CONCERNING HARNACK'S THEORY OF IMPROPER DEFINITE INTEGRALS*

BY

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INTRODUCTION.

In this paper I consider the improper simple definite integrals of HARNACK (1883, 1884). In the introduction I wish to characterize somewhat clearly the theories of the improper simple and multiple integrals recently given by JORDAN (1894) and STOLZ (1898, 1899), and in this introductory paragraph I summarize the contents of the whole introduction. These theories for the simple integrals have intimate relations with the HARNACK theory. The definition adopted for the multiple integrals is more exacting than that for the simple integrals. The multiple integrals converge or exist (as limits) only absolutely. For the simple integrals we have then two theories, on the one hand, of the integrals with the milder definition, and, on the other hand, of the integrals with the stronger definition and so with a larger body of properties. The first class of integrals includes the second class of integrals. The HARNACK theory relates to the first and general class of integrals; this theory has not received systematic development; however, for the theory of the absolutely convergent HARNACK integrals this is not true, and these integrals constitute the second and special class of integrals. I discuss both classes of simple integrals simultaneously and by uniform process; this is made possible by suitable determinations of the definitions; the absolute convergence of the integrals of the second class appears only at the conclusion, and hence it is desirable to introduce terms of discrimination connoting the two definitions, the milder and the stronger; the terms chosen, "narrow," "broad," connote the geometric form of the definitions, and likewise the fact that the class of narrow integrals has a less extensive body of properties than the (included) class of broad integrals. There has been a tendency to do away with the non-absolutely convergent HARNACK integrals; I hope to show that this tendency rests upon misconceptions.—The theory of DE LA VALLÉE POUSSIN (initiated in 1892) is in form distinct from the HARNACK theory and

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relates primarily to absolutely convergent integrals. It is not involved in the present paper.

HARNACK, in papers* published in 1883 and 1884, in volumes 21 and 24 of the Mathematische Annalen, first gave a definition for the notion of an improper definite integral from a to b of a real function $F(x)$ having these two properties:

(1) The function $F(x)$ assumes values indefinitely great in the neighborhood of points ζ constituting a point-set † Z lying on the interval ab and of content zero; and

(2) The function $F(x)$ is properly integrable in the neighborhood of every other point x of ab, or what is equivalent, it is properly integrable from a' to b' where $a'b'$ is any interval of ab, which contains no point ζ.

HARNACK's definition was formally a definition not so much of the definite integral

$$\int_a^b F(x)\,dx \tag{1}$$

itself as of the definite integral function

$$J(X) = \int_a^X F(x)\,dx \qquad (X \text{ of } ab).$$

Indeed his definition, in common with many limit-definitions of that period and of earlier periods, was not expressed in explicit form. From the context one may infer that HARNACK was so desirous of passing to the less immediate applications of the notion that he was unwilling to attend to the systematic exposition of the fundamental elements of the theory. Thus he was led into error,—with respect to the general theory, notably in the theorem ‡:

If the set Z of singularities of the function $F(x)$ is reducible (and so of content zero), and a continuous function $\phi(x)$ of x on ab exists, for which

$$\phi(x'') - \phi(x') = \int_{x'}^{x''} F(x)\,dx$$

for every interval $x'x''$ of ab, containing no singularity ζ, then the integral

Anwendung der Fourier'schen Reihe auf die Theorie der Functionen einer complexen Veränderlichen; vol. 21 (1883), pp. 305–326; cf. pp. 324–326. *Die allgemeinen Sätze über den Zusammenhang der Functionen einer reellen Variabelen mit ihren Ableitungen, II. Theil;* vol. 24 (1884), pp. 217–252; cf. p. 220 fg.

† It is convenient to use *set* as the equivalent of *Menge* and *ensemble.*—For CANTOR's *Punktmenge* OSGOOD has used the English term *Cantor's set.*

‡ Loc. cit., vol. 24, p. 222, theorem 3. This theorem depends on theorem 2, which is likewise erroneous (cf. § 5 6°).

$$\int_a^x F(x)\,dx \qquad (x \text{ of } ab)$$

exists * or converges and has the value

$$\phi(x) - \phi(a).$$

This theorem was designed to show that the improper definite integrals defined by HÖLDER (Mathematische Annalen, vol. 24 (1884), p. 190 fg.) are special cases of the HARNACK integrals, the set Z being reducible. And this does follow, by the elucidations of HÖLDER as to the uniqueness of his definite integrals, whenever the corresponding HARNACK integrals exist. Thus the HARNACK integrals with reducible sets of singularities are HÖLDER integrals, and the HÖLDER integrals for which the corresponding HARNACK integrals (with reducible sets of singularities) exist are these HARNACK integrals. By examples in § 5 (4°, 9°) I show that each system of integrals is more extensive than their common subsystem.

I accept the sharp formulation of the limit-definition of the integral (1) given by STOLZ † (1898) as doubtless a correct expression of the content of HARNACK's meaning.

As SCHOENFLIES ‡ has remarked, the generalized simple definite integrals of JORDAN (*Cours d'analyse*, ed. 2, vol. 2 (1894), pp. 46, 50 fg.) are § HARNACK's improper integrals, although there is no reference to HARNACK and the form of exhibition is different. JORDAN (loc. cit., p. 46) selects two properties of the

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* I think of the integral as a limit, that is, as a certain number obtained by a certain limiting process, and prefer to say in general that the limit exists rather than that it converges; the limit- and expression converges to the number in question as its limit.

† The full reference is given below.

‡ Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8: *Bericht* (1900), p. 176.—SCHOENFLIES (loc. cit., p. 186) gives suggestively but not explicitly a definition of the HARNACK integrals in terms involving the set of non-overlapping intervals of ab which enclose no point ζ and every point not-ζ enclosed by ab. One may form three explicit definitions in these terms. The third definition agrees with the definition given by STOLZ, and the second is equivalent to it, while the first, which alone fully expresses the implications to me of the language used by SCHOENFLIES, is a definition considerably milder.

§ At least if the content of the set Γ of points c (loc. cit., p. 50) is zero. SCHOENFLIES suggests that a condition to this effect may have been unintentionally omitted. But JORDAN defines (loc. cit., p. 76) double integrals without the insertion of the analogous condition; for (contrary to a statement of SCHOENFLIES) he speaks merely of the *interior* content of the region of integration. However, in the case of the simple integrals, if this condition is omitted, the theorem cited in the text will fail, whenever the condition for $\int_a^b F(x)\,dx$ is not satisfied, even for all cases in which $G(x)$ is a non-zero constant. It may be noted here that JORDAN's set Γ is the set of all points c in whose neighborhood the function $F(x)$ is not capable of proper integration. Thus, the set Γ includes HARNACK's set Z. But we have the theorem that the set Γ is HARNACK's set Z, in case Γ has content zero and the improper definite integral exists. Cf. theorem VIIA of § 3.

proper integrals which serve to suggest the explicit definition of the generalized integrals. But the theorems of the sequel and the reference theorem implied (p. 226, ll. 9, 10) in the proof of the second mean value theorem for the generalized integrals, certainly need fuller proofs; indeed, the theorem* that $\int_a^b \big(F(x)+G(x)\big)\,dx$ exists and

$$\int_a^b \big(F(x)+G(x)\big)\,dx = \int_a^b F(x)\,dx + \int_a^b G(x)\,dx$$

in case the integrals on the right exist, is not in general true. In § 5 8° I give a simple example in which the integral in question does not exist.

Quite recently HARNACK's theory (without reference to JORDAN's theory) has been considered systematically and critically by STOLZ, first in 1898 in a paper † entitled "*Zur Erklärung der absolut convergenten uneigentlichen Integrale*," and then in 1899 (with but slight modifications) in appendix III to volume 3 of his *Grundzüge der Differential- und Integralrechnung*, and further immediately thereafter, in 1899, in a paper entitled "*Über die absolute Convergenz der uneigentlichen bestimmten Integrale*." †

STOLZ gives ‡ an explicit definition of the HARNACK integral (1) as a certain limit, and then, considering the theory of these integrals, he affirms ‡ that it fails to justify for the limit in question the notation and designation *definite integral* —that one has not even the property that if $\int_a^b F(x)\,dx$ exists, so do all the integrals $\int_a^c F(x)\,dx$ $(a<c<b)$. Contenting himself at this point with an affirmation, STOLZ turns to the theory of the absolutely convergent integrals (1), that is, those for which also the corresponding integral $\int_a^b |F(x)|\,dx$ exists, and finds § that these integrals deserve the name, since they possess certain four fundamental properties.

The definition given by STOLZ has application with respect to any point-set Ξ lying on the interval ab and of content zero, which contains the set Z used by HARNACK, and in so far his definition is formally more comprehensive than HARNACK's definition. This extension was necessary for the formulation of the third and fourth of the four fundamental propositions which are as follows:

Under the hypothesis that $F(x)$ is absolutely integrable from a to b with respect to a point-set Ξ of content zero:

I. $F(x)$ is likewise integrable from a to c and from c to b, where c is any point of ab, and

* Theorem 3, § 58, p. 56.—This theorem would follow directly from HARNACK's erroneous theorem 2 cited above.

† Wiener Berichte, vol. 107_{IIa}, pp. 207–224; vol. 108_{IIa}, pp. 1234–1238.

‡ Wiener Berichte, vol. 107, pp. 207, 211. *Grundzüge*, vol. 3, p. 277.

§ Wiener Berichte, vol. 107, p. 211, § 2, and pp. 216–221, § 5. *Grundzüge*, vol. 3, pp. 277, 279–284.

$$\int_a^b F(x)\,dx = \int_a^c F(x)\,dx + \int_c^b F(x)\,dx.$$

II. For every positive ϵ there is a positive δ_ϵ such that for any set of a finite number (n) of non-overlapping intervals $a_k b_k$ $(a_k < b_k;\ k = 1, \cdots, n)$ of the interval ab of total length less than δ_ϵ the sum of the n integrals $\int_{a_k}^{b_k} F(x)\,dx$ is in absolute value less than ϵ.

III. $F(x)$ is absolutely integrable from a to b with respect to the point-set $\Xi + \mathrm{H}$ obtained by extending Ξ by any set H of content zero, and the integrals with respect to Ξ and to $\Xi + \mathrm{H}$ are equal.

IV. If $G(x)$ is absolutely integrable from a to b with respect to the point-set H of content zero, then $F(x) + G(x)$ is likewise integrable with respect to the aggregate set $\Xi + \mathrm{H}$, and

$$\int_a^b \big(F(x) + G(x)\big)\,dx = \int_a^b F(x)\,dx + \int_a^b G(x)\,dx.$$

As to these theorems it is to be noted that the third is an immediate consequence of the first two, and that the fourth is an immediate consequence of the third.

