DETERMINATION OF AN ABSTRACT SIMPLE GROUP
OF ORDER $2^7 \cdot 3^6 \cdot 5 \cdot 7$

HOLOEDRICALLY ISOMORPHIC WITH A CERTAIN ORTHOGONAL GROUP
AND WITH A CERTAIN HYPERABELIAN GROUP*

BY

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1. Among the known simple groups† occur an orthogonal group and a hyper-
abelian group of the same order $2^7 \cdot 3^6 \cdot 5 \cdot 7$. They are shown to be holoe-derically
isomorphic in this paper. We first determine in §§ 2–14 an abstract group $\Gamma$
(§§ 4, 2) simply isomorphic with the orthogonal group. This is accomplished
by means of a rectangular array, a direct method of procedure employed by
the writer in two recent papers in the Proceedings of the London Mathe-
matical Society (vol. 31, p. 30; vol. 31, p. 351).

§§ 2, 3. Rectangular array for the orthogonal group $O_{6,3}$ with respect to
the subgroup $O_{5,3}$.

2. The general orthogonal group of senary linear substitutions of modulus 3
having determinant unity has a subgroup $H$ of index two.‡ The group $H$ has
as maximal invariant subgroup the group of order two generated by the substi-
tution $G$ which changes the signs of the six indices. The quotient group is a
simple group $O_{5,3}$ of order $2^7 \cdot 3^6 \cdot 5 \cdot 7$. The substitutions of $H$ which affect
only the indices $\xi_1, \ldots, \xi_5$ form a simple group $O_{5,3}$ of order $2^7 \cdot 3^6 \cdot 5$ and of
index $2 \cdot 126$ under $H$.

In a paper communicated December 28, 1899, to the London Mathematical
Society, the writer has shown that $O_{5,3}$ is simply isomorphic with the abstract
group $G$ generated by the operators $E_1$, $E_2$, $E_3$, $B_1$, $W$ subject to the genera-
tional relations:

(1) $E_1^3 = E_2^2 = E_3^2 = B_1^2 = W^3 = I$,
(2) $(E_1E_2)^3 = (E_2E_3)^3 = (B_1E_1)^3 = (E_1E_3)^2 = (B_1E_3)^2 = I$,
(3) $WE_1 = B_1E_2E_1W$, $WE_2 = B_1W$, $WB_1 = B_3E_2W$.

* Presented to the Society (Chicago) April 14, 1900. Received for publication January 15, 1900.
† See the list of known simple groups in the Bulletin of the American Mathematical Society for July, 1899.

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(4) \[ WB_4 = B_4 B_2 E_2 E_1 W_1, \]
(5) \[ WE_3 E_2 E_1 W E_3 = E_1 E_2 E_1 W E_3 E_2 E_1 W, \]
where for brevity we have set

(6) \[ B_2 = E_1 B_1 E_1^3, \quad B_3 = E_2 E_1 E_2^3 B_1 E_1 E_1^2 E_1, \quad B_4 = E_2 E_3 B_3 E_2 E_2. \]

From the above relations we derive the following:

(7) \[ E_2 E_1 W = B_3 W E_1 = W E_2 E_1, \]
(8) \[ W E_2 = E_2 W B_1, \quad E_1 B_1 B_2 = B_1 E_1. \]

Indeed, we have

\[ WE_2 = WE_2 E_2 E_1 E_2 = E_2 E_1 E_2 E_1 W E_2 = E_1 E_2 E_1 W B_1, \]
\[ E_1 B_1 B_2 = E_1 B_1, \quad E_1 B_1 E_1^2 = (E_1 B_1)^3 B_1 E_1 = B_1 E_1. \]

The isomorphism is defined by the following correspondences:

(9) \[ E_1 \sim E_1' = (\xi_1 \xi_2 \xi_3), \quad E_2 \sim E_2' = (\xi_1 \xi_2) (\xi_3 \xi_5), \quad E_3 \sim E_3' = (\xi_1 \xi_2) (\xi_4 \xi_5), \]
(10) \[ W \sim W' = \left\{ \begin{array}{l} \xi_1' = \xi_1 - \xi_2 - \xi_3 - \xi_4, \\
\xi_2' = \xi_1 - \xi_2 + \xi_3 + \xi_4, \\
\xi_3' = \xi_1 + \xi_2 - \xi_3 + \xi_4, \\
\xi_4' = \xi_1 + \xi_2 + \xi_3 - \xi_4 \end{array} \right\}, \]
(11) \[ B_1 \sim C_1 C_2, \quad B_2 \sim C_2 C_3, \quad B_3 \sim C_3 C_1, \quad B_4 \sim C_4 C_5, \]

where \( C_i \) denotes the orthogonal substitution changing the sign of \( \xi_i \).

3. The group \( O_{6,3} \) is extended to the orthogonal group \( H \) by the substitution

\[ F' = (\xi_1 \xi_2) (\xi_3 \xi_5). \]

The substitutions of \( H \) replace \( \xi_6 \) by every one of the \( 2 \cdot 126 = 3^3 + 2^2 \) linear functions

\[ (f') \sum_{i=1}^{6} a_i \xi_i \quad \quad \quad \left[ \sum_{i=1}^{6} a_i^2 = 1 \pmod{3} \right]. \]

It follows that all the substitutions of \( H \) are given by the formula

\[ S_i O_{6,3} \quad (i = 1, 2, \cdots, 2 \cdot 126), \]

where the \( 2 \cdot 126 \) substitutions \( S_i \) replace \( \xi_6 \) by the \( 2 \cdot 126 \) distinct linear functions \((f')\). In the quotient group \( O_{6,3} \), \( S_i \) and \( CS_i \equiv S_i C \) become identical. Denote by \( S_i' \) \((i = 1, \cdots, 126)\) the corresponding distinct substitutions of \( O_{6,3} \).

