suppose that $g$ contains $e$. Without loss of generality we can assume that the vertices of $g$ in the natural order are $o', o, o, o, \ldots$. Since $g$ is a geodesic, $e$ appears in $g$ only once, and $\text{dist}_{M'}(o, o) = \text{dist}_{M}(o, o)$, and $\text{dist}_{M'}(o, o) = \text{dist}_{M}(o, o) + 1$. By the choice of the pair $(o, o_{i+1})$, $f(i) \in \{0, 1, -1\}$. Since $\text{dist}_{M}(o, o) \geq \text{dist}_{M'}(o, o)$, we can deduce that $\text{dist}_{M}(o, o) = \text{dist}_{M'}(o, o)$, so there exists a geodesic $g'$ in $M'$ connecting $o'$ and $o$ and avoiding $e$. Hence $\text{dist}_{M'}(o', o) = \text{dist}_{M}(o', o)$.

This implies that we can cut all faces of degree $\geq 7$ by diagonals into several parts so that the resulting map on $\mathbb{R}^2$ satisfies all five properties of the lemma. □

Lemma 3.6 implies that for the map $M'$ the growth function $\gamma_{S, o}$ of every $c$-separated set $S$ of vertices with respect to vertex $o$ is at most quadratic if $c$ is large enough (because every $c$-separated set $S$ of vertices in $M'$ is $c$-separated in $M$ by property (4), and the functions $\gamma_{S, o}(r)$ for $M'$ and $M$ are the same by property (3)). To obtain a contradiction with this quadratic growth, Lemma 3.6 allows us to assume from now on that the degrees of all faces in $M$ are at most 6.

**Lemma 3.7.** For every $r > 0$ there exists a simple submap $N = N(r)$ of $M$ containing the vertex $o$, such that $\text{dist}_{M}(o, \partial(N)) \geq r$ and the maximal distance (in $M$) from $o$ to an exterior vertex of $N$ is at most $r + 2$.

**Proof.** Let $N$ be the smallest (with respect to the length of the boundary) submap of $M$ containing all faces $\Pi$ of $M$ with $\text{dist}_{M}(o, \Pi) \leq r - 1$. Such submap $N$ exists since $M$ is locally finite. We claim that the boundary path of $N$ has no cut points. Indeed, let $o'$ be a cut point on $\partial(N)$, and $o'$ subdivides $N$ into two submaps $N_1, N_2$ containing faces, $N_1 \cap N_2 = \{o'\}$, $o \in N_1$. Suppose that $N_2$ contains a face $\Pi$ at distance (in $M$) at most $r - 1$ from $o$. Let $g$ be a geodesic connecting $o$ and $\Pi$ in $M$. Then every vertex on $g$ is at distance (in $M$) at most $r - 1$ from $o$. Hence every face of $M$ having a common vertex with $g$ is in $N$. Thus $g$ is a path in the interior of $N$. Since $g$ connects a vertex in $N_1$ with a vertex in $N_2$, $g$ must contain $o'$. Hence $o'$ is an interior vertex of $N$, a contradiction.

Since the boundary path $\partial N$ contains no cut points and has minimal length, it is simple.

There are no vertices $o' \in \partial(N)$ at distance (in $M$) at most $r - 1$ from $o$. Indeed, otherwise every face of $M$ containing $o'$ would be at distance $\leq r - 1$ from $o$, and would be contained in $N$, hence $o$ would be an interior vertex in $N$, a contradiction.

Suppose that $N$ contains an exterior vertex $o'$ at distance (in $M$) at least $r + 3$ from $o$. Then $o'$ belongs to an exterior face $\Pi$ of $N$. Since $d(\Pi) \leq 6$, we have $\text{dist}_{M}(o, \Pi) \geq r$. Therefore, if we remove $\Pi$ from $M$ together with the longest subpath of $\partial(\Pi)$ containing $o'$ and contained in $\partial(N)$, we get a smaller submap $N'$ of $N$ containing all faces of $M$ at distance $\leq r - 1$ from $o$, a contradiction. Hence $\text{dist}(o, o') \leq r + 2$ for every $o' \in \partial(M)$. □

Let $\Phi(r)$ be the number of vertices $o'$ of $M$ with $\text{dist}_{M}(o', o) \leq r$.