As to the corresponding theorems for the HARNACK integrals in general I shall show (§ 3 V, VIII) that the first is in fact true, notwithstanding the statement* of STOLZ to the contrary; the second is the erroneous theorem 2 of HARNACK and the fourth is the erroneous theorem 3 of JORDAN. Indeed, the validity of the second theorem for a HARNACK integral implies its absolute convergence (cf. § 2, def. 1, note 1, footnote, and § 4 III). The second theorem is in effect a generalization of the uniform continuity of the definite integral function $J(x)$; for the non-absolutely convergent HARNACK integrals the uniform continuity of $J(x)$ holds (§ 3 IX, § 5 6°).

The limit-definition of the integral (1) relates to the various interval-sets I of a finite number of intervals enclosing all points ζ or ξ of the set Z or Ξ; in notation $I = I(\mathrm{Z})$ or $I(\Xi)$. According to the implication of HARNACK and of STOLZ and the explicit statement of JORDAN (loc. cit., p. 51, top) every interval of I encloses (at least) one point ζ or ξ.

JORDAN'S definitions† for the improper simple and multiple integrals relate to closed measurable regions D lying within the bounded or limited region E

* This statement of STOLZ is reaffirmed by PRINGSHEIM (Münchener Berichte, 1900, p. 220) in his recent paper on the second mean value of the integral calculus, and it is the basis of PRINGSHEIM'S rejection of HARNACK'S extension (Annalen, vol. 21, p. 326) of that theorem to the general improper integrals; the theorem as extended is, however, correct, and the following development of the general theory makes HARNACK'S elegant proof effective.

† loc. cit., pp. 51, 76, and 87. The theory of the improper upper and lower multiple integrals is first developed.

of integration and containing no singularity c. For the simple integrals the complementary region $E - D$ is the interval-set $I(\Xi) = I(\Gamma)$ each of whose intervals encloses a point c. For the multiple integrals the corresponding condition is not imposed, and accordingly the condition for the existence of the improper integral is stronger, and JORDAN proves* that the multiple integrals as defined exist only absolutely. For the simple integrals the corresponding theorem does not hold, and JORDAN (p. 87) remarks that the loss of the theorem is due to the fact that the definition for the simple integrals is less exacting, although he expresses himself somewhat obscurely, in terms of one dimension not capable of immediate extension to more dimensions.

STOLZ† develops the JORDAN definition of improper double integrals for the case‡ of a region E with ordinary boundaries on which all the singularities c lie. The one-dimensional analogue of this case is that of a finite interval E with singular extremities, and for the simple integrals D is a sub-interval of E. In his last paper§ STOLZ, following WIRTINGER, notices that if (as in two dimensions) the D is allowed to be an interval-set the improper integrals exist only absolutely. (This, we have seen, was the implication of the remark of JORDAN.) And, further, he likewise notices (as a generalization for the special case of the theorem of JORDAN) that the double integrals would still exist only absolutely even if they were given a less exacting definition, the region D being required to be (as in the case of the simple integrals) connected‖ or of one piece.

To revert to the general case of multiple integrals, it is now apparent that for a definition formally less exacting than JORDAN's we may impose on the region D which converges to E the conditions: (1) the region $E - D$ consists of one or more regions, each being of one piece and each enclosing a singular point c; (2) the region D consists of one or more regions, each being of one piece and no two being capable of union as parts of an including connected region D; (3) the region D consists of a finite number of connected regions. And these conditions on the region D may be imposed or not imposed independently of one another.

In this paper I confine attention to the simple integrals over a finite interval $E = ab$; the conditions (1) and (2) then become identical; we impose the condition (3), and thus have the theory of the narrow or of the broad simple inte-

* loc. cit., pp. 80, 87.

† *Grundzüge*, vol. 3 (1899), p. 122 fg.

‡ This case is hardly equivalent to the general case, although STOLZ considers that it is (cf. loc. cit., p. 122). The proof (p. 141 fg.) of the theorem of the absolute existence of the improper double integrals I find inconclusive.

§ Wiener Berichte, vol. 108 (1899), pp. 1234–1238.

‖ SCHOENFLIES (loc. cit., pp. 203–205) in his report on this paper of STOLZ seems to confuse definitional conditions for the existence of an integral and properties resulting from its existence.

grals, according as the interval-set $E - D$ is not or is required to have on its every interval a singularity c. By developing the two theories simultaneously we obtain a new insight into the varying properties of the general and the absolutely convergent HARNACK integrals.

In my judgment the general integrals—even the non-absolutely convergent integrals—deserve to be classed with the proper definite integrals under the designation definite integrals. To be sure they lack in general many fundamental properties of the absolutely convergent integrals. We may however look at these matters in such a way that the properties of the different classes are more nearly equivalent. Let us think not of the proper or improper definite integral but, more generally, of the definite integral with respect to a point-set Ξ of content zero and thus write for instance $\int_{a\,\Xi}^{b} f(x)\,dx$, and speak of the Ξ-integrals. Then for every Ξ there is a theory of the narrow and of the broad Ξ-integrals, and for the various sets Ξ these theories, in so far as they relate to a single set Ξ or to a single set Ξ and its subsets Ξ_0 including the set Z of singularities of the integral function $F(x)$, are to a large extent the same. (I remark in passing that the desirable theorem (§ 2 V): if $F(x)$ is Ξ-integrable from a to b and the set Z is non-existent, then $F(x)$ is properly integrable from a to b and $\int_a^b F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx$, holds only if the content of the set Ξ is zero. And in this fact I see one of the strongest reasons for considering only such sets Ξ.)

In the sequel the theories of the narrow and the broad HARNACK integrals are developed from this point of view and otherwise essentially in the spirit of the original HARNACK papers. The necessary fixed hypotheses, etc., being introduced in § 1, I give in § 2 for the Ξ-integrals the definitions and a number of fundamental theorems, and then in § 3 I develop the properties of a function $F(x)$ Ξ-integrable from a to b. I bound the inquiry of § 3 in effect by the form of the four fundamental theorems of STOLZ. Then in § 4 the questions of absolute convergence are introduced. And in § 5 (added July 1, 1901) after exhibiting an important condition necessary and sufficient for the existence of the general or narrow HARNACK integral, I construct for the general closed point-set Ξ on ab of content zero a Ξ-integral from a to b for which the set Ξ is the set Z of singularities, and which is near every point $x = \xi$ essentially narrow or non-absolutely convergent; and in connection with the simplest case, $\Xi = (b)$, I exhibit the examples needed to show the error of various statements already referred to.

§ 1.

FUNDAMENTAL HYPOTHESES, DEFINITIONS, AND NOTATIONS.

1°. We consider the finite interval ab of values of the real variable x. In the proofs of the propositions we shall consider the case $a < b$, to which the case $a > b$ is immediately reducible.

2°. We consider further a point-set Ξ of points ξ which is of content zero. As explained in § 2 the set Ξ will later be supposed to be closed; this supposition involves no restriction of generality. The aggregate or sum $\Xi_1 + \Xi_2$ of two sets Ξ_1, Ξ_2, each closed and of content zero is again a set of the same kind.

3°. On the interval ab the real function $F(x)$ is supposed to be single valued wherever defined, and it is supposed to be defined certainly at all points x not of Ξ. Moreover it is supposed that $F(x)$ is capable of proper definite integration from a' to b' where the interval $a'b'$ is any interval lying on the interval ab and containing no point ξ and no limit-point ξ' of Ξ.

4°. With respect to $F(x)$ it is convenient to separate points $x = x_0$ into two classes. The point $x = x_0$ is *regular* if near x_0 (i. e., on some interval $x_0 - \delta \cdots x_0 + \delta$) $F(x)$ is everywhere defined, single valued, and limited; and otherwise it is *singular*. A singular point $x = \zeta$ may be both progressively and regressively singular or merely progressively ($\zeta = \zeta_+$) or regressively ($\zeta = \zeta_-$) singular. We speak of the singularities ζ and of the singular point-set Z of all points ζ. The set Z is closed. Obviously every singular point ζ is a point ξ or a limit-point ξ' of Ξ; if the set Ξ is closed, the set Z is a subset of it.

5°. A finite number of intervals such that no two have a common point is called an interval-set. Denoting by I an interval-set we denote its length, the sum of the lengths of its intervals, by D_I.

6°. Two intervals $i_1 i_2$ having a common inner point determine a definite interval $i_{12} = i_{21}$ common to i_1 and i_2. Two interval-sets $I_1 I_2$ having a common inner point determine a definite interval-set $I_{12} = I_{21}$ which may be called the set of intervals common to $I_1 I_2$, that is, every i_{12} of I_{12} is the interval common to certain two intervals i_1 of I_1, i_2 of I_2. The set common to a set I and an interval $x_1 x_2$ is denoted by $I_{x_1}^{x_2}$.

For brevity, especially in partitioning interval-sets, it is sometimes convenient to use these notations, even when the interval-sets in question do not exist; for example, an interval-set I is, by an interval $x_1 x_2$ separated into two interval-sets $I_{x_1}^{x_2}$, I' (of which one may not exist), and we write $I = I_{x_1}^{x_2} + I'$.

7°. An interval-set I is said to contain any point-set S, all of whose points are points of I; and in this case S is said to be of or to lie on I. Further an interval-set is said to enclose a point-set S if every point s and likewise every limit-point s' of S lies within some interval of I; and in this case S is said to be an inner set of or to lie within I. The interval-set I is said to contain or to enclose the point-set S narrowly, and S to lie respectively on or within I narrowly, if furthermore every interval of I contains or encloses at least one point s of S. It is convenient to denote by $I(S)$ an interval-set which encloses S narrowly, and by $I\{S\}$ one which encloses S not necessarily narrowly, or, say, one which encloses S broadly. Thus the broad is the generic enclosure, and the narrow is a specific enclosure.

8°. The set Ξ being by hypothesis of content zero, for any positive number ϵ there an interval-set I_ϵ enclosing Ξ narrowly and of length $D_{I_\epsilon} < \epsilon$.

9°. On the fixed interval ab in connection with any interval-set I (not necessarily lying on the interval ab) we introduce a function $F_I(x)$ by the following stipulation: according as the point x of ab lies or does not lie on I, $F_I(x)$ has the value 0 or $F(x)$, with the understanding that, if I is the symbol of a non-existent interval-set, $F_I(x) = F(x)$ for every x of ab. It is to be noticed that if I encloses Ξ, then $F_I(x)$ is on the interval ab everywhere defined, single valued, limited, and capable of proper definite integration from a to b.