A rectangular array of the substitutions of \( O_{6,3} \) is therefore given by the formula:

\[
S'_i O_{6,3} \quad (i=1, 2, \ldots, 126).
\]

To determine the \( S'_i \), we note that the linear functions \((f')\) are of the forms:

\[
\pm \xi_1, \pm \xi_2, \pm \xi_3 \pm \xi_4, \pm \xi_5.
\]

Since \(-f\) is derived from \(f\) by the substitution \(C\), we take only one of each pair \(\pm f\) in determining the \(S'_i\). For six of the \(S'_i\) we may take

\[
(a) \quad I, \ F', \ E'_2F', \ E'_1E'_2F', \ E'_2E'_1F', \ E'_3E'_2E'_1F',
\]

which replace \(\xi_0\) by \(\xi_0, \xi_1, \xi_2, \xi_3, \xi_4\), respectively.

The substitution \(W'F'\) replaces \(\xi_0\) by \(w = \xi_1 - \xi_2 - \xi_3 - \xi_4\). Consider the 2⁵ products \(K'\) of an even number of the \(C_i (i = 1, \ldots, 6)\). If a particular \(K'\) replace \(w\) by \(w\), the four substitutions \(KK', \ C_iC_iK', \ C_iC_3C_5C_4K'\) replace \(w\) by either \(w\) or \(-w\). Hence, of the 2⁵ substitutions \(K'\), we need only consider 2³ representatives, as

\[
\begin{align*}
K'_1 &= I, \quad K'_1 = C_iC_i (i=2, \ldots, 5), \quad K'_6 = C_iC_3C_4C_5, \\
K'_2 &= C_iC_2C_4C_5, \quad K'_3 = C_iC_3C_1C_4.
\end{align*}
\]

The substitutions \(K'_j W'F'\) may therefore be taken as eight of the \(S'_i\), distinct from the above six.

Using for the moment the notation 1234 for \(\xi_1 + \xi_2 + \xi_3 + \xi_4\), we find that the substitutions

\[
I, \ E'_3, \ E'_1E'_3, \ E'_1E'_2E'_3, \ E'_1E'_2F', \ F', \ E'_2F', \ E'_1E'_2F', \ E'_1E'_2E'_1F',
\]

respectively replace 1234 by 1234, 1235, 1245, 1345, 2345, 1346, 1246, 1236, 1356, 2456, 1256, 2356, 3456, 1456, giving each of the 15 combinations of 1, 2, 3, 4, 5, 6 four at a time.

It follows that we may take as our 126 substitutions \(S'_i\) the six substitutions \((a)\) and the 120 obtained by multiplying the 16 substitutions \((b)\) on the right hand by \(K'_j W'F'\) \((j = 1, 2, \ldots, 8)\). We have therefore explained the origin of the table given in § 7. Indeed, from that table we obtain a rectangular array for \(O_{6,3}\) by replacing the group \(G\) by \(O_{3,3}\) and accenting all the letters.

\[\text{§§ 4–14. Determination of the abstract group } \Gamma.\]

4. Consider the abstract group \(\Gamma\) obtained by the extension of \(G\) by an operator \(F\) subject to the relations:

\[
F^2 = I, \quad (E_iF)^3 = I,
\]

\[
B_iF = FB_iB_i,
\]

We readily verify that the generators $E', E_2', E_3'$, $B', W', F'$ of $O_{6,3}$ satisfy the relations (12) ... (18).* Note that $V \sim V' \equiv (t_2 t_3)(t_5 t_6)$. The order of $\Gamma$ is therefore at least as great as the order of $O_{6,3}$. To complete the proof of their simple isomorphism it remains only to prove that the order of $\Gamma$ is at most as great as the order of $O_{6,3}$.

5. From the relations (1), ..., (8), (12), ..., (18), we proceed to derive a number of relations needed below. From (14) and (15) we find

$$E_3 E_1 E_2 E_1 F = F E_2 E_2', E_1' E_2 = F E_3 E_2.$$  

Taking the reciprocal of (19) and multiplying on the right and left by $F$, we get

$$FF_2 E_3 = E_1' E_2' E_3' E_3 F = E_2' E_3' E_3 F.$$  

From (20) we find

$$E_1 = E_2 F E_3 E_2' E_3 E_2.$$  

From (12) and (14) we get

$$BE_1 E_2 E_1 F = F E_1 E_2; E_1' E_2 F = F E_1 E_1 F E_2 E_1 = E_1' E_2 E_1.$$  

From (12) and (7) we find

$$E_3 E_2 E_1 F E_1 E_2 E_3 = E_3 E_2 E_1 F E_3 E_2 E_3 = E_3 F E_1 E_2 E_3 = E_3 F E_1 E_2 E_3,$$

$$E_1 V = E_3 E_2 E_1 F E_3 E_2,$$

$$E_1 V E_1 V \equiv (E_3 E_2 E_1 F E_1 E_2 E_3)^2 = I.$$  

Taking the reciprocal of (13) and applying $B_i' = I$, $B_i B_i = B_i B_i$, we find

$$F B_i = B_i B_i F = B_i F B_i B_i = B_i F B_i B_i;$$

$$B_i F = F B_i.$$  

*In regard to the relation corresponding to (18) it should be remarked that for orthogonal substitutions there occurs an additional factor $C$ in one member.
Transforming (13) by $E_2 E_1$, which by (15) transforms $F$ into itself, we get

$$B_1 B_2 F = FB_1 B_2 B_4.$$  

By (13), $B_1 B_2 F = B_2 FB_1 B_4$. Hence from (26) we find

$$B_2 F = FB_2.$$  

Applying (1), (2), (6), (8) and (14), we find

$$E_2 E_1 B_4 F = B_4 B_2 B_3 E_2 E_1 F = B_4 B_2 B_3 E_2 E_1 F = B_4 B_2 B_3 E_2 E_1 F = B_4 B_2 B_3 E_2 E_1 F = B_4 B_2 B_3 E_2 E_1 F = B_4 B_2 B_3 E_2 E_1 F.$$  

Equating these products by virtue of (13), we find

$$B_1 B_2 B_3 F = FB_1 B_2 B_3.$$  

Combining (28) with (26) and (28) with (25), we find respectively

$$B_2 F = FB_2,$$

$$B_1 B_2 B_3 F = FB_1 B_2 B_3.$$  

Hence $F$ transforms any product formed from $B_1$, $B_2$, $B_3$, $B_4$ into another such product.