**Lemma 3.8.** The function $\Phi$ is superquadratic, i.e., $\lim_{r \to \infty} \Phi(r)/r^2 = \infty$. 


Proof. Let us denote by $\phi(r)$ the minimum of the numbers of vertices on the boundaries of the finite submaps $Q$ with the property that $Q$ is simple and $\text{dist}_M(o, \partial Q) \geq r$. For any $Q$ with this property, let $N$ be the component of the interior $Q^0$ containing $o$. Since every (exterior) face of $Q$ has degree at most 6, the boundary $\eta_N$ of $N$ satisfies inequality $|\eta_N| \geq \phi(r-3)$. If $f$ is the boundary path of $Q$, then by Lemma 2.6, $|f| \geq |\eta_N| + J + 4$, where $J = -I_f(Q) - 2I_f^0(Q)$. Since $J$ is not less than the number $K = K(Q)$ of nonflat faces in $Q$ plus the number of interior nonflat vertices in $Q$, we have $|f| > |\eta_N| + K \geq \phi(r-3) + K$, and so $\phi(r) > \phi(r-3) + K$.

If a nonflat face $\Pi$ (or vertex $o'$) lies in $M$ at a distance $\leq r-1$ from $o$, then it belongs to $Q$ (resp., it is interior in $Q$). Indeed, if $\Pi$ is not in $Q$ or $o'$ is not an interior vertex of $Q$, then any path connecting $\Pi$ or $o'$ with $o$ has to intersect $\partial Q$, which contradicts the property $\text{dist}_M(o, \partial Q) \geq r$. Hence, $K \geq \phi(r-1)$, where $\phi(r-1)$ is the number of nonflat vertices and faces of $M$ at the distance $\leq r-1$ from $o$. Therefore, $\phi(r) > \phi(r-3) + \psi(r-1)$.

Since $\psi(r) \to \infty$ as $r \to \infty$, we have $\phi(r)/r \geq \frac{1}{r} \sum_{0 \leq i < r} \psi(r-1-3i) \geq \psi([r/2]-1)/6 \to \infty$.

Since the boundaries of the maps $N(r)$ and $N(r')$ from Lemma 3.7 do not intersect if $|r-r'| \geq 3$, we have

\[ \Phi(r)/r^2 \geq \frac{1}{r^2} \sum_{0 \leq i \leq r/6} \phi(r-2-3i) \geq \frac{1}{6r} \phi([r/2] - 2) \to \infty. \]

\[ \Omega \]

Lemma 3.9. For an arbitrary integer $c \geq 1$, there is a superlinear function $\alpha_c(r) = \alpha_c(r)$ such that the boundary of every submap $N(r)$ satisfying the condition of Lemma 3.7 contains a c-separated set of vertices $S$ with $|S| \geq \alpha_c(r)$.

Proof. We may assume that $r > c$. The simple boundary path $f$ of $N(r)$ has winding number $\pm 1$ around the vertex $o$, and it is a sequence of edges, i.e., of subpaths of length $1 \leq c$. Therefore, there is the smallest $t \geq 1$ and the vertices $o_1, \ldots, o_t$ of $f$ such that $\text{dist}_M(o_i, o_{i+1}) \leq c$ (indices modulo $t$) and the geodesic paths $\bar{z}_i = o_i - o_{i+1}$ (in $M$) form a product $\bar{z} = \bar{z}_1 \cdots \bar{z}_t$ with nonzero winding number around $o$. (Self-intersections are allowed for $\bar{z}_i$, but no $\bar{z}_i$ goes through $o$ since $|\bar{z}_i| \leq c < r \leq \text{dist}_M(o, o_i)$, and so the winding number is well defined.)