§2.

THE DEFINITE INTEGRALS NARROW AND BROAD

$$\int_{a\,(\Xi)}^{b} F(x)\,dx, \qquad \int_{a\,\{\Xi\}}^{b} F(x)\,dx$$

OF THE FUNCTION $F(x)$ FROM a TO b WITH RESPECT TO THE SET Ξ: DEFINITIONS, CONDITIONS NECESSARY AND SUFFICIENT FOR EXISTENCE, AND FUNDAMENTAL THEOREMS.

We consider simultaneously the two cases:

(1) the narrow Ξ-integral, $\displaystyle\int_{a\,(\Xi)}^{b} F(x)\,dx$;

(2) the broad Ξ-integral, $\displaystyle\int_{a\,\{\Xi\}}^{b} F(x)\,dx$.

When these integrals are considered simultaneously and disjunctively (as in §§ 2, 3) we speak simply of

(3) the Ξ-integral: $\displaystyle\int_{a\,\Xi}^{b} F(x)\,dx$.

In case I denotes an interval-set enclosing Ξ narrowly or broadly the function $F_I(x)$ is capable of proper definite integration from a to b. We consider the proper definite integrals:

(4) $$\int_a^b F_I(x)\,dx,$$

for the various interval-sets I enclosing Ξ $^{\text{narrowly}}_{\text{broadly}}$, and denote by the respective limit-notations:

(5) $$\mathop{\mathrm{L}}_{D_I=0}^{I\,(\Xi)} \int_a^b F_I(x)\,dx; \quad \mathop{\mathrm{L}}_{D_I=0}^{I\,\{\Xi\}} \int_a^b F_I(x)dx,$$

certain finite constant limits, if existent. Then we give those limits the specific Ξ-integral notations:

$$\int_{a\,(\Xi)}^{b} F(x)dx; \quad \int_{a\,\{\Xi\}}^{b} F(x)\,dx. \tag{6}$$

Thus we have the explicit

Definition * 1. *The (existent)* $\begin{smallmatrix}\textit{narrow}\\ \textit{broad}\end{smallmatrix}$ *Ξ-integral:*

$$\begin{aligned} \int_{a\,(\Xi)}^{b} F(x)\,dx &= \mathop{\mathrm{L}}_{D_I=0}^{I\,|\,I(\Xi)} \int_a^b F_I(x)\,dx; \\ \int_{a\,\{\Xi\}}^{b} F(x)\,dx &= \mathop{\mathrm{L}}_{D_I=0}^{I\,|\,I\{\Xi\}} \int_a^b F_I(x)\,dx; \end{aligned} \tag{7}$$

is a certain finite constant such that for every positive number ϵ there exists a positive number† δ_ϵ such that

$$\left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi} F(x)\,dx \right| < \epsilon \tag{8}$$

for every interval-set‡ I enclosing Ξ $\begin{smallmatrix}\textit{narrowly}\\ \textit{broadly}\end{smallmatrix}$ *and of length $D_I < \delta_\epsilon$.*

Note 1. The broad Ξ-integrals constitute a special type of the narrow Ξ-integrals §. In this connection the adjectives *narrow* and *broad* may connote the fact that the body of properties of the narrow integrals is less extensive than and included in the body of properties of the broad integrals. The essentially narrow integrals are the not-broad narrow integrals.

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* This definition (of the narrow Ξ-integral) seems to express (for $\Xi = Z$) exactly HARNACK'S meaning in form as well as in content.

HARNACK defines the improper definite integral over a certain interval as a certain limit of the proper definite integral over the same interval of a modified function. There are certain advantages (cf. the remark of § 3 VI) in this type of definition. A different definition of this type is that of DE LA VALLÉE POUSSIN for the absolutely convergent generalized definite integrals. (With respect to this matter SCHOENFLIES (loc. cit., pp. 186, 187) has erred in setting in contrast the two limiting processes in question.)

† To the various δ's related in this paper to the arbitrary positive ε I give the notations $\delta_\epsilon, \delta'_\epsilon, \delta^1_\epsilon, \delta^2_\epsilon$, etc., where the superscripts are discriminating affixes and not exponents.

‡ This fundamental definitional property of a function $F(x)$ Ξ-integrable from a to b is (with modification of the δ_ϵ) considerably extended in theorem XIV of § 3.

§ That the narrow Ξ-integral (1) be a broad Ξ-integral (2) it is obviously necessary and sufficient that for every ε a δ^3_ϵ exist such that, in the notation (11),

$$\left| \int_J^{\cdot} F(x)\,dx \right| < \varepsilon$$

for every interval-set J lying on ab and containing no point ξ and of length $D_J < \delta^3_\epsilon$. Indeed this condition (cf. JORDAN, loc. cit., § 74, p. 77) is necessary and sufficient for the existence of the broad integral (2). The reader will compare this remark with § 3 XIII′.

Note 2. From the definition it is apparent that the Ξ-integral and the $(\Xi + \Xi')$-integral, where Ξ' is the set of all limit-points ξ' of Ξ, coexist and are equal; that is, if either exists, so does the other with the same value.—The set $\Xi + \Xi'$ is closed and of content zero.—*We suppose hereafter that the set Ξ is closed.* This supposition involves no essential loss of generality and it facilitates the phrasing of many proofs.

If this Ξ-integral exists we say that $F(x)$ is Ξ-integrable from a to b.

Definition 2. *The function $F(x)$ is progressively* $\substack{\textit{narrowly}\\ \textit{broadly}}$ *Ξ-integrable at $x = x_0$ if there is an interval $x_0 \cdots x_0 + \epsilon\,(\epsilon > 0)$ such that the* $\substack{\textit{narrow}\\ \textit{broad}}$ *Ξ-integral from x_0 to $x_0 + \epsilon$ exists.—Regressive Ξ-integrability at a point is similarly defined.*

Definition 3. *The function $F(x)$ is* $\substack{\textit{narrowly}\\ \textit{broadly}}$ *Ξ-integrable on ab $(a < b)$ if at a it is progressively, at b it is regressively, and at every other point of ab it is both progressively and regressively* $\substack{\textit{narrowly}\\ \textit{broadly}}$ *Ξ-integrable.*

Remark. It will appear (§ 3, theorem V, corollary) that if $F(x)$ is Ξ-integrable on ab it is Ξ-integrable from a to b.

One proves by the usual limit-considerations the following two theorems:

I. *The Ξ-integral if existent is uniquely existent.*

II. *For the existence of the definite* $\substack{\textit{narrow}\\ \textit{broad}}$ *Ξ-integral*

$$\int_{a\,\Xi}^{b} F(x)dx \tag{9}$$

it is necessary and sufficient that for every ϵ a δ_ϵ^1 exists such that

$$\left| \int_a^b F_{I_1}(x)dx - \int_a^b F_{I_2}(x)dx \right| < \epsilon \tag{10}$$

for every pair of interval-sets $I_1 I_2$ each enclosing Ξ $\substack{\textit{narrowly}\\ \textit{broadly}}$ *and of length less than δ_ϵ^1.*

Remark. If the Ξ-integral (9) exists, the number $\delta_{\epsilon/2}$ is effective as a number δ_ϵ^1 of the necessary condition (10). And if the sufficient condition (10) is fulfilled the number $\delta_{\epsilon/2}^1$ is effective as a number δ_ϵ connected with the conclusion that the Ξ-integral (9) exists.

III. *If the function $F(x)$ is Ξ-integrable from a to c and from c to b, where c lies between a and b, then the function $F(x)$ is Ξ-integrable from a to b, and*

$$\int_{a\,\Xi}^{b} F(x)dx = \int_{a\,\Xi}^{c} F(x)dx + \int_{c\,\Xi}^{b} F(x)dx.$$

One proves this theorem by use of the corresponding theorem for proper definite integrals together with theorem I. The converse of this theorem is theorem VIII of § 3.

Notation. For a function $f(x)$ properly integrable on the various intervals $a_l b_l$ $(a_l < b_l;\ l = 1, 2, \cdots, n)$ of an interval-set J we introduce the notation

$$\int_J f(x)\,dx, \tag{11}$$

by the definitional equation:

$$\int_J f(x)\,dx = \sum_{l=1}^{n} \int_{a_l}^{b_l} f(x)\,dx; \tag{12}$$

and, in case J is the symbol of a non-existent interval-set, it is convenient to understand by the notation (11) the constant zero.

Lemma I. If the function $f(x)$ is properly integrable from a to b and J is any interval-set of the interval ab, then

$$\left|\int_J f(x)\,dx\right| < D_J\, C,$$

where C is any positive number greater than $|f(x)|$ for every point x of the interval ab.

This is a fundamental theorem of the theory of proper definite integrals.

Lemma II. If the function $f(x)$ is properly integrable from a to b, then for every ϵ there is a δ'_ϵ such that

$$\left|\int_J f(x)\,dx\right| < \epsilon$$

for every interval-set J lying on ab and of length $D_J < \delta'_\epsilon$.

Lemma II follows from lemma I immediately. Lemma II is a generalization of the theorem of the uniform continuity on the X-interval ab of the definite integral function $\int_a^X f(x)\,dx$.

IV. *If the function $F(x)$ is properly integrable from a to b, then with respect to every set Ξ of content zero the function $F(x)$ is Ξ-integrable from a to b, and*

$$\int_{a\,\Xi}^{b} F(x)\,dx = \int_a^b F(x)\,dx.$$

*Corollary.** If the function $F(x)$ is properly integrable from a to b and a function $G(x)$ is defined and equal to $F(x)$ on the interval ab, except perhaps at points of a set Ξ of content zero, then $G(x)$ is Ξ-integrable from a to b, and

$$\int_{a\,\Xi}^{b} G(x)\,dx = \int_a^b F(x)\,dx.$$

* Compare the remark of STOLZ, *Grundzüge*, vol. 3 (1899), pp. 283, 284.

Taking a definite set Ξ we denote by I any interval-set enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$; by J the interval-set I_a^b, and by K the interval-set making up with J the interval ab. Either J or K may be non-existent, but one at least is existent. In these notations we have evidently

$$\int_a^b F(x)\,dx = \int_J F(x)\,dx + \int_K F(x)\,dx\,,$$

$$\int_a^b F_I(x)\,dx = \int_K F(x)\,dx\,,$$

and so

$$\int_a^b F_I(x)\,dx - \int_a^b F(x)\,dx = -\int_J F(x)\,dx\,.$$

From this equality, since $D_J \leqq D_I$, the truth of theorem IV appears by the use of lemma II, the definition of the Ξ-integral, and theorem I.

