A NEW DEFINITION OF THE GENERAL ABELIAN LINEAR GROUP

BY

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1. We may give a striking definition of the general Abelian group, making use of the fruitful conception of the "compounds of a given linear homogeneous group," introduced in recent papers by the writer.† In § 3 we determine the multiplicity of the isomorphism of a given linear homogeneous group to its compound groups. This result is applied in § 4 to show the simple relation of the Abelian group to the general linear homogeneous group in the same number of variables. In § 5 it is shown that the simple groups of composite order which are derived from the decompositions of the quaternary Abelian group and the quinary orthogonal group, each in the $GF'[p^n], p > 2,$ are simply isomorphic. The investigation affords a proof of the simple isomorphism between the corresponding ten-parameter projective groups without the consideration of their infinitesimal transformations.

2. It will be convenient to introduce a notation more compact than that usually employed‡ for the substitutions of the general Abelian group $A_{2m, p^n}$ on $2m$

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* Presented to the Society at the Columbus meeting, August 26, 1899, and in abstract at the meeting of February, 1899, under the title, Concerning the Abelian and hypoabelian groups. Received for publication August 26, 1899.

† Concerning a linear homogeneous group in $C_n,$ $q$ variables isomorphic to the general linear homogeneous group in $m$ variables, Bulletin, Dec., 1898.


Instead of considering the cogredient linear substitutions leaving invariant (up to a factor $a$) the usual bilinear function it is convenient to consider here the substitutions $A$ leaving invariant the function

$$\psi \equiv \sum_{j=1}^{m} \frac{\xi_{uj} - 1}{\xi_{uj}^{p^n}} - \frac{\xi_{uj} - 1}{\xi_{uj}^{p^n}},$$

The conditions that $A$ shall leave $\psi$ invariant are seen to be (1). The hyperabelian group of linear homogeneous substitutions in the $GF[p^n]$ on $2m$ indices which leave $\psi$ invariant has been studied by the writer in an article presented to the London Mathematical Society.
indices in the Galois field of order \( p^n \). The conditions that a substitution

\[
\xi_i = \sum_{j=1}^{2m} a_{ij} \xi_j \quad \text{(i = 1, \ldots, 2m)}
\]

shall be Abelian are the following:

\[
\sum_{i=1}^{2m} \begin{vmatrix} a_{2j-1, i} & a_{2j-1, k} \\ a_{2j, i} & a_{2j, k} \end{vmatrix} = a \epsilon_{ik} \quad \text{(i, k = 1, \ldots, 2m; i < k)}
\]

where \( a \) is a parameter \( \neq 0 \) depending upon the particular substitution \( A \) and where every \( \epsilon_{ik} = 0 \) unless \( k = i + 1 \) = even, when

\[
\epsilon_{2i-1, 2i} = 1 \quad \text{(i = 1, \ldots, m)}.
\]

The second compound \( C_{2m, 2} \) of the \( 2m \)-ary group \( A_{2m, p^n} \) is formed by the substitutions

\[
Y_{i_1, i_2} = \sum_{j_1 < j_2} \begin{vmatrix} a_{i_1, j_1} & a_{i_1, j_2} \\ a_{i_2, j_1} & a_{i_2, j_2} \end{vmatrix} Y_{j_1, j_2} \quad \text{((i_1, i_2 = 1, \ldots, 2m; i_1 < i_2))}.
\]

We readily verify that the group \( C_{2m, 2} \) has the relative invariant

\[
Z = \sum_{i=1}^{m} Y_{2i-1, 2i}.
\]

Indeed, in virtue of the relations (1), we have, on applying to \( Z \) the substitution (2),

\[
\sum_{i=1}^{m} Y_{2i-1, 2i} = \sum_{j_1, j_2} \left\{ \sum_{i=1}^{m} \begin{vmatrix} a_{2i-1, j_1} & a_{2i-1, j_2} \\ a_{2i, j_1} & a_{2i, j_2} \end{vmatrix} \right\} Y_{j_1, j_2} = a \sum_{i=1}^{m} Y_{2i-1, 2i}.
\]

Inversely, if the substitution (2) multiply the function \( Z \) by a constant \( a \), the relations (1) hold true. We have proved the result:

**Theorem.**—The general Abelian group \( A_{2m, p^n} \) is the largest \( 2m \)-ary linear homogeneous group whose second compound has as a relative invariant the linear function \( Z \).