Suppose there is a pair of distinct vertices $o_i, o_j$, where $i < j$ and $i - j \neq \pm 1$ (mod $t$), with $\text{dist}(o_i, o_j) \leq c$. Then a geodesic path $\bar{z}_i = o_i - o_j$ defines two new closed paths $\bar{z}' = \bar{z}_1 \cdots \bar{z}_j \bar{z}_j^{-1} \cdots \bar{z}_i$ and $\bar{z}'' = \bar{z}_1 \cdots \bar{z}_j^{-1} \bar{z}_i$ with numbers of factors less than $t$. Since one of the paths $\bar{z}'$ and $\bar{z}''$ has nonzero winding number with respect to $o$, this contradicts the minimality in the choice of $t$. Hence $\text{dist}(o_i, o_j) > c$, and so one can chose the set $S$ of cardinality $\geq (t-1)/2$.

Assume now that $t < \sqrt[4]{r^2 \Phi(r-c/2)}$, where $\Phi(r)$ is defined before Lemma 3.8. Note that $o$ belongs to a submap $L$ bounded by a simple closed path $\mathcal{W}$, which is a product of some pieces of the paths $\bar{z}_i$. Therefore $|\mathcal{W}| \leq |\bar{z}| \leq tc$, and by Lemma 3.5, area$(L) \leq k(ct)^2 \leq Dr \sqrt{\Phi(r-c/2)}$, where $D = kc^2$.

On the other hand, $\text{dist}_M(o', o) \geq r-c/2$ for every vertex $o'$ of $\mathcal{W}$ since $\bar{z}_i \leq c$ for every $i$. Therefore area$(L) \geq \Phi(r-c/2)$ by Lemma 3.6. We obtain the inequality $\Phi(r-c/2) \leq Dr \sqrt{\Phi(r-c/2)}$, which can hold only for finitely many values of $r$ since the function $\Phi$ is superquadratic by Lemma 3.8. Thus $t = t(r) \geq \sqrt{t^2 \Phi(r-c/2)}$ for every $r \geq r_0$, and since $|S| \geq (t-1)/2$, one can define $\alpha(r)$ to be equal to $\frac{1}{2}(\sqrt{t^2 \Phi(r-c/2)} - 1)$ if $r \geq r_0$ and $\alpha(r) = 1$ if $r < r_0$. \[ \square \]
Now we can prove that for any given integer \( c \geq 1 \), the map \( M \) contains an infinite \( c \)-separated set \( S \) of vertices which grows superquadratically with respect to the vertex \( o \), which would give the desired contradiction. The boundary \( \partial N(r) \) has a \( c \)-separated subset \( S_r \) with at least \( \alpha(r) \) vertices. Since the distance between \( \partial N(r) \) and \( \partial N(r') \) is greater than \( c \) for \( r - r' \geq c + 3 \), the union \( S(r) = S_r \cup S_{r-(c+3)} \cup S_{r-(2c+3)} \cup \cdots \) is \( c \)-separated and

\[
|S(r)|/r^2 \geq \frac{1}{r^2} \sum_{0 \leq i \leq \frac{r}{2(c+3)}} \alpha(r - i(c + 3)) \leq \frac{1}{2(c+3)r} \min_{0 \leq i \leq \frac{r}{2(c+3)}} \alpha(r - i(c + 3)) \to \infty
\]
as \( r \to \infty \), since \( r - i(c + 3) \geq r/2 \) and the function \( \alpha \) is superlinear by Lemma 3.9.

Remark 3.10. The proof of Theorem 3.1 for \((4, 4)\)-maps can be easily adapted for \((6, 3)\)- and \((3, 6)\)-maps. The “only if” part needs virtually no modification.