V. *If the function $F(x)$ exists on the interval ab as a single valued limited function of x, and if it is with respect to a certain set Ξ of content zero Ξ-integrable from a to b, then it is likewise properly integrable from a to b, and hence, by theorem IV,*

$$\int_a^b F(x)\,dx = \int_{a\,\Xi}^b F(x)\,dx\,.$$

This theorem is a corollary of the known

Lemma III. A necessary and sufficient condition for the proper integrability from a to b of a function $F(x)$ existent on the interval ab as a single valued limited function is this, that, with respect to a closed point set S of content zero, the function $F(x)$ be properly integrable from a' to b' where the interval $a'b'$ is any interval of ab containing no point s of the set S.

The theorem of lemma III was developed by PASCH * for the proper definite integral as defined by RIEMANN, and later independently by STOLZ † for the integral as defined by PEANO.‡ The two definitions are equivalent in content.

The lemma-theorem is a corollary of a theorem of DINI (1878: DINI-LÜROTH, § 185), which has close relations to the PEANO definition, and which is likewise given by STOLZ § independently, but with reference to another theorem of DINI (DINI-LÜROTH, § 184), a particular case of the one here referred to.

VI. *If $F(x)$ is Ξ-integrable from a to b and if Ξ_0 is any subset of Ξ, such that for every interval-set I_0 enclosing Ξ_0 the function $F_{I_0}(x)$ is properly integrable from a to b, then $F(x)$ is Ξ_0-integrable from a to b, and*

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* Mathematische Annalen, vol. 30 (1887), p. 147.

† Wiener Berichte, vol. 107 (1898), p. 736; *Grundzüge*, vol. 3 (1899), p. 272.

‡ *Lezioni di Analisi infinitesimale* (1893), § 103; Annali di Matematica, ser. 2, vol. 23 (1895), p. 157.

§ Monatshefte für Mathematik und Physik, vol. 7 (1896), p. 293.

$$(13) \qquad \int_{a\,\Xi_0}^{b} F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx.$$

Remark. The set Ξ_0 includes the subset Z_{ab} lying on ab of the singular set Z of the function $F(x)$ (cf. § 1 4°), and it will appear by theorem VIIA of § 3 that Ξ_0 may be any such subset of Ξ.

We shall prove that

$$(14) \qquad \left| \int_a^b F_{I_0}(x)\,dx - \int_{a\,\Xi}^{b} F(x)\,dx \right| < \epsilon$$

for every interval-set I_0 enclosing Ξ_0 $\substack{\text{narrowly}\\ \text{broadly}}$ and of length $D_{I_0} < \frac{1}{2}\delta_{\epsilon/2}$.

We extend I_0 by an interval-set J so that the interval-set I, $I = I_0 + J$, encloses Ξ $\substack{\text{narrowly}\\ \text{broadly}}$. Since Ξ has content zero and in accordance with the present hypothesis, in view of an obvious extension of lemma II from an interval ab, to an interval-set (in the present application, to the interval-set extending I_{0a}^{b} to the interval ab), we are able to choose J so that $D_J < \frac{1}{2}\delta_{\epsilon/2}$ and

$$(15) \qquad \left| \int_{J_a^b} F(x)\,dx \right| < \tfrac{1}{2}\epsilon.$$

Then $D_I < \delta_{\epsilon/2}$, and hence

$$(16) \qquad \left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi}^{b} F(x)\,dx \right| < \tfrac{1}{2}\epsilon.$$

Then the desired inequality (14) follows from (15), (16) since

$$\int_a^b F_I(x)\,dx + \int_{J_a^b} F(x)\,dx = \int_a^b F_{I_0}(x)\,dx.$$

§ 3.

Fundamental properties of a function $F(x)$ Ξ-integrable from a to b.

Fixed hypothesis. The following theorems concerning the function $F(x)$ involve the fixed hypothesis that the $\substack{\text{narrow}\\ \text{broad}}$ Ξ-integral

$$\int_{a\,\Xi}^{b} F(x)\,dx$$

exists.

I. *The function $F(x)$ is Ξ-integrable from b to a, and*

$$\int_{b\,\Xi}^{a} F(x)\,dx = -\int_{a\,\Xi}^{b} F(x)\,dx.$$

II. *The function $cF(x)$, where c is any constant, is Ξ-integrable from a to b, and*

$$\int_{a\,\Xi}^{b} cF(x)\,dx = c\int_{a\,\Xi}^{b} F(x)\,dx.$$

These two theorems follow immediately from the corresponding theorems concerning proper definite integrals together with the definition of the Ξ-integral.

III. *For every ϵ there is a δ_ϵ^1 such that the inequality:*

$$\left|\int_a^b F_{I_1}(x)\,dx - \int_a^b F_{I_2}(x)\,dx\right| < \epsilon,$$

holds for every pair of integral-sets $I_1 I_2$, each enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than δ_ϵ^1.

This theorem is the necessary condition given in theorem II of § 2.

IV. *For every ϵ there is a δ_ϵ^2 such that*

$$\left|\int_{x_1}^{x_2} F_{I_1}(x)\,dx - \int_{x_1}^{x_2} F_{I_2}(x)\,dx\right| < \epsilon \tag{1}$$

for every two points $x_1 x_2$ of the interval ab and every pair of interval sets I_1, I_2 each enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than δ_ϵ^2.

This theorem, whose proof will be given in connection with theorem X, of which it is a particular case, affirms that with respect to any two points x_1, x_2 of the interval ab the condition sufficient for the existence* of the Ξ-integral $\int_{x_1\,\Xi}^{x_2} F(x)\,dx$ is satisfied, and we obtain immediately theorems V and VI.

V. *The definite integral*

$$\int_{x_1\,\Xi}^{x_2} F(x)\,dx$$

exists where $x_1 x_2$ are any two points of the interval ab, and indeed uniformly† on the set of all such point-pairs $x_1 x_2$.

Corollary.‡ If the function $F(x)$ is Ξ-integrable on ab, then it is Ξ-integrable from a to b.

VI. *For every ϵ and every two points $x_1 x_2$ of the interval ab*

$$\left|\int_{x_1}^{x_2} F_I(x)\,dx - \int_{x_1\,\Xi}^{x_2} F(x)\,dx\right| \leqq \epsilon \tag{2}$$

*At this point one observes that our definition of the integral $\int_{a\,\Xi}^{b} F(x)\,dx$ is more convenient than a definition applicable merely to cases in which the point-set Ξ lies on the interval ab.

† In that δ_ϵ^2 is independent of $x_1 x_2$.

‡ Cf. definition 3 and theorem III of § 2. The proof is indirect and quite analogous to the proof of the corresponding theorem as to proper definite integrals.

for every interval-set I enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than δ_ϵ^2.

Remark. Theorem VI is useful especially in case $x_1 = a$, $x_2 = X$, when it establishes a relation on the X-interval ab between the various definite integral functions:

$$\int_{a\,\Xi}^{X} F(x)\,dx, \qquad \int_{a}^{X} F_I(x)\,dx,$$

where I denotes an interval-set enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$.

VII A. *If $F(x)$ is Ξ-integrable from a to b and if Ξ_0 is any subset of Ξ which contains the subset Z_{ab} lying on ab of the singular set Z, then $F(x)$ is Ξ_0-integrable from a to b, and*

$$\int_{a\,\Xi_0}^{b} F(x)\,dx = \int_{a\,\Xi}^{b} F(x)\,dx = \int_{a\,Z}^{b} F(x)\,dx.$$

VII B. *If the function $F(x)$ is integrable from a to b with respect to each of two point-sets Ξ_1, Ξ_2 of content zero, then it is likewise with respect to their common subset Ξ_{12} and with respect to their aggregate set $\Xi_1 + \Xi_2$, and the four integrals are equal:*

$$\int_{a\,\Xi_1+\Xi_2}^{b} F(x)\,dx = \int_{a\,\Xi_1}^{b} F(x)\,dx = \int_{a\,\Xi_2}^{b} F(x)\,dx = \int_{a\,\Xi_{12}}^{b} F(x)\,dx.$$

The theorem VII A is a generalization of theorem VI of § 2, and, in the light of § 3 V and § 2 V, it is a corollary of that theorem, the function $F_{I_0}(x)$ for any interval-set I_0 enclosing Ξ_0 being on ab a single valued limited function of x, as one proves from the definition of the singular set Z (§ 1, 4°) indirectly by the usual interval-halving process.

As to the theorem VII B, the sets Ξ_1, Ξ_2, Ξ_{12} have the set Z of essential singularities as a common subset. Hence by VII A the Ξ_{12}-integral exists and we have:

$$\int_{a\,\Xi_1}^{b} F(x)\,dx = \int_{a\,\Xi_2}^{b} F(x)\,dx = \int_{a\,\Xi_{12}}^{b} F(x)\,dx.$$

Denoting for the moment by δ_ϵ the least of the δ_ϵ's related to these three integrals, we consider an interval-set I enclosing $\Xi_1 + \Xi_2$ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length $D_I < \delta_{\epsilon/3}$. Then, setting in a definite way:

$$I = I_{12} + I_1 + I_2,$$

where I_{12} encloses Ξ_{12} narrowly, and $I_{12} + I_1$ encloses Ξ_1 $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$, and $I_{12} + I_2$ encloses Ξ_2 $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$, we have obviously, in view of the equality of the three integrals, the three inequalities:

$$\left| \int_{a}^{b} F_J(x)\,dx - \int_{a\,\Xi_{12}}^{b} F(x)\,dx \right| < \tfrac{1}{3}\epsilon \quad (J = I_{12},\ I_{12} + I_1,\ I_{12} + I_2),$$

from which, in view of the fact that $F(x)$ is properly integrable on I_1 and I_2, we deduce readily the inequality:

$$\left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi_{12}}^b F(x)\,dx \right| < \epsilon,$$

needed to complete the proof of the theorem.

VIII. *The relation:*

$$\int_{x_1\,\Xi}^{x_2} F(x)\,dx + \int_{x_2\,\Xi}^{x_3} F(x)\,dx = \int_{x_1\,\Xi}^{x_3} F(x)\,dx, \tag{3}$$

holds for every three points $x_1 x_2 x_3$ of the interval ab.

This follows easily from the corresponding theorem concerning proper definite integrals. We have for every interval-set I enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$

$$\int_{x_1}^{x_2} F_I(x)\,dx + \int_{x_2}^{x_3} F_I(x)\,dx = \int_{x_1}^{x_3} F_I(x)\,dx. \tag{4}$$

For an interval-set I of length $D_I < \delta_\epsilon^2$ the two sides of the equality (4) differ respectively from the two corresponding sides of (3) by less than 2ϵ. Hence the two sides of (3) differ by less than 4ϵ and so are indeed equal.