6. Corresponding to the orthogonal substitutions $K_i^j$ defined by formulae (k) of § 3, we have the following operators of $\Gamma$:

$$K_1 = I, \quad K_2 = B_1, \quad K_3 = B_2 B_3, \quad K_4 = B_1 B_2 B_3,$$

$$K_5 = B_1 B_2 B_3 B_4, \quad K_6 = B_1 B_2 B_3 B_4, \quad K_7 = B_1 B_2 B_3 B_4, \quad K_8 = B_1 B_2 B_3.$$  

Any product derived from $E_1$, $E_2$, $E_3$, $B_1$ transforms any product derived from $B_1$, $B_2$, $B_3$, $B_4$, $B_4$ into a product of the $B_i$, a statement made evident by considering the corresponding substitutions of the isomorphic group $O_{5,2}$. In virtue of the theorem at the end of § 5, a like result holds when the transformer is any operator (for example, $V$) derived from $F$, $E_1$, $E_2$, $E_3$, $B_1$.

Since $B_1 B_3 W = WB_1 B_3$, we have on applying (30),

$$B_1 B_3 WFG = WFB_1 B_2 B_3 B_4 = WFG.$$  

Hence the products $B WFG$, where $B$ runs through the 16 distinct products of the $B_1$, $B_2$, $B_3$, $B_4$, reduce to the eight distinct products $K_j WFG$ ($j = 1, \ldots, 8$). For example, $B_2 WFG = K_4 WFG$.

* In establishing identities between operators of $G$ it is frequently simpler to work with the corresponding substitutions of the simply isomorphic group $O_{5,2}$.  

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7. Consider the following set of operators belonging to $\Gamma$:

\[
R_0 = G \quad R_1 = FG \quad R_2 = E_2FG \\
R_3 = E_1FG \quad R_4 = E_2E_1FG \quad R_5 = E_3E_2E_1FG \\
R_6 = K_j WFG \quad R_7 = E_3K_j WFG \quad R_8 = E_2E_3K_j WFG \\
R_9 = E_1E_2E_3K_j WFG \quad R_{10} = E_2E_3K_j WFG \quad R_{11} = E_3E_1E_2K_j WFG \\
R_{12} = E_1E_2E_3K_j WFG \quad R_{13} = E_2E_3E_1K_j WFG \quad R_{14} = E_3E_1E_2K_j WFG
\]

where $j = 1, 2, \ldots, 8$.

From the developments given below it will follow that this table is a rectangular array of the operators of the abstract group $\Gamma$ with $G$ as first row and therefore that $\Gamma$ is holomorphically isomorphic with $O_{6,3}$. Indeed, in §§10–14, we prove that the 126 rows of our table are merely permuted amongst themselves on applying as a left hand multiplier any of the generators $E_1$, $E_2$, $E_3$, $B_1$, $W$, $F$ and therefore for an arbitrary operator of $\Gamma$. Since the row $R_0$ contains the identity, it will follow that an arbitrary operator of $\Gamma$ belongs to the above table. The order of $\Gamma$ is thus not greater than that of $O_{6,3}$.

8. Lemma I.—The rows $R_{1j}$ ($j = 1, \ldots, 8$) are merely permuted upon applying as a left hand multiplier either $E_1$ or $E_2$.

\[
E_2R_{1j} = E_2K_j WFG = K_j E_2WFG = K_j B_1E_2FG \\
= K_j WFB_1E_2G = K_j WFG \equiv R_{1u}.
\]

By the preceding results. Hence

\[
E_1R_{1j} = E_1K_j WFG = K_j WFG \equiv R_{1u}.
\]

9. Lemma II.—The rows $R_{ij}$ are permuted upon applying as a left hand multiplier the operator $V$ defined by (17).

\[
VR_{ij} = VK_j WFG = K_j VWFG \quad \text{[by §6]} \\
= K_j WFG = K_j WFE_3E_1E_2E_1E_3 \quad \text{[by (17)]} \\
= K_j WFG \equiv R_{1i}.
\]
10. Theorem.—The application of $E_2$ as a left hand multiplier permutes the 126 rows.

By inspection we see that $E_2$ interchanges $R_1$ with $R_2$, $R_3$ with $R_4$, $R_5$ with $R_6$, $R_7$ with $R_8$, $R_9$, with $R_{10}$, $R_{11}$ with $R_{12}$, $R_{13}$ with $R_{14}$, $R_{15}$ with $R_{16}$. Furthermore,

$$E_2R_3 = E_2E_1E_2^2E_1^2FG = E_2E_3E_2E_2^2E_1^2FG = E_2E_3E_2^2E_1^2$$

$$= E_3E_2E_2^2E_3E_2F = E_3E_2E_2^2E_3F = R_3$$  [by use of (15)].

$$E_2R_4 = E_2E_1E_2E_3K_2WFG = E_1E_2E_3E_2K_2WFG = R_5,$$

upon applying lemma I to replace $E_1R_2$ by $R_2$.