3. To establish the more important theorem of § 4, we determine the multiplicity of the isomorphism of a given \( m \)-ary linear homogeneous group \( G_m \) to its \( q \)-th compound \( C_{m, q} \), supposing that \( q < m \). To the substitution

\[
(a_q) \quad \text{(i, j = 1, \ldots, m)}
\]

of \( G_m \) there corresponds the following substitution of \( C_{m, q} \):

* We employ Sylvester's umbral notation for determinants.
Let $j$ be an integer such that $q < j \equiv m$. Consider the matrix $J$ of certain coefficients of the substitution $[a_j]_q$, viz.,

\[
\begin{vmatrix}
1 & 2 & \cdots & q - 1 & q \\
1 & 2 & \cdots & q - 1 & j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & \cdots & q - 1 & q - 1 \\
\end{vmatrix}
- \begin{vmatrix}
1 & 2 & \cdots & q - 1 & q \\
1 & 2 & \cdots & q - 1 & j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & \cdots & q - 1 & q - 1 \\
\end{vmatrix}
\cdots \\
\begin{vmatrix}
(-1)^q & 1 & 2 & \cdots & q - 1 & j \\
1 & 2 & \cdots & q - 1 & j \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 2 & \cdots & q - 1 & j \\
\end{vmatrix}
\cdot (-1)^{q+1}
\begin{vmatrix}
2 & 3 & \cdots & q - 1 & j \\
1 & 2 & \cdots & q - 1 & j \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & \cdots & q - 1 & j \\
\end{vmatrix}
\]

Consider also the matrix $A$ of determinant $\Delta$,

\[
A = \begin{bmatrix}
a_{ij} & a_{iq} & \cdots & a_{jq}
\end{bmatrix}
\]

The composition of the matrices $J$ and $A$ gives the result

\[
JA = \begin{bmatrix}
\Delta & 0 & \cdots & 0 \\
0 & \Delta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Delta \\
\end{bmatrix}
\]

We seek those substitutions of $G_m$ which correspond to the identity in $C_{m,q}$. Suppose, therefore, that $[a_j]_q$ reduces to the identical substitution, so that the matrix $J$ is the identity.

In this case we have

\[
\Delta^{q+1} = \Delta, \ a_{ik} = 0, \ a_{ii} = \Delta \quad (i, k = 1, 2, \ldots, q, j; i \neq k).
\]

Taking in turn $j = q + 1, q + 2, \ldots, m$, we have the result

\[
(a_q) = \begin{bmatrix}
\Delta & 0 & \cdots & 0 \\
0 & \Delta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Delta \\
\end{bmatrix}
\]

Hence $\Delta = 0$ and therefore $\Delta^q = 1$. Inversely, every such substitution of $G_m$ corresponds to the identity in $C_{m,q}$.
Theorem.—The continuous group of all $m$-ary linear homogeneous substitutions is $(q, 1)$-fold isomorphic to its $q^h$ compound ($q < m$).

For linear substitutions in the $GF[p^n]$, we have

$$\Delta'^t = 1, \quad \Delta'^{p^n-1} = 1.$$ 

Thus we have the analogous.

Theorem.—The group of all $m$-ary linear homogeneous substitutions in the $GF[p^n]$ is $(g, A)$-fold isomorphic to its $q^h$ compound, $g$ being the greatest common divisor of $q$ and $p^n - 1$.

4. From the results of § § 2–3, we derive immediately the

Theorem.—According as $p = 2$ or $p > 2$, the general Abelian group $A_{2m, p^n}$ is holoedrically or hemiedrically* isomorphic to that subgroup of the second compound of the general $2m$-ary linear homogeneous group in the $GF[p^n]$ which has as a relative invariant the linear function $Z$.