IX. *The definite integral function:*

$$\phi(X) = \int_{a\,\Xi}^{X} F(x)\,dx,$$

is a continuous function of the variable upper limit X on the X-interval ab.

Introducing with respect to an ϵ an interval-set I enclosing Ξ narrowly and of length $D_I < \delta_{\epsilon/2}^2$, we have by the use of theorems VIII and VI the relation:

$$|\phi(X_1) - \phi(X_2)| < \epsilon, \tag{5}$$

where X_1 and X_2 are any two points of the X-interval ab such that

$$|X_1 - X_2| < \delta'_{\epsilon/2},$$

the $\delta'_{\frac{1}{2}\epsilon}$ being the number related to the properly integrable function $F_I(x)$ quâ $f(x)$ by lemma II of § 2. And this relation establishes the uniform continuity of $\phi(X)$ on the X-interval ab. This theorem is generalized in theorems XIII′ and XIII″.

Remark. Up to this point of the present development the theories of the narrow and of the broad Ξ-integrals are equivalent. From this point on they diverge; I secure for the narrow Ξ-integrals conclusions analogous to the conclusions for the broad Ξ-integrals, by inserting an additional hypothesis in the first case of every theorem.

X. *For every* ϵ *there is a* δ_ϵ^2 *such that for every interval-set* J *lying on the interval* ab

$$\left| \int_J F_{I_1}(x)\,dx - \int_J F_{I_2}(x)\,dx \right| < \epsilon \tag{6}$$

for every pair $I_1 I_2$ *of interval-sets satisfying the following conditions:*

(1) *Each interval-set* $I_1 I_2$ *encloses* Ξ $\begin{smallmatrix}\textit{narrowly}\\ \textit{broadly}\end{smallmatrix}$;

(2) *Each length* $D_{I_1} D_{I_2}$ *is less than* δ_ϵ^2;

and, moreover, in the first case, whenever J *consists of more than a single interval,*

(3) *Every interval* i *of* I_1 *or* I_2 *which encloses an interval* l *joining two consecutive intervals of* J *has with respect to every such enclosed interval* l *the following property: the interval* i *contains either on the interval* l *a point* ξ *or on each of the adjoining intervals of* J *a point* ξ.

Remark. In case J consists of a single interval, theorem X becomes theorem IV; and in case J consists of more than a single interval, the third condition is fulfilled in particular if there is imposed on J an obvious restriction, and likewise, if there are imposed on I_1 and I_2 obvious restrictions, which have reference to the interval-set J in question. In the theorem as here stated, the uniformity with respect to the set of all interval-sets I_1, I_2, J of the nature specified is especially important for the sequel.

We use the determination

$$\delta_\epsilon^2 = \tfrac{1}{2}\delta_\epsilon^1, \tag{7}$$

and have from theorem III the inequality:

$$\left| \int_a^b F_{I_1}(x)\,dx - \int_a^b F_{I_2}(x)\,dx \right| < \epsilon, \tag{8}$$

valid for every ϵ and for every pair of interval-sets $I_1 I_2$ each enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than $2\delta_\epsilon^2$.

For the purposes of an indirect proof of theorem X, we suppose that the inequality:

$$\left| \int_J F_{I_1}(x)\,dx - \int_J F_{I_2}(x)\,dx \right| \geqq \epsilon, \tag{9}$$

holds for a certain ϵ, a certain interval-set J lying on the interval ab, and a certain pair of interval-sets $I_1 I_2$ each enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than δ_ϵ^2, and, in the first case, satisfying the condition (3) of X, and proceed to exhibit for this ϵ two interval-sets $I_3 I_4$ each enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than $2\delta_\epsilon^2$ for which, in contradiction with (8), the inequality:

$$\left| \int_a^b F_{I_3}(x)\,dx - \int_a^b F_{I_4}(x)\,dx \right| \geqq \epsilon, \tag{10}$$

has validity.

It is easy to see how the inequality (9) may be transformed into an inequality of the general form (10). For denoting by K the interval-set obtained by excising J from the interval ab, we have for every pair of sets $I_3 I_4$ enclosing Ξ, the equation:

$$(11)\qquad \int_a^b F_{I_3}(x)\,dx - \int_a^b F_{I_4}(x)\,dx = \begin{cases} \left(\int_J F_{I_3}(x)\,dx - \int_J F_{I_4}(x)\,dx\right) \\ + \left(\int_K F_{I_3}(x)\,dx - \int_K F_{I_4}(x)\,dx\right), \end{cases}$$

since the functions $F_{I_3}(x)$, $F_{I_4}(x)$ are properly integrable from a to b. Hence from (9) we obtain the relation (10) for every pair of sets $I_3 I_4$ each enclosing Ξ and such that

$$(12)\qquad \begin{aligned} F_{I_3}(x) &= F_{I_1}(x), \qquad F_{I_4}(x) = F_{I_2}(x) && (x \text{ within } J), \\ F_{I_3}(x) &= F_{I_4}(x) && (x \text{ within } K), \end{aligned}$$

the various proper definite integrals having values independent of the values of the respective integrand functions at the extremities of the intervals of J and K.

Thus we seek to find two sets $I_3 I_4$ each enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than $2\delta_\epsilon^2$, and, moreover, such that they agree on J with $I_1 I_2$ respectively and on K with each other.

We denote by L the complete x-axis with the omission of the inner points of J. Thus L contains K. And we denote by $I_J I_L$ respectively the J-section and the L-section of an interval-set I, so that

$$(13)\qquad I = I_J + I_L.$$

With respect to J and L an interval i may lie on J or L and so be itself an interval i_J or i_L. Otherwise the interval i is by J and L partitioned into a sequence of two or more parts i_J, i_L, two adjoining parts being a part i_J and a part i_L. The two extreme parts may or may not be undivided intervals j or l. The intermediate parts are undivided intervals j or l.

We must for an interval-set I enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ distribute the intervals i_L of I_L into two complementary classes.

An interval i_L is an interval i''_L if it contains no point ξ and is either an extreme part of the interval i of I from which it is derived or an intermediate part each of whose adjoining parts i_J contains a point ξ. An interval i_L is an interval i'_L if it either contains a point ξ or is of its interval i an intermediate part one (at least) of whose adjoining intervals i_J contains no point ξ.

Thus we have two interval-sets $I'_L I''_L$ such that $I_L = I'_L + I''_L$.

In case I encloses Ξ narrowly we notice two facts which are of use in the immediate sequel. An interval i which lies on J or L is an interval i_J or i'_L. An interval i undergoing partition with respect to J and L contains at least one part i_J and the sum of its parts i_J, i'_L is an interval-set every interval of which contains a point ξ. These intervals of i are the intervals arising by the excision from i of its various parts i''_L. The interval i itself contains a point ξ. If it contains no part i''_L it itself is the interval in question. If, however, the interval i contains one or more parts i''_L, we consider any such part i''_L. If i''_L is an extreme part the adjoining part i_J contains a point ξ or has in progressive consecution an adjoining part i'_L, since an adjoining part i''_L would be an extreme part, and the interval i consisting of these three parts would have no point ξ. If this third part has no point ξ, there is a fourth part, i_J. If this part i_J has no point ξ, there is a fifth part i'_L. Thus the parts i_J, i'_L enter in alternation and the interval consisting of this complete and unbroken sequence of parts i_J, i'_L contains a point ξ. If on the other hand i''_L is an intermediate part each adjoining part i_J contains a point ξ.

In these notations we take $I_3 I_4$ as follows:*

$$I_3 = I_{1J} + I'_{1L} + I'_{2L}, \quad I_4 = I_{2J} + I'_{1L} + I'_{2L}. \tag{14}$$

And these sets satisfy the prescribed conditions:

(1) Each set is of length less than $2\delta_\epsilon^2$;

(2) $$I_{3J} = I_{1J}, \quad I_{4J} = I_{2J};$$

(3) $$I_{3L} = I_{4L} = I'_{1L} + I'_{2L};$$

(4) Each set encloses Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$.

We need to prove that I_3 and so I_4 encloses Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$. The set Ξ is by hypothesis closed.

We consider first any particular point ξ. If ξ lies within J or L it is enclosed by an interval of I_{1J} or I'_{1L}, and so by an interval of I_3. If, however, ξ is common to J and L it is an extremity of an interval of each set; the point ξ lies within a certain interval of I_1, and it is the point of junction of two intervals of $I_{1J} I'_{1L}$ respectively, and in $I_3 = I_{1J} + I'_{1L} + I'_{2L}$ these two intervals perhaps with others form an interval i_3 enclosing the point ξ. Thus the interval-set I_3 encloses Ξ at least broadly and theorem X is proved for the second case.

For the first case we consider any particular interval i_3 of I_3 and now need to prove merely that it contains a point ξ. The interval i_3 arises by the union of certain intervals of $I_{1J} I'_{1L} I'_{2L}$. In the introduction of the notation I'_L

* In the second case, one may determine suitable interval-sets I_3, I_4 more simply as follows:

$$I_3 = I_{1J} + I''', \quad I_4 = I_{2J} + I''',$$

where I''' is the interval-set common to I_{1L} and I_{2L}.

for an interval-set $I = I(\Xi)$ it was noted that every interval i_J or i'_L forms part of a sequence of intervals $i_J i'_L$ constituting an interval containing a point ξ. Thus i_3 contains a point ξ or it consists of a single interval i'_{2L} which contains no point ξ. But this latter case is impossible. For an interval i'_{2L} containing no point ξ is an intermediate part of an interval i_2 one of whose adjoining parts i_{2J} contains no point ξ. But this is impossible, by virtue of the third hypothesis of theorem X.

Thus, indeed, every interval i_3 contains a point ξ. And now theorem X is proved also for the first case, and, hence, completely.

XI. *For every interval-set J lying on the interval ab there exists a constant, in notation,*

$$\int_{J\Xi} F(x)\,dx, \tag{15}$$

the integral on the interval-set J with respect to the point-set Ξ of the function $F(x)$, such that for every ϵ the relation:

$$\left| \int_J F_I(x)\,dx - \int_{J\Xi} F(x)\,dx \right| \leqq \epsilon, \tag{16}$$

holds for every interval-set I satisfying the following conditions:

(1) *The set I encloses Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$;*

(2) *The length D_I is less than δ_ϵ^2;*

and, moreover, in the first case, whenever J consists of more than a single interval,

(3) *Every interval i of I which encloses an interval l joining two consecutive intervals of J has with respect to its every such enclosed interval l the following property: the interval i contains either on the interval l a point ξ or on each of the adjoining intervals of J a point ξ.*

This theorem follows directly from the preceding theorem by the usual limit-considerations, wherein it needs to be noticed that interval-sets $I_1 I_2$ satisfying the conditions 1, 2, 3 of that theorem actually exist for every ϵ and interval-set J of the interval ab.