The condition for the identity $E_2^2R_{1+} = R_{1+}$ is

$$E_2E_1E_3FK_2WFG = E_2E_3FK_1WFG,$$

or

$$FE_1E_2E_3FK_2WFG = K_1WFG,$$

which is satisfied in virtue of lemma II.

Likewise, the condition for the identity $E_2R_{1+} = R_{1+}$ is

$$(E_1E_2E_3F)^{-1}E_2(E_1E_2E_3F)K_2WFG = K_1WFG,$$

which is satisfied by lemma II since we have

$$FE_1E_2E_3E_2^2E_1E_2E_3F = FE_1E_3E_2E_3F = V.$$

11. Theorem.—The application of $E_3$ as a left hand multiplier permutes the 126 rows.

By inspection $E_3$ interchanges $R_1$ with $R_2$, $R_3$ with $R_4$, $R_5$ with $R_6$, $R_7$ with $R_8$, $R_9$ with $R_{10}$, $R_{11}$ with $R_{12}$, $R_{13}$ with $R_{14}$.

$$E_3R_1 = E_3FG = E_3FE_1E_2E_3G = E_3FG = R_2$$  [by (21)].

$$E_3R_2 = E_3E_2E_3E_2^2E_1^2FG = E_3E_2E_3E_2^2E_1^2FG = E_3E_2E_3E_2^2E_1^2FG = R_3.$$  [by lemma I].

$$E_3R_{1+} = E_3E_1E_2E_3K_2WFG = E_3E_2E_3E_2K_2WFG = R_{1+},$$

upon applying (14) and lemma I.

The condition for the identity $E_3R_{1+} = R_{1+}$ is

$$(E_1E_2E_3F)^{-1}E_3(E_1E_2E_3F)K_2WFG = K_1WFG.$$
and it is satisfied in virtue of lemma II since
\[FE_1E_2E_3E_4E_5F = FE_1E_2E_3E_4E_5F = FE_1E_2E_3E_4E_5F = V.\]

\[E_3R_{14} \equiv E_3E_4E_5FK_jWFG = E_4E_5E_6FK_jWFG \]
\[= E_4E_5E_6FK_jWFG \quad \text{[by lemma I and (14)]} \]
\[= E_4E_5E_6FK_jWFG = R_{14}. \]

The condition for the identity \(E_3R_{14} = R_{14}\) is
\[(E_1E_2E_3F)^{-1}E_3(E_4E_3F)K_jWFG = K_jWFG,\]
and it is satisfied by lemmas I and II since we have
\[FE_3E_4E_1E_2E_3E_4E_5F = FE_4E_4E_1E_2E_3E_4E_5F \]
\[= FE_4E_4E_1E_2E_3E_4E_5F = FE_3E_4E_5E_6E_7E_8E_9F \]
\[= FE_3E_4E_5E_6E_7E_8E_9F = V. \]

12. Theorem.—The application of \(F\) as a left hand multiplier permutes the 126 rows.

\[FR_0 = R_1. \]
\[FR_{14} = R_{14}. \]
\[FR_2 = FE_2FG = E_2^2FE_2E_1G = E_2^2FG \equiv R_3 \quad \text{[by (22)].} \]
\[FR_4 = FE_4E_2E_1^2FG = E_2E_1FE_2FG \quad \text{[by (14)]} \]
\[= E_2E_1^2E_2E_1^2FG = E_2E_1^2E_2E_1^2FG \quad \text{[by (12)].} \]
\[= E_2^2E_1^2E_2E_1^2E_2E_1^2FG = E_2^2E_1^2E_2E_1^2E_2E_1^2FG = R_4 \quad \text{[by (14)].} \]
\[FR_5 = FE_3E_2E_1^2FG = E_3E_2E_1^2FG \quad \text{[by (21)]} \]
\[= E_3E_2E_1^2FG = E_3E_2E_1^2FG \quad \text{[by lemma I].} \]

The condition \(FR_{14} = R_{14}\) or
\[(E_1E_2E_3F)^{-1}F(E_2E_3)K_jWFG = K_jWFG \]
is satisfied by lemma II, since we have, by (20),
\[FE_3E_4E_1E_2E_3F = FE_3E_4E_1E_2E_3F = FE_3E_4E_1E_2E_3F = V. \]
\[FR_{14} = FE_3E_4E_1E_2E_3F = E_3E_4E_1E_2E_3F = R_{14}. \]

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since by (14) and (15) we have
\[ FE_1^2 E_2 E_3 = E_2 E_1 FE_3 = E_2 E_1 E_3 E_1 FE_1^2 = E_2 E_3 FE_1^2. \]

The condition for the identity \( FR_{ij} = R_{ij} \) is satisfied by lemmas I and II:

\[ (E_1 E_2 E_3)^{-1} FE_1 E_2 E_3 K_i WFG = E_1 V K_i WFG = K_i WFG \quad \text{[by (23)]}. \]

\[ FR_{ij} = FE_1^2 E_1 E_2 K_i WFG = E_1^2 E_1 E_2 E_1 K_i WFG \quad \text{[by (22)]} \]

since
\[ FE_1^2 E_1^2 F = E_1^2 E_2 E_1 F = E_1^2 E_1 E_1^2 F = E_2 E_1^2 F E_2 E_1^2 = E_2 E_1^2 F E_2 E_1^2. \]