To prove the “if” part for \((3, 6)\)-maps by contradiction, again one can use Lemma 3.6 to uniformly bound the degrees of all faces. Then one can subdivide nonflat faces by diagonals and obtain a quasi-isometric \((3, 6)\)-map \( M' \), where all the faces have degree 3. If two distinct triangles of \( M' \) share an edge \( e \), we say that they form a diamond with the hidden edge \( e \). We can view diamonds as new faces, and build analogues of corridors made of diamonds, where the gluing edges of a diamond are not adjacent. The additional requirement is that the hidden edges of neighbor diamonds in a corridor have no common vertices (see Figure 14). The vertices on sides \( q \) and \( q' \) of a corridor \( B \) have degrees at most 4 in \( B \). Then the statement of Lemma 3.2 holds since every exterior vertex (except for one) of the submap \( N \) should have degree \( \geq 4 \). Hence, one obtains the notion of infinite and semi-infinite corridors. Lemma 3.4 reduces the task to a map \( M \) with single nonflat vertex, and the rest of the proof of Theorem 3.1 is as above: the quadrangles \( e_i f_i g_i e_{i+1} \) should be replaced by diamonds, and the corridors \( B, C_1, C'_1, \ldots \) are now built from diamonds. If one now erases the hidden edges of all these diamonds in the quadrant \( Q_1, \ldots \), then the obtained quadrants \( Q'_i \) are \((4, 4)\)-maps quasi-isometric to \( Q_i \). So our task is reduced to the case of \((4, 4)\)-maps.

The case of \((6, 3)\)-map \( M \) can be easily reduced to \((3, 6)\). For this goal, one bounds the degrees of faces as above, then chooses a new vertex inside of every face \( \Pi \) and connects it with the vertices of \( \partial \Pi \). The resulting map is a \((3, 6)\)-map which is quasi-isometric to \( M \) and has finitely many nonflat vertices and faces.

4. Maps with angle functions

Let \( M \) be a map with an angle function (for the definition, see section 1).

For every face \( \Pi \) (vertex \( o \)) we denote by \( \Sigma_{\Pi} \) (resp., \( \Sigma_o \)) the sum of the angles of the corners of \( \Pi \) (resp., corners at \( o \)). Note that if there are no corners at a vertex \( o \), then \( \Sigma_o = 0 \). We define the curvature \( \text{curv}(\Pi) \) of a face \( \Pi \) with degree \( d = d(\Pi) \)
as \( \Sigma_{\Pi} - \pi(d - 2) \). The curvature \( \text{curv}(o) \) of a vertex \( o \) is defined as \( (2 - \mu(o))\pi - \Sigma_o \), where, as before, \( \mu(o) \) is the multiplicity of \( o \) in the boundary path of \( M \).

We denote by \( I_f \) (by \( I_v \)) the sum of the curvatures of the faces (vertices) of a finite map \( M \). The following discrete analogue of the Gauss–Bonnet formula is well known, but we include its proof here anyway.

**Lemma 4.1.** Let a map \( M \) with angle function have at least one edge. Then \( I_f + I_v = 2\pi \).

**Proof.** Let \( V, E, \) and \( F \) be the numbers of vertices, nonoriented edges, and faces in \( M \), respectively, and let \( n \) be the perimeter of \( M \). It was observed in the proof of Lemma 2.2 that \( n = \sum_o \mu(o) \) (the sum over all vertices in \( M \)). Since \( \sum_{\Pi} d(\Pi) \) (the sum over all faces in \( M \)) is equal to the number of exterior edges of the faces in \( M \) plus twice the number of the interior edges in \( M \), we have \( 2E = \sum_{\Pi} d(\Pi) + n = \sum_{\Pi} d(\Pi) + \sum_o \mu(o) \). Hence,

\[
I_f + I_v = \sum_{\Pi} (2 - d(\Pi))\pi + \sum_o (2 - \mu(o))\pi + \left( \sum_{\Pi} \Sigma_{\Pi} - \sum_o \Sigma_o \right)
\]
\[
= \sum_{\Pi} 2\pi + \sum_o 2\pi - \pi \left( \sum_{\Pi} d(\Pi) + \sum_o \mu(o) \right) + 0
\]
\[
= 2\pi F + 2\pi V - 2\pi E = 2\pi. \quad \square
\]

In the next lemma, \( I_v^i \) (resp., \( I_v^e \)) is the sum of the curvatures of interior (exterior) vertices in \( M \).