Remark. If J is a single interval $a'b'(a' < b')$ then obviously

$$\int_{J\Xi} F(x)\,dx = \int_{a'\Xi}^{b'} F(x)\,dx. \tag{17}$$

This remark is generalized in the following theorem:

XII. *For every interval-set J of the interval ab,*

$$\int_{J\Xi} F(x)\,dx = \sum_{l=1}^{n} \int_{a_l\Xi}^{b_l} F(x)\,dx, \tag{18}$$

where the intervals $a_l b_l (a_l < b_l;\ l = 1, \cdots, n)$ are the intervals of J.

For every interval-set I enclosing Ξ we have, by the definition (11) of § 2,

$$\int_J F_I(x)\,dx = \sum_{l=1}^{n} \int_{a_l}^{b_l} F_I(x)\,dx. \tag{19}$$

We introduce a set I which satisfies with respect to J and so with respect to every interval $a_l b_l$ of J the conditions of theorem XI for $\epsilon = \epsilon'/(n+1)$ where ϵ' is any particular positive constant. Then by theorem XI we have

$$\left| \int_J F_I(x)\,dx - \int_{J\Xi} F(x)\,dx \right| \leqq \frac{\epsilon'}{n+1}, \tag{20}$$

$$\left| \int_{a_l}^{b_l} F_I(x)\,dx - \int_{a_l\Xi}^{b_l} F(x)\,dx \right| \leqq \frac{\epsilon'}{n+1} \qquad (l = 1, 2, \cdots, n), \tag{21}$$

and then from (19), (20), (21) we have

$$\left| \int_{J\Xi} F(x)\,dx - \sum_{l=1}^{n} \int_{a_l\Xi}^{b_l} F(x)\,dx \right| \leqq \epsilon', \tag{22}$$

from which the theorem follows at once.

Remark. Theorem XI is to be understood as applying also to the case in which the symbol J denotes a non-existent interval-set, the constant in question having the value 0. Then in view of theorem XII it is evident that with respect to an interval-set J lying on the interval ab and a function $F(x)$ Ξ-integrable from a to b a definition of the symbol $\int_{J\Xi} F(x)\,dx$ might be given similar to that given in § 2 for the symbol $\int_J f(x)\,dx$ with respect to a function $f(x)$ properly integrable from a to b.

XIII′. *For every ϵ there exists a δ_ϵ^3 such that*

$$\left| \int_{J\Xi} F(x)\,dx \right| < \epsilon \tag{23}$$

for every interval-set J satisfying the following conditions:

(1) *The interval-set J lies on the interval ab;*

(2) *The length D_J is less than δ_ϵ^3;*

and, moreover, in the first case, whenever J consists of more than a single interval,

(3′) *The interval-set J has with respect to every interval l joining two consecutive intervals of J the following property: either the interval l contains a point ξ or each of the adjoining intervals of J contains a point ξ.*

XIII″. *In the first case: For every two positive numbers $\epsilon\epsilon'$ there exists a $\delta_{\epsilon\epsilon'}^3$ such that*

$$\left| \int_{J\Xi} F(x)\,dx \right| < \epsilon \tag{23}$$

for every interval-set J satisfying the following conditions:

(1) *The interval-set J lies on the interval ab;*

(2) *The length D_J is less than $\delta^3_{\epsilon\epsilon'}$;*

(3″) *Whenever J consists of more than a single interval, and has an interval l joining two consecutive intervals of J, and lying within the ϵ'-neighborhood of a point ξ, it has with respect to every such interval l the following property: either the interval l contains a point ξ or each of the adjoining intervals of J contains a point ξ.*

Note 1. The condition 3′ of XIII′ is satisfied in particular by interval-sets J satisfying the condition:

(3‴) *Every interval of J contains a point ξ.*

Note 2. For an interval-set I enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length less than δ^3_ϵ the inequality

$$\left| \int_{I^b_a \Xi} F(x)\,dx \right| < \epsilon$$

holds. This is a particular case of XIII′. With δ^3_ϵ replaced by δ_ϵ, it follows from the definition of the Ξ-integral from a to b, in view of the relation:

$$\int_{I^b_a \Xi} F(x)\,dx = \int_{a\Xi}^{b} F(x)\,dx - \int_a^b F_I(x)\,dx,$$

derived from theorem XII, together with § 2 IV.

Theorem XIII is for functions Ξ-integrable a generalization* of the second lemma of § 2 for functions properly integrable and it is proved by means of that lemma and theorem XI.

We take a particular interval-set I enclosing Ξ $\begin{smallmatrix}\text{narrowly}\\ \text{broadly}\end{smallmatrix}$ and of length D_I less than $\delta^2_{\epsilon/2}$, and moreover, in connection with theorem XIII″, of intervals each of length at most ϵ'. The function $F_I(x)$ is properly integrable from a to b.

An effective number δ^3_ϵ, or, for the second theorem of the first case, $\delta^3_{\epsilon\epsilon'}$, is the smaller of certain numbers: viz., the number $\delta'_{\epsilon/2}$ connected with this function $F_I(x)$ quâ $f(x)$ by the second lemma of § 2, and if I^b_a consists of more than a single interval, the various lengths of the intervals joining two consecutive intervals of I^b_a. Thus an interval j of an interval-set J of length $D_J < \delta^3_\epsilon$ or $\delta^3_{\epsilon\epsilon'}$ has points in common with at most one interval i of I.

This number δ^3_ϵ or $\delta^3_{\epsilon\epsilon'}$ is indeed effective. For the inequality (23) follows at once from the simultaneous inequalities:

$$\left| \int_J F_I(x)\,dx \right| < \tfrac{1}{2}\epsilon, \tag{24}$$

* The inequality (23) of XIII holds if on the right ε be replaced by 2ε and on the left the absolute value of the Ξ-integral over the interval-set J be replaced by the sum of the absolute values of the integrals over the constituent intervals of J. This is seen by considering separately the intervals yielding positive integrals and those yielding negative intergals. Cf. JORDAN, loc. cit., p. 226, ll. 9, 10; pp. 224–5.

$$(25) \qquad \left| \int_J F_I(x)\,dx - \int_{J\,\Xi} F(x)\,dx \right| \leqq \tfrac{1}{2}\epsilon,$$

where J is an interval-set satisfying the conditions of theorem XIII. The inequality (24) is by virtue of the second lemma of § 2, since $D_J < \delta'_{\epsilon/2}$. The inequality (25) is by virtue of theorem XI.

And theorem XI is, indeed, applicable. For J lies on the interval ab, and I encloses Ξ $\substack{\text{narrowly}\\ \text{broadly}}$ and is of length D_I less than $\delta^2_{\epsilon/2}$. Thus the double theorem XIII is proved, in the second case, and also in the first case, if J consists of a single interval.

In the first case, if J consists of more than a single interval, we have further to prove that condition (3) of the hypothesis of theorem XI is satisfied. We consider the interval-sets J and I with respect to an interval l joining two consecutive intervals j', j'' of J. If l contains a point ξ or if j' and j'' each contain a point ξ, then an interval i of I enclosing l, if one there be, has with respect to l the property of condition (3) of theorem XI, that is, the interval i contains either on the interval l a point ξ or on each interval j', j'' a point ξ. For each point ξ lies in some interval of I, and since j' and j'' intersect i neither intersects a second interval of I. If, however, l contains no point ξ and either j' or j'' contains no point ξ, then no interval i encloses l; for we have now to do with XIII″, and by hypothesis (3″), the interval l lies within the ϵ'-neighborhood of no point ξ, while every interval i contains a point ξ and is of length at most ϵ'.

XIV. *The function $F_I(x)$ is properly integrable from a to b and the inequality:*

$$(26) \qquad \left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi}^b F(x)\,dx \right| < \epsilon,$$

holds for every interval-set I whose length D_I is less than δ^3_ϵ or $\delta^3_{\epsilon\epsilon'}$, and moreover, in the first case, whose interval-set I^b_a, quâ interval-set J satisfies the condition* (3′) or (3″) of theorem* XIII, *provided moreover (in each case) that the interval-set I contains the singular point-set* Z *of the function $F(x)$ in such a way that if a singular point ζ is an extremity of an interval i of I it is singular merely towards the interior of i.*

Note. This theorem is, in accordance with note 2 of theorem XIII, *of the nature of an extension of the definitional property of the function $F(x)$ Ξ-integrable from a to b.*—In particular, we see that *the relation* (26) *holds for every interval-set I of length $D_I < \delta^3_\epsilon$ whose corresponding function $F_I(x)$ is properly integrable from a to b, which contains Ξ, and, in the first case, whose every*

* The $\delta^3_{\epsilon\epsilon'}$ enters only in the first case and then only if the hypothesis imposes (merely) the condition (3″) of theorem XIII″.

interval contains a point ξ *of* Ξ. Such an interval-set I obviously may perhaps not enclose Ξ, for an interval i may have a limit-point ξ' of Ξ as an extremity.

We proceed to prove theorem XIV. The singular point-set Z was defined in § 1 4°. From the final condition as to the interval-set I it follows by the usual process of indirect proof that the function $F_I(x)$ is on the interval ab everywhere defined, single valued, and limited. It is moreover Ξ-integrable from a to b, by virtue of theorems III and IV of § 2, and V and VIII of § 3. Thus by theorem V of § 2 it is properly integrable from a to b. Further by these theorems and theorem XII we have the equality:

$$(27)\qquad \int_a^b F_I(x)\,dx = \int_{a\,\Xi}^b F_I(x)\,dx = \int_{a\,\Xi}^b F(x)\,dx - \int_{I_a^b\,\Xi} F(x)\,dx,$$

from which the theorem follows by virtue of theorem XIII.

XV. *If the function* $F(x)$ *is integrable from* a *to* b *with respect to a closed point-set* Ξ *of content zero, and if* H *is any point-set of content zero, such that* $\Xi + \mathrm{H}$ *is closed, and which, moreover, in the first case, satisfies the condition specified below, then the function* $F(x)$ *is* $(\Xi + \mathrm{H})$*-integrable from* a *to* b, *and*

$$(28)\qquad \int_{a\,\Xi+\mathrm{H}}^b F(x)\,dx = \int_{a\,\Xi}^b F(x)\,dx.$$

The set H *in the first case has the following property: there exist on the interval* ab *at most a finite number of intervals* $a'b'$ *enclosing no point* ξ *and no point* η *and having for extremities points* η *which are not points* ξ.