The condition for the identity \( FR_{ij} = R_{ij} \), viz.,

\[ (E_1 E_2 E_3 F)^{-1} F (E_1 E_2 E_3 F) K_j WFG = K_j WFG, \]

is seen to be satisfied in virtue of lemmas I and II as follows:
\[ FE_1 E_2 E_1 E_2 E_3 F = E_1 E_2 E_3 E_1 E_2 E_3 F E_1 E_2 E_3 F = E_1 E_2 E_3 E_1 E_2 E_3 F = E_1 E_2 E_3 E_1 E_2 E_3 F = E_1 E_2 E_3 E_1 E_2 E_3 F = E_1 E_2 E_3 E_1 E_2 E_3 F \]

is seen to be expressed as a product of the \( E_1, E_2, E_3, F \). Hence by

\[ 13. \quad \text{By (21), } E_i \text{ is expressed as a product of the } E_1, E_2, E_3, F \text{. Hence by } \quad \text{§§ 10–12, } E_i \text{ permutes the 126 rows when applied as a left hand multiplier.}
\]

From the following relations given under formulae (8), (2), (13) above,

\[ B_1 E_1 = E_1 B_1 B_2, \quad B_1 E_2 = E_2 B_1, \quad B_1 E_3 = E_3 B_1, \quad B_1 F = B_1 B_4, \]

and from the remarks in § 6 concerning the \( K_j (j = 1, \ldots, 8) \), it follows that

\[ B_1 \text{ applied as a left hand multiplier permutes the 126 rows. Then by (6) a like result holds for } B_2, B_3 \text{ and } B_4. \]

14. Theorem.— The application of \( W \) as a left hand multiplier permutes the 126 rows.

\[ WR_0 = R_0. \]

\[ WR_1 = WFG \equiv R_{11}. \]

\[ WR_2 = WE_2 FG = B_3 WFG = B_1 WB_1 B_2 FG = K_2 WFG = R_{12}. \]
\[ WR_3 \equiv WE_1^2FG = E_1^2WB_1FG = E_1^2WFG = K_3WFG \equiv R_{15} \]

\[ WR_4 = WE_2E_1^2FG = E_2E_1WE_1FG = E_2E_1B_3E_2E_2WFG \]

\[ = B_4E_2E_1E_2E_1WFG = B_4WE_1^2E_2FG = K_4WFG \equiv R_{14}. \]

The condition for the identity \( WR_3 = R_5 \) is that \((E_3E_2E_1)WE_3E_2E_1F \) shall belong to \( G \). We shall verify that it equals \( E_1E_2E_3E_2WE_2E_3E_2E_1^2 \).

The condition for this equality may be written

\[ (E_1E_2E_3E_2)^{-1}FE_1E_2E_3WE_2E_2E_1^2E(E_2E_2E_2E_1^2)^{-1} = W. \]

By (23), the left member is equal to

\[ E_2E_1VWE_1VE_2 = E_2E_1WVE_1VE_2 \quad \text{[by (17)]} \]

\[ = E_2E_1WE_1E_2 = W \quad \text{[by (24) and (7)].} \]

Each row \( R_\psi \) is of the form \( AKWFG \), where \( A \) denotes a product built from \( E_1, E_2, E_3, F \). But \( E_1, E_2, E_3 \) each transform \( K_j \) into some \( K_i \), a statement made evident by considering the isomorphic orthogonal substitutions. Furthermore, by \$6, F \) transforms \( K_j \) into some \( K_i \). Hence each row \( R_\psi \) may be given the form \( K_iA WFG \).

In virtue of the following relations between orthogonal substitutions,

\[ W' C_1C_2 = C_3C_4E_2'W', \quad W' C_1C_3 = C_2C_4E_2'E_1W', \]

\[ W' C_1C_4 = C_2C_4E_1'E_2'E_1W', \quad W' C_1C_2C_3C_5 = C_1C_2E_1'E_2'E_1W', \]

\[ W' C_1C_2C_4C_5 = C_1C_2E_1'E_2'E_1W', \quad W' C_1C_3C_4C_5 = C_1C_3E_1'E_2'E_1W'. \]

we have in the isomorphic group \( G \) the general relation

\[ WK_\psi = K_iA W^{-1} \]

where \( A' \) is derived from \( E_1 \) and \( E_2 \). Hence

\[ WR_\psi = WK_iA WFG = K_iA W^{-1} A WFG. \]

But, by \$10 and 13, \( K_i \) or \( A' \) when applied as left hand multiplier permutes the 126 rows. Hence it remains only to prove that \( W \) (and hence also \( W^{-1} \)) permutes the rows \( R_{11} \equiv A WFG \) when applied as a left hand multiplier.

\[ WR_{11} \equiv W^2FG = B_1B_3B_3B_4WB_1B_3B_3B_4FG = B_1B_3B_3B_4WFG \]

\[ = K_5WFG = R_{15}. \]

\[ WR_{61} \equiv WFWFG = FWFWG = FWFG = R_{61} \quad \text{[by (16)].} \]
The condition for the identity $WR_{21} = R_{33}$ is that the operator

$$(E_2E_3K_3WF)^{-1}WE_3WF = F(W^2K_3E_2E_2WE_3W)F$$

shall belong to $G$. To it corresponds in the orthogonal group a substitution which corresponds to

$$S = B_2B_3E_1E_2E_3E_3W^2E_2E_2E_1^2.$$ 

We proceed to prove that in the abstract group $\Gamma$:

$$FSF = W^2K_3E_3E_2WE_3W = T_1.$$ 

Now

$$FSF = B_3FE_1E_2E_3E_3W^2E_2E_2E_1^2F$$

$$= B_3E_1E_2E_3W^2E_2E_2E_1^2F$$  \[\text{by (23)}\]

$$= B_3E_1E_2E_3VW^2E_1^2E_2E_3E_2E_1^2F$$  \[\text{by (24) and (7)}\]

$$= B_3E_1E_2E_3W^2E_3E_3E_1^2FE_2E_3E_3E_2E_2E_1^2F$$  \[\text{by (17) and (23)}\]

$$= B_3E_1E_2E_3W^2E_3E_3E_1^2FE_3$$

$$= B_3E_1E_2E_3W^2E_3E_3E_1^2FE_3 = T_2$$  \[\text{by (15)}\].