The writer has shown (Bulletin, l. c.) that this second compound leaves invariant the Pfaffian

$$F_{2m} = [1, 2, \cdots, 2m].$$

Hence $A_{2m, p^n}$ is isomorphic to a linear homogeneous group in $m (2m - 1)$ variables $Y_0$ with coefficients in the $GF[p^n]$ and having as relative invariants the functions

$$F_{2m}'$, $Z = \sum_{i=1}^{2m} Y_i^{2i-1} x_i$$

To the subgroup* of the latter which leaves these functions absolutely invariant there corresponds a self-conjugate subgroup of $A_{2m, p^n}$, which leaves the customary bilinear function absolutely invariant. This subgroup, containing only substitutions of determinant $\pm 1$, may be designated as the special Abelian group $A_{2m, p^n}'$. It has the self-conjugate substitution which changes the signs of all the $2m$ indices. Except for $(2m, p^n) = (2, 2), (2, 3)$ and $(4, 2)$, the quotient group $H_{2m, p^n}$ is simple.†

5. Theorem.—For $p > 2$, the simple group $H_{4, p^n}$, having the order

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* Since the Abelian group contains the substitution $e_i = -e_i (i = 1, 2, \cdots, 2m)$.
† This group has been studied by the writer in the Proceedings of the London Mathematical Society, l. c., §§ 22–33.
‡ Dickson, A triply infinite system of simple groups, The Quarterly Journal of Mathematics, July, 1897; ibid., April, 1899, for the cases $p = 2, 2m = 4, n > 1$, previously unconsidered.
\( \frac{1}{2}(p^n - 1)(p^{2n} - 1)p^n, \)

is simply isomorphic to the simple subgroup of equal order of the quinary orthogonal group in the GF\([p^n]\).

On introducing the invariant \( Z \equiv Y_{12} + Y_{24} \) as a new variable in place of \( Y_{34} \), the general substitution [see (2)] of the second compound of \( A'_{4,p^n} \) becomes, for \( p > 2 \):

\[
\begin{align*}
Y_{12} - \frac{1}{2} Z &= 2 \begin{pmatrix} 2 & 12 & -1 & 12 & 12 & 12 \end{pmatrix} \\
Y_{13}' &= 2 \begin{pmatrix} 2 & 12 & -1 & 13 & 14 \end{pmatrix} \\
Y_{14}' &= 2 \begin{pmatrix} 2 & 12 & -1 & 14 & 14 \end{pmatrix} \\
Y_{23}' &= 2 \begin{pmatrix} 2 & 12 & -1 & 23 & 24 \end{pmatrix} \\
Y_{24}' &= 2 \begin{pmatrix} 2 & 12 & -1 & 24 & 24 \end{pmatrix}
\end{align*}
\]

It is therefore a substitution on five indices leaving absolutely invariant the function

\[ \varphi \equiv (\frac{1}{2} Z)^2 - [1234] \equiv (Y_{12} - \frac{1}{2} Z)^2 + Y_{13}Y_{24} - Y_{14}Y_{23}. \]

This second compound is simply isomorphic to \( H_{4,p^n} \). Indeed, the former is hemiedrically isomorphic to \( A'_{4,p^n} \) by §3; while to the substitution changing the sign of every index there corresponds the identity in the second compound.

By a simple transformation of indices* the function \( \varphi \) can be given the form

\[ \sum_{i=1}^{5} x_i^2. \]

Hence the second compound is simply isomorphic to a subgroup \( O_{5,p^n} \) of the total orthogonal group \( O \) of determinant unity. From the result of §16 of the paper in the Proceedings of the London Mathematical Society, above cited, it follows that \( O_{5,p^n} \) does not contain the substitution which extends the simple subgroup of \( O \) of order

*See the first pages of the article, Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant, American Journal of Mathematics, July, 1899.
to the total group $O$. Hence $O_{n,p^n}$ is this simple subgroup.

This investigation also proves the theorem due to Lie: The projective group of a linear complex in space of three dimensions is isomorphic to the projective group of a non-degenerate surface of the second order in space of four dimensions, each group having ten parameters.

UNIVERSITY OF CALIFORNIA, August 20, 1898.