In the second case this theorem is evident, since the interval-sets I enclosing $\Xi + \mathrm{H}$ broadly are amongst those enclosing Ξ broadly.

We proceed to prove the theorem in the first case. An interval-set K enclosing $\Xi + \mathrm{H}$ narrowly is with respect to Ξ separated into an interval-set I enclosing Ξ narrowly and a complementary interval set J perhaps not existent.

With respect to a number ϵ we shall prove that

$$(29)\qquad \left| \int_a^b F_K(x)\,dx - \int_{a\,\Xi}^b F(x)\,dx \right| < \epsilon$$

for every such interval-set K of length $D_K < \delta_\epsilon^\iota$, where δ_ϵ^ι is the least of the three numbers $\delta_{\epsilon/2}$, ϵ', and $\delta^\iota_{\epsilon/2\,\epsilon'}$, where ϵ' is a certain positive number to be specified in the sequel.

Indeed the inequality (29) follows at once from the fact that the relations:

$$(30)\qquad \int_a^b F_K(x)\,dx = \int_a^b F_I(x)\,dx - \int_{J_a^b\,\Xi} F(x)\,dx,$$

$$(31) \qquad \left| \int_a^b F_I(x)\,dx - \int_{a\,\Xi}^b F(x)\,dx \right| < \tfrac{1}{2}\,\epsilon,$$

$$(32) \qquad \left| \int_{J_a^b\,\Xi} F(x)\,dx \right| < \tfrac{1}{2}\,\epsilon,$$

hold simultaneously. Of these relations the first and second are clear, and the third follows from theorem XIII'', in connection with the stipulations $D_{J_a^b} \leqq D_K < \delta_\epsilon^1 < \epsilon'$, $\delta_{\epsilon/2\,\epsilon'}^3$. In order to secure this application of theorem XIII'' we are to determine the positive number ϵ' so that every interval l joining two consecutive intervals of J_a^b either contains a point ξ or lies within the ϵ'-neighborhood of no point ξ. This determination is made by means of the final condition of the theorem.

We denote by H* the set of points $\eta = \eta^*$ which lie on the interval ab and which are not points ξ; by H_* the set obtained by adding to H* the points a and b; and for brevity by J_0 the set J_a^b.

An interval j of J contains a point η and no point ξ. An interval j_0 of J_0 contains a point η_* and no point ξ. Two consecutive intervals j_0', j_0'' of J determine an interval $a'b'$ where a' is that point η_* of j_0' nearest to j_0'', and where similarly b' is that point η_* of j_0'' nearest to j_0'. These points $a'b'$ are definite points, for the sets H_* on the two intervals $j_0' j_0''$ are closed sets. This interval $a'b'$ encloses no ξ and no η,—except perhaps within the interval l joining j_0' and j_0''. Now if l encloses no point ξ it encloses no interval i of I, and hence, since $j_0' j_0''$ are consecutive intervals of J_0, it encloses no point η. Thus the two consecutive intervals $j_0' j_0''$ of J_0 determine a joining-interval l containing a point ξ or else an interval $a'b'$ enclosing no point ξ and no point η and having for extremities points η_*.

Now a enters as an η_*-extremity of such an interval at most once, and likewise b enters at most once. And by the present hypothesis at most a finite number of such intervals $a'b'$ with η^*-extremities exist.

We denote by $3\epsilon'$ the least of the lengths of the various intervals $a'b'$ with η_*-extremities, setting $3\epsilon' = 1$, if no such interval exists. Thus ϵ' is a definite positive number.

Then, since $D_K < \epsilon'$, the intervals $j_0' j_0''$ are each of length less than ϵ', while their interval $a'b'$, whenever it contains no point ξ, is of length at least $3\epsilon'$ and hence their joining-interval l, whenever it contains no point ξ, is of length greater than ϵ' and so lies within the ϵ'-neighborhood of no point ξ.—Thus theorem XIII'' is available for the completion of the proof of theorem XVII.

XVI. *In the first case: If the function $F(x)$ is Ξ-integrable from a to b, and hence* Z-integrable, if however it is not* (Z + H)*-integrable, and hence**

* Theorem VIIA.

not $(\Xi + \mathrm{H})$*-integrable, where* H *is a closed point-set of content zero, then there is at least one point* $\zeta = \zeta_0$ *of* Z *on* ab *at which* $F(x)$ *fails either of progressive or of regressive* $(\mathrm{Z} + \mathrm{H})$*- and so of* $(\Xi + \mathrm{H})$*-integrability. Hence, by theorem* XV, *in every progressive or in every regressive neighborhood of such a point* ζ_0 *there is an infinitude of intervals* $a'b'$ *enclosing no point* ξ *and no point* η *and having for extremities points* η *which are not points* ξ.

For if there were no such point ζ_0, then $F(x)$ would be $(\mathrm{Z} + \mathrm{H})$-integrable on ab and hence from a to b. We need to recall the statements of 4° of § 1, the definitions 2 and 3 and theorem IV of § 2, and the corollary of theorem V of § 3.

Remark. According to theorem XV the hypotheses of theorem XVI imply that the Ξ-integral of $F(x)$ is essentially narrow (that is, not broad). It has not been proved however that, in this case, there exists a point-set H for which the narrow $(\Xi + \mathrm{H})$-integral of $F(x)$ is non-existent.

XVII. *If with respect to two closed point-sets* Ξ, H *of content zero two functions* $F(x)$, $G(x)$ *are respectively integrable from* a *to* b, *and if, in the first case,* Ξ, H *are so related (for instance, as indicated in theorem* XV) *that each function is* $(\Xi + \mathrm{H})$*-integrable from* a *to* b, *then* $F(x) + G(x)$ *is* $(\Xi + \mathrm{H})$*-integrable from* a *to* b, *and*

$$\int_{a\,\Xi}^{b} F(x)\,dx + \int_{a\,\mathrm{H}}^{b} G(x)\,dx = \int_{a\,\Xi+\mathrm{H}}^{b} \big(F(x) + G(x)\big)\,dx.$$

This theorem is easily seen to be true.

§ 4.

Concerning the absolutely convergent $\substack{\text{narrow}\\\text{broad}}$ Ξ-integrals.

Definition. The $\substack{\text{narrowly}\\\text{broadly}}$ existent integral $\int_{a\Xi}^{b} F(x)\,dx$ is said to converge or exist (as a limit) *absolutely* in case the integral $\int_{a\Xi}^{b} |F(x)|\,dx$ exists $\substack{\text{narrowly}\\\text{broadly}}$.

Making suitable use of the references to the work of Jordan and Stolz given in the introduction, the reader will readily construct the proofs of the following theorems.

I. *The upper limits*

$$\underset{I|I(\Xi)}{\overline{\mathrm{L}}} \int_a^b F_I(x)\,dx, \quad \underset{I|I\{\Xi\}}{\overline{\mathrm{L}}} \int_a^b F_I(x)\,dx$$

with respect to a non-negative function $F(x)$ *on the set of all the interval-sets* I *enclosing* Ξ *respectively narrowly or broadly are equal. If they are finite, the* Ξ*-integral*

$$\int_{a\,\Xi}^{b} F(x)\,dx$$

exists broadly and is equal to the common value of the upper limits. Conversely, if the Ξ-integral exists (even) narrowly, then the upper limits are finite, and the Ξ-integral exists broadly.

II. *If the integral*

$$\int_{a\,\Xi}^{b} |F(x)|\,dx$$

exists (if narrowly, then broadly), then the integral

$$\int_{a\,\Xi}^{b} F(x)\,dx$$

exists broadly, and hence as a broad integral it converges absolutely. Conversely, if the latter integral exists absolutely (even) narrowly, it exists also absolutely broadly.

III. *The broad and absolutely convergent Ξ-integrals are in fact identical; hence, the essentially narrow and the non-absolutely convergent Ξ-integrals are in fact identical.*

§ 5.*

Concerning the general or narrow Ξ-integrals

$$\int_{a\,(\Xi)}^{b} F(x)\,dx.$$

1°. The following determination of the general point-set Ξ lying on ab closed and of content zero is well known.

Every finite or numerably infinite set H of intervals $h_\nu = a_\nu b_\nu$ ($\nu = 1, 2, \cdots, n$ or $\nu = 1, 2, 3, \cdots$) lying everywhere densely on ab and no two having a common inner point determines a closed nowhere dense point-set Ξ; and, conversely, every closed nowhere dense point-set Ξ of ab is so determinable by its "point-free" intervals h_ν. Every inner point x of ab not a point ξ lies lies within a definite interval h_ν. The points ξ together with a and b are the extremities $a_\nu b_\nu$ of the intervals h_ν and the limit-points of such extremities.—For later use I introduce subintervals $h'_\nu = a'_\nu b'_\nu$ of the respective intervals $h_\nu = a_\nu b_\nu$, with the understanding that $a'_\nu = a_\nu$ if $a_\nu = a =$ not-ξ; $b'_\nu = b_\nu$ if $b_\nu = b =$ not-ξ; and otherwise $a_\nu < a'_\nu < b'_\nu < b_\nu$.

The content of the set Ξ so determined is $|a - b| - \sum_\nu |a_\nu - b_\nu|$, so that for our purposes

$$\sum_\nu |a_\nu - b_\nu| = |a - b|. \tag{1}$$

2°. We consider any such set Ξ and have the following important theorem:

* Addition of July 1, 1901.

THEOREM. *For the existence of the narrow Ξ-integral*

$$\int_{a(\Xi)}^{b} F(x)\,dx \tag{2}$$

a necessary and sufficient (twofold) condition is this, that

(α) *The narrow Ξ_ν-integrals*

$$\int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx \qquad (\Xi_\nu = (a_\nu, b_\nu)); \tag{3}$$

exist;

and, if Ξ contains an infinitude of points,

(β) *The infinite series*

$$\sum_{\nu=1}^{\infty} O_\nu \tag{4}$$

converges, where for every O_ν denotes the oscillation on the X-interval $a_\nu b_\nu$ of the continuous definite integral function

$$J_\nu(X) = \int_{a_\nu(\Xi_\nu)}^{X} F(x)\,dx. \tag{5}$$

This sufficient twofold condition being satisfied, the integral (2) *has the value*

$$\int_{a(\Xi)}^{b} F(x)\,dx = \sum_{\nu} \int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx. \tag{6}$$

As to the final statement, it is obvious that the terms of the series on the right of (6) are by (α) definite numbers, and that the series (6) if infinite is convergent by (β).