It remains to prove that the products denoted by $T_1$ and $T_2$ are equal. Each belongs to the group $G$; they will therefore be identical if the corresponding orthogonal substitutions are identical. But to both $T_1$ and $T_2$ there corresponds the same orthogonal substitution, viz.,

$$\xi'_1 = \xi_1, \quad \xi'_2 = -\xi_2 + \xi_3 - \xi_4 + \xi_5, \quad \xi'_3 = \xi_2 - \xi_3 - \xi_4 + \xi_5,$$

$$\xi'_4 = -\xi_2 - \xi_3 - \xi_4 - \xi_5, \quad \xi'_5 = -\xi_2 - \xi_3 + \xi_4 + \xi_5.$$ 

The condition for the identity $WR_{101} = R_{107}$ is that the product

$$(E_3FB_4WF)^{-1}WE_3WF$$

shall belong to $G$. It is satisfied in virtue of relation (18).

In view of the following relations derived from (3) and (8),

$$WE_2 = B_3W, \quad WE_1 = B_3E_2E_1W, \quad WE_1^2 = E_1^2B_3E_2W, \quad WE_2 = B_3W, \quad WE_1 = B_3E_2E_1W, \quad WE_1^2 = E_1^2B_3E_2W,$$

$WR_{31}$, $WR_{41}$ and $WR_{51}$ are each of the form

$$DWE_3WFG = DWR_{21} = DR_{33},$$

where $D$ is derived from $B_2$, $E_1$, $E_2$. Also $WR_{71}$, $WR_{81}$, $WR_{91}$ are each of the form

$$DWFWFG = DWR_{61} = DR_{61}.$$
Finally, the products \( WR_{i1} (i = 11, 12, 13, 14, 15) \) are each of the form

\[ D W E_2 F W F G = D W R_{101} = D R_{107}. \]

By the results of §§10 and 13, each of the products \( D R_{33}, D R_{61}, D R_{107} \) equals some row \( R_i \) or \( R_{ij} \).

§§15–17. \textit{Isomorphism and correspondence of generators between the orthogonal and hyperabelian groups.}

15. The simple group \( H_{4,3} \) of order 25920, which is derived from the decomposition of the Abelian group of modulus 3 on four indices, is simply isomorphic with the simple subgroup \( O_{5,3} \) of the quinary orthogonal group of modulus 3.*

We proceed to determine the operators of the former group which correspond to the generators \( E'_1, E'_2, E'_3, E', C_1 C_2, W' \) of the latter. We first determine the operators of \( O_{5,3} \) which correspond to the generators \( B_1, B_2, B_3, B_4, B_5, B \) of \( H_{4,3} \) given on pages 65 and 67 of volume 31 of the \textit{Proceedings of the London Mathematical Society}. Of the two possible forms for \( B \), we choose that one given by \( \gamma_i \equiv 1 \) (mod 3), viz.

\[
B = \pm \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}.
\]

By the general correspondence set up in §5 of the paper cited in the foot-note, we find that \( B \) corresponds to the substitution

\[
\begin{align*}
\xi'_1 & = \begin{bmatrix} 1 \end{bmatrix} \\
Y'_{13} & = \begin{bmatrix} 0 \end{bmatrix} \\
Y'_{14} & = \begin{bmatrix} 0 \end{bmatrix} \\
Y'_{23} & = \begin{bmatrix} 0 \end{bmatrix} \\
Y'_{24} & = \begin{bmatrix} 0 \end{bmatrix}
\end{align*}
\]

leaving invariant modulo 3 the function

\[
\phi \equiv \xi^2_1 + Y_{13} Y_{24} - Y_{14} Y_{23}.
\]

---

* \textit{Transactions of the American Mathematical Society}, vol. 1, p. 93.

† The substitutions \( B_i \) enter §15 alone and are to be distinguished from the earlier \( B_i \).
We introduce the new indices
\[
\xi_2 = -Y_{13} - Y_{21}, \quad \xi_3 = Y_{13} + Y_{14} + Y_{25} - Y_{24}, \\
\xi_4 = -Y_{13} + Y_{14} + Y_{22} + Y_{24}, \quad \xi_5 = -Y_{14} + Y_{23}.
\]
Then
\[
\phi \equiv \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 \pmod{3}.
\]
Solving modulo 3 the relations (32), we find
\[
Y_{13} \equiv \xi_2 + \xi_3 - \xi_4, \quad Y_{14} \equiv \xi_3 + \xi_4 + \xi_5, \\
Y_{24} \equiv \xi_2 - \xi_3 + \xi_4, \quad Y_{25} \equiv \xi_3 + \xi_4 - \xi_5.
\]
Expressing in terms of the new indices \(\xi_i\) the substitution (31) to which \(B\) corresponds, we obtain the result:
\[
(33) \quad \tilde{B} \sim B' \equiv C_i C_j (\xi_k \xi_l \xi_m).
\]
Proceeding similarly with the substitutions \(B_1, \ldots, B_5\), we find that
\[
(34) \quad B_1 \sim B'_1 \equiv C_1 C_2 E_3 E_4 E_5 E_6 E_7 W' E_3' E_4' E_5' E_6' E_7' C_8 C_9, \\
(35) \quad B_2 \sim B'_2 \equiv C_1 C_2 C_3 C_4, \\
(36) \quad B_3 \sim B'_3 \equiv C_3 C_4 B'_1 C_3 C_4, \\
(37) \quad B_4 \sim B'_4 \equiv C_1 C_5 E_3 B'_1 E_3' C_4 C_5, \\
(38) \quad B_5 \sim B'_5 \equiv C_1 C_3 C_4 C_5,
\]
where the substitutions \(B'_1, B'_4\) are in matricular notation
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & -1 \\
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & -1 & 1 \\
1 & 0 & -1 & 1 & 1 \\
-1 & 0 & 1 & 1 & 1
\end{pmatrix}, \quad \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & -1 \\
0 & 1 & 1 & 1 & 1 \\
0 & -1 & 1 & 1 & 1
\end{pmatrix}.
\]
We proceed next to express the generators \(E'_1, E'_2, E'_3, C_1 C_2, W'\) of the orthogonal group \(O_{5,3}\) in terms of \(B', B'_1, \ldots, B'_5\), which correspond to the generators of \(H_{4,3}\),
\[
W' = B'_2 B'_3 B'_4 B'_5, \quad E'_3 = B' B'_1 B' W' B'_2 W' B' , \quad E'_2 = C_1 C_3 W' B'_1 W', \\
C_1 C_5 = B' W' B'_2 W' E'_3, \quad E'_1 = C_3 C_4 B'_1 B'_2 B' E'_3 C_1 C_2 C_1 C_5, \\
C_1 C_5 = B'_2 B'_5, \quad C_1 C_5 = E'_3 C_3 C_4 E'_3, \quad C_3 C_4 = B'_2 C_1 C_3.
\]
The corresponding Abelian substitutions have corresponding relations. The substitutions \( W, W^2 = W^{-1} \); \( E_2, E_1, E_1; C_1C_5, C_3C_4, C_5C_2, C_1C_2, C_2C_3 \) are found to be the following:

\[
\begin{bmatrix}
0 & -1 & 0 \\
0 & 1 & 1 \\
-1 & -1 & 1 \\
1 & 1 & 0
\end{bmatrix}
\quad\quad
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
1 & 1 & 0 \\
-1 & -1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 \\
-1 & 0 & 0 & -1
\end{bmatrix}
\quad\quad
\begin{bmatrix}
1 & 0 & 0 & -1 \\
1 & -1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}
\quad\quad
\begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\quad\quad
\begin{bmatrix}
-1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & -1 & 1
\end{bmatrix}
\]

These calculations have been checked in several ways.

16. Theorem.—If the hyperabelian group \( HA_{4,3} \) be isomorphic with the orthogonal group \( O_{6,3} \) in such a manner that the correspondences of §15 hold between the operators of their respective subgroups \( H_{4,3} \) and \( O_{6,3} \), then the substitution of the group \( HA_{4,3} \),

\[
(39) \quad \bar{T} \equiv \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I^{-3} & 0 & 0 \\
0 & 0 & I^{-1} & 0 \\
0 & 0 & 0 & I^{3}
\end{bmatrix}
\]
in which the mark $I$ is a suitably chosen root of the congruence

$$x^2 \equiv x + 1 \pmod{3},$$

must correspond to the substitution $C_1 C_2 (E_1 E_2) (E_3 E_4)$ which extends $O_{5,3}$ to $O_{6,3}$.

In the quotient group $HA_{4,3}$, every hyperabelian substitution multiplying all four indices by the same factor corresponds to the identity, viz., the powers of

$$\begin{bmatrix}
I^2 & 0 & 0 & 0 \\
0 & I^2 & 0 & 0 \\
0 & 0 & I^2 & 0 \\
0 & 0 & 0 & I^2
\end{bmatrix} \equiv \begin{bmatrix}
i & 0 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & i
\end{bmatrix} = 1,$$

where we have set

$$I^2 \equiv i, \quad I \equiv i - 1, \quad i^2 \equiv -1 \pmod{3}.$$  

Then since $I^8 = 1$, one has easily

$$I^2 = C_2 C_3 C_4 C_5,$$

so that $I$ is of period four. We readily verify the following relations:

$$\begin{align*}
I C_3 C_4 &= C_3 C_4 I, \\
I C_1 C_2 &= C_1 C_2 I^{-1}, \\
I C_1 C_5 &= C_1 C_5 I^{-1}, \\
I C_2 C_3 &= C_2 C_3 I^{-1}, \\
E_2 E_3 I E_2 E_3 &= I, \\
E_2 E_1 I E_2 E_1 &= C_2 C_3 I^{-1}.
\end{align*}$$

Suppose that $O_{6,3}$ contains a substitution $I'$ which combines with the generators $E_1', E_2', E_3', C_1' C_2, W'$ of $O_{5,3}$ according to the same laws by which $I$ combines with $E_1, E_2, E_3, C_1 C_2, W$ of $H_{4,3}$. Assume for $I'$ the most general form possible, viz.,

$$I' = \pm \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{16} \\
a_{21} & a_{22} & \cdots & a_{26} \\
\vdots & \vdots & \ddots & \vdots \\
a_{61} & a_{62} & \cdots & a_{66}
\end{bmatrix},$$

the simultaneous change of sign of every coefficient leaving $I'$ unchanged.

In virtue of the relation corresponding to (42), viz.,

$$I'^{-1} = I' C_2 C_3 C_4 C_5,$$

we find that
According to the sign ±, we find for \( I^+ \), \( I^- \) the respective values:

\[
\begin{pmatrix}
  a_{11} & a_{21} & \cdots & a_{61} \\
  a_{12} & a_{22} & \cdots & a_{62} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{16} & a_{25} & \cdots & a_{65} \\
  a_{16} & a_{26} & \cdots & a_{66}
\end{pmatrix} = \pm
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{16} \\
  -a_{21} & -a_{22} & \cdots & -a_{25} \\
  \vdots & \vdots & \ddots & \vdots \\
  -a_{51} & -a_{52} & \cdots & -a_{56} \\
  a_{61} & a_{62} & \cdots & a_{66}
\end{pmatrix}
\]