**Lemma 4.2.** Let \( M \) be a map of perimeter \( n \geq 1 \) with angle function. Assume that the curvatures of the faces and of the interior vertices of a map \( M \) are nonpositive. Then \( n\pi \geq -I_f - I_v^i + 2\pi \).

**Proof.** On the one hand, it follows from the definition that

\[
I_v^i = \sum_{o \in \partial M} \left( (2 - \mu(o))\pi - \Sigma_o \right) \leq 2n\pi - n\pi - \sum_{o \in \partial M} \Sigma_o \leq n\pi.
\]

On the other hand, Lemma 4.1 gives us \( I_v^i + I_v^e + I_f = 2\pi \). Therefore, we have \( n\pi + I_f + I_v^i \geq 2\pi \), as required. \( \square \)

Recall that a map \( M \) is called \((\delta, b)\)-map for some \( \delta > 0 \) and a natural number \( b > 0 \) if

1. the curvature of every nonflat vertex or face does not exceed \(-\delta \), and
2. the degree of every face and of every vertex in \( M \) is at most \( b \).

We denote by \( B(d, o) \) the ball of radius \( d \) centered at \( o \) in a graph \( G \), i.e., the set of vertices \( o' \) of \( G \) such that \( \text{dist}(o', o) \leq d \).

The following lemma is well known and obvious.

**Lemma 4.3.** The inequality \( |B(d, o)| \leq b^d + 1 \) holds for any graph where degrees of all vertices are at most \( b \) (hence for \((\delta, b)\)-maps).

**Proof of Theorem 1.3.** Let \( V \) be the set of vertices of \( M \) which are either exterior or nonflat or belong to a nonflat face. From the \((\delta, b)\)-condition and Lemma 4.2 one deduces that

\[
|V| \leq n + \delta^{-1}(-I_v^i) + \delta^{-1}b(-I_f) \leq n\pi + \delta^{-1}bn\pi = (\delta^{-1}b + 1)n\pi.
\]
For an arbitrary vertex $o \in \mathcal{V}$, we consider the ball $B(o, r)$. By the assumption of the theorem, every vertex $o'$ of $M$ belongs in one of these balls. Therefore by Lemma 4.3, $\text{area}(M) \leq |\mathcal{V}| \times (b' + 1) \leq (1 + \delta^{-1}b)n\pi(b' + 1)$. Since every vertex belongs to the boundaries of at most $b$ faces, the inequality $\text{Area}(M) \leq Ln$ follows, provided $L \geq \pi b(1 + \delta^{-1}b)\sum_{b'} + 1)$. $\square$

**Remark 4.4.** Theorem 1.5 generalizes Theorem 1.1. Indeed, it is enough to establish Theorem 1.1 for simple maps. As explained in section 2.3, every simple $(p, q)$-map $M$ can be modified so that the new $(p, q)$-map $M'$ satisfies condition (B) from section 2.3. Condition (B) implies condition (2) above with $b \geq 11$. Moreover, the area of $M'$ is not smaller than the area of $M$, the perimeter of $M'$ is the same as the perimeter of $M$, and the maximal distance from a vertex to an exterior vertex or nonflat vertex or face in $M'$ does not exceed that for $M$. The $(p, q)$-map $M'$ can be naturally viewed as a map with angle function which assigns the angle $\frac{\pi(d-2)}{d}$ to every corner of a $d$-gon face. Again, condition (B) implies condition (1) above with $\delta > \frac{\pi}{21}$. It remains to note that the function $L(r)$ in Theorem 1.5 is exponential, as in Theorem 1.1.

**Acknowledgments**

The authors thank the referee for many useful remarks.

**References**


**Department of Mathematics, Vanderbilt University, Nashville, Tennessee; and Department of Higher Algebra, MEHMAT, Moscow State University, Moscow, Russia**

*Email address: alexander.olshanskiy@vanderbilt.edu*

**Department of Mathematics, Vanderbilt University, Nashville, Tennessee**

*Email address: m.sapir@vanderbilt.edu*