The twofold condition is necessary. In fact, § 3 V gives (α) at once, while (β) follows indirectly* easily from § 3 XIII′. For the oscillation O_ν may be expressed in the form

$$O_\nu = \int_{X_\nu''(\Xi_\nu)}^{X_\nu'} F(x)\,dx = \int_{X_\nu''(\Xi)}^{X_\nu'} F(x)\,dx, \tag{7}$$

where $X_\nu' X_\nu''$ are points of $a_\nu b_\nu$ at which $J_\nu(X)$ has its maximum and minimum values respectively. Now we suppose that the infinite series (4) diverges, and we consider any two positive numbers $\epsilon\,\delta$. In view of (1) there is an integer ν_δ such that

$$\sum_{\nu > \nu_\delta} \left| a_\nu - b_\nu \right| < \delta. \tag{8}$$

* The direct proof is equally simple.

Then in connection with the divergent series

$$\sum_{\nu>\nu_\delta} O_\nu \tag{9}$$

there is an integer $\nu_{\delta\epsilon} > \nu_\delta$ such that

$$\sum_{\nu=\nu_\delta+1}^{\nu=\nu_{\delta\epsilon}} O_\nu > 2\epsilon. \tag{10}$$

The terms O_ν of (10) separate into two sets according as $X'_\nu > X''_\nu$ or $X'_\nu < X''_\nu$, and thereby determine two interval-sets J_1, J_2 of intervals $X''_\nu X'_\nu$, $X'_\nu X''_\nu$, each lying on ab and of length less than δ, and each having the property that every interval l joining two of its intervals contains a point ξ, and furthermore either J_1 or J_2 quâ J having the property that

$$\left| \int_{J(\Xi)} F(x)\,dx \right| > \epsilon. \tag{11}$$

And this result is in contradiction with § 3 XIII′.

It remains to prove that the twofold condition is sufficient. For a finite set Ξ this follows at once from § 2 III. We consider* then an infinite set Ξ, so that $\nu = 1, 2, \cdots$. For every ϵ a δ_ϵ must be exhibited such that for every interval-set I enclosing Ξ narrowly and of length $D_I < \delta_\epsilon$ we have

$$\left| \int_a^b F_I(x)\,dx - \sum_\nu \int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx \right| < \epsilon. \tag{12}$$

We denote by J the complementary interval-set on ab of I_a^b; J consists then of say m intervals $a'_{\nu'} b'_{\nu'}$ $(\nu' = \nu_l;\ l = 1, 2, \cdots, m)$; thus we have

$$\int_a^b F_I(x)\,dx = \int_J F(x)\,dx = \sum_{\nu'} \int_{a'_{\nu'}}^{b'_{\nu'}} F(x)\,dx. \tag{13}$$

Here we have denoted certain m indices ν by ν'. Denoting the remaining indices ν by ν'', we have the desired inequality (12) in the more convenient form

$$\left| \sum_{\nu'} \left(\int_{a_{\nu'}(\Xi_{\nu'})}^{b_{\nu'}} - \int_{a'_{\nu'}}^{b'_{\nu'}} \right) F(x)\,dx + \sum_{\nu''} \int_{a_{\nu''}(\Xi_{\nu''})}^{b_{\nu''}} F(x)\,dx \right| < \epsilon. \tag{14}$$

The series (4) converges, and there is an integer ν_ϵ such that

$$\sum_{\nu>\nu_\epsilon} O_\nu < \frac{\epsilon}{4}. \tag{15}$$

* By slight modifications the proof of the sequel may be made to cover also the case of the finite set Ξ.

The ν_ϵ integrals

$$\int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx \qquad (\nu=1, 2, \cdots, \nu_\epsilon)$$

exist, and there is a number δ_ϵ such that

$$(16) \qquad \delta_\epsilon < |a_\nu - b_\nu|, \quad \left|\left(\int_{a_\nu(\Xi_\nu)}^{b_\nu} - \int_{a'_\nu}^{b'_\nu}\right) F(x)\,dx\right| < \frac{\epsilon}{2\nu_\epsilon} \qquad (\nu=1, 2, \cdots, \nu_\epsilon)$$

if throughout

$$(17) \qquad |a_\nu - b_\nu| - |a'_\nu - b'_\nu| < \delta_\epsilon.$$

This number δ_ϵ is the number desired: if I is any interval-set enclosing Ξ narrowly and of length $D_I < \delta_\epsilon$ the inequality (14) will hold. For

$$(18) \qquad D_I = \sum_{\nu'} (|a_{\nu'} - b_{\nu'}| - |a'_{\nu'} - b'_{\nu'}|) + \sum_{\nu''} |a_{\nu''} - b_{\nu''}|,$$

and so

$$(19) \qquad |a_{\nu'} - b_{\nu'}| - |a'_{\nu'} - b'_{\nu'}| < \delta_\epsilon, \quad |a_{\nu''} - b_{\nu''}| < \delta_\epsilon.$$

By comparison of (16) and (19) we see that the indices ν' include the first ν_ϵ indices ν, $\nu = 1, 2, \cdots, \nu_\epsilon$. Denoting the left member of (14) by L, we have

$$L \leqq \sum_{\nu=1}^{\nu_\epsilon} \left|\left(\int_{a_\nu(\Xi_\nu)}^{b_\nu} - \int_{a'_\nu}^{b'_\nu}\right) F(x)\,dx\right| + \sum_{\nu>\nu_\epsilon} \left|\int_{a_\nu(\Xi_\nu)}^{b_\nu} F(x)\,dx\right|$$

$$+ \sum_{\nu'>\nu_\epsilon} \left|\int_{a'_{\nu'}}^{b'_{\nu'}} F(x)\,dx\right|,$$

so that by (15) and (16) in view of (19) we have indeed the desired inequality (14)

$$L < \epsilon.$$

3°. The preceding theorem reduces the problem of construction of all narrow Ξ-integrals essentially to the corresponding problem for the case $\Xi = (b)$, and this remark holds likewise if the integrand function $F(x)$ is required to be on ab continuous except at points $x = \xi$.

In 5° I exhibit by a known process an essentially narrow or non-absolutely convergent integral

$$(20) \qquad \int_{a((b))}^{b} F(x)\,dx,$$

and in connection with it develop examples designed to show the error of certain statements referred to in the introduction.

In 4° I use such an integral (20), for the case $ab = 01$, as the element for a simple construction according to the conditions of 2° of a Ξ-integral for the general set Ξ of ab, non-absolutely convergent in the neighborhood of every point ξ.

4°. We denote by $F(x) = \psi_{01}(x)$ a function on the x-interval 01 continuous except at $x = 1$ for which the integral

$$\int_{0\,((1))}^{1} F(x)\,dx$$

converges non-absolutely. A function $F(x) = \psi_{ab}(x)$ of similar character on the interval ab $(a \gtrless b)$ is given by the transformation

$$\psi_{ab}(x') = \psi_{01}(x), \qquad x' = (b - a)x + a.$$

Denoting the middle point of ab by c, we denote by $\chi_{ab}(x)$ the function on ab everywhere continuous except at a and b which on ac is $\psi_{ca}(x)$ and on cb is $\psi_{cb}(x)$. The functions $\chi_{ab}(x)$ are derivable from $\chi_{01}(x)$ by the transformation used above. The integral

$$\int_{a\,((a,\,b))}^{b} \chi_{ab}(x)\,dx$$

converges, non-absolutely near a and b.

Then with respect to any set Ξ (1°) we construct a function $F(x)$ everywhere continuous on ab except at its singularities ξ and such that the integral

$$\int_{a\,(\Xi)}^{b} F(x)\,dx$$

converges non-absolutely near every point ξ. The function $F(x)$ is on $a_\nu b_\nu$ the function $\chi_{a_\nu b_\nu}(x)$, or ψ_{ab_ν} if $a_\nu = a =$ not-ξ, or ψ_{ba_ν} if $b_\nu = b =$ not-ξ.

The effectiveness of this construction follows easily from the theorem of 2° since the oscillations O_{ab}, O_{01}, O'_{ab}, of the integrals

$$\int_{a\,((b))}^{X} \psi_{ab}(x)\,dx, \quad \int_{0\,((1))}^{X} \psi_{01}(x)\,dx, \quad \int_{a\,(a,\,b)}^{X} \chi_{ab}(x)\,dx$$

on the respective intervals ab, 01, ab have the relations

$$O_{ab} = O_{01}|a - b|, \quad O'_{ab} \leqq O_{ca} + O_{cb} = O_{ab}.$$

5°. *Construction of a function $F(x)$ everywhere continuous on ab except at b, for which the integral*

$$\int_{a\,((b))}^{b} F(x)\,dx \tag{20}$$

converges non-absolutely.

We introduce any non-absolutely convergent series

$$\sum_{\nu=1}^{\infty} u_\nu, \tag{21}$$

—for instance, the modified harmonic series $(u_\nu = -(-1)^\nu/\nu)$,—and further any infinite sequence H of intervals $h_\nu = a_\nu b_\nu (\nu = 1, 2, 3, \cdots)$ lying progressively on ab, two consecutive intervals having no common inner point, and b being the limit-point of each of the sequences $(a_1, a_2, \cdots)$, $(b_1, b_2, \cdots)$.

Then we determine the function $F(x)$ as follows. On the interval h_ν $F(x)$ is everywhere continuous, does not change sign, and vanishes at the extremities $a_\nu b_\nu$, and further

$$\int_{a_\nu}^{b_\nu} F(x)\,dx = u_\nu, \tag{22}$$

—for instance, $y = F(x)$ is given graphically by the two sides of the isosceles triangle of (signed) height $u_\nu/2(b_\nu - a_\nu)$ on the base $a_\nu b_\nu$. At points x of ab but of no interval $a_\nu b_\nu$ $F(x)$ vanishes.

Then indeed $F(x)$ is everywhere continuous on ab except at $x = b$. In view of (22) $|F(x)|$ has on h_ν an upper limit greater than $|u_\nu|/|a_\nu - b_\nu|$, and hence, since the series (21) is non-absolutely convergent, $F(x)$ has values indefinitely great positive and negative in every neighborhood of $x = b$.

The narrow (b)-integral (20) is by definition (§ 2)

$$\mathop{\mathrm{L}}_{X=b}^{X \text{ on } ab} \int_a^X F(x)\,dx. \tag{23}$$

Now we have