By the relation corresponding to the first relation (43), \( I^- \) must be commutative with \( C_3 C_4 \). We find in consequence for \( I^+ \), \( I^- \) the values:

\[
\begin{pmatrix}
  a_{11} & 0 & 0 & 0 & 0 & a_{16} \\
  0 & a_{23} & a_{24} & a_{25} & 0 \\
  0 & -a_{23} & 0 & a_{34} & a_{35} & 0 \\
  0 & -a_{24} & -a_{34} & 0 & a_{45} & 0 \\
  0 & -a_{25} & -a_{35} & -a_{45} & 0 & 0 \\
  a_{16} & 0 & 0 & 0 & 0 & a_{66}
\end{pmatrix}
\pm
\begin{pmatrix}
  0 & 0 & 0 & 0 & 0 & a_{16} \\
  0 & a_{22} & a_{23} & a_{24} & a_{25} & 0 \\
  0 & a_{33} & a_{34} & a_{35} & 0 \\
  0 & a_{43} & a_{44} & a_{45} & 0 \\
  0 & a_{53} & a_{54} & a_{55} & 0 \\
  -a_{16} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The resulting substitution \( I^+ \) does not satisfy the relation corresponding to (43),

\[ I^+ C_1 C_2 = C_1 C_2 I^{-1} \]

and is therefore excluded. And \( I^- \) satisfies it if and only if \( a_{25} = 0 \).

The relation \( I^+ C_1 C_5 = C_1 C_5 I^{-1} \) is then satisfied by \( I^- \). But the relation:

\[ I^+ C_1 C_3 = C_1 C_3 I^{-1} \]

requires that \( a_{33} = a_{44} = 0 \) in \( I^- \). The relations:

\[ E_2 E_3 I^- E_2 E_3 = I^- \quad E_2 E_1 I^- E_2 E_1 = C_3 C_5 I^{-1} \]
require in succession that \(a_{34} = a_{35}, a_{34} = -a_{22}\). Evidently \(a_{16} \neq 0\), so that 
\(a_{16} \equiv \pm 1 \pmod{3}\). By suitable choice of the sign \(\pm\) in front of the matrix for \(I'_-\), we may take \(a_{16} = +1\). Setting, for brevity, \(a_{34} = \gamma\), we have

\[
I'_- \equiv \pm \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -\gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma & 0 & 0 \\
0 & 0 & \gamma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \gamma & 0 \\
-1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Hence, according as \(\gamma = \pm 1\), we have

\[
I_{-}^{-1} = C_1 C_2 (\xi_3 \xi_6) (\xi_3 \xi_4).
\]

But, by (39),

\[
I^{-1} \equiv I^3 = \begin{bmatrix}
I^3 & 0 & 0 & 0 \\
0 & I^{-9} & 0 & 0 \\
0 & 0 & I^{-3} & 0 \\
0 & 0 & 0 & I^9
\end{bmatrix}
\]

is derived from \(I\) by replacing \(I\) by \(I^3\). But \(I\) is defined as one root of the irreducible congruence:

\[
I^2 \equiv I + 1 \pmod{3},
\]

whose second root is \(I^3\). Hence, by a proper choice of notation for the root \(I\), we may set

\[
I'_- = C_1 C_2 (\xi_3 \xi_6) (\xi_3 \xi_4) = C_1 C_2 F' E'^2 E'_2.
\]

Hence

\[
F' = C_1 C_2 I'_- E'_2 E'_1.
\]

It follows that to \(F'\) must correspond the hyperabelian substitution

\[
F = C_1 C_2 I E'_2 E'_1 \equiv \begin{bmatrix}
0 & 0 & -I & 0 \\
0 & 0 & -I & I \\
I^3 & 0 & 0 & 0 \\
I^3 - I^3 & 0 & 0
\end{bmatrix}.
\]
17. Theorem.—The hyperabelian group $HA_{4,3}$ is holoedrically isomorphic* with the abstract group $\Gamma$.

From the simple isomorphism of $H_{4,3}$ with $G$, it follows that the operators $E_1$, $E_2$, $E_3$, $B_1 \equiv C_1 C_2$, $W$ satisfy the generational relations (1), \ldots, (6) of $G$, a result capable of verification by direct calculation. It therefore remains only to prove that the operators $F$, $E_1$, $E_2$, $B_1 \equiv C_1 C_2$, $W$ satisfy the generational relations (12), \ldots, (18). This result may be verified by simple calculations. We note the auxiliary formulae:

\[
C_4 C_5 = \begin{pmatrix}
-1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix}, \quad E_3 W E_3 = \begin{pmatrix}
-1 & 0 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 0 \\
1 & 1 & 0 & -1
\end{pmatrix},
\]

\[
W^2 C_1 C_2 C_4 C_5 = \begin{pmatrix}
1 & 1 & -1 & 1 \\
0 & 0 & 1 & 1 \\
1 & -1 & -1 & -1 \\
-1 & -1 & 0 & 0
\end{pmatrix}, \quad V = \begin{pmatrix}
1 & 0 & -i & 0 \\
1 & -1 & -i & i \\
i & 0 & -1 & 0 \\
i & -i & -1 & 1
\end{pmatrix}.
\]

We have therefore proved that the simple groups $HA_{4,3}$ and $O_{6,3}$ are holoedrically isomorphic.

University of Texas, January 12, 1900.

*Addition: May 5, 1900. In the May number of the Bulletin of the Society the writer establishes the holoedric isomorphism of $O_{6,3}$ and $HA_{4,3}$ for any $p^n$ of the form $4f - 1$. As the method there used consists in the transformation of the defining invariant of the former group into that of the second compound of the latter group, it gives no direct knowledge of the correspondences of the generators of the isomorphic groups. For $p^n = 3$, \S\ 15-16 of the present paper enable us to pass readily from an arbitrary substitution of either group to the corresponding substitution of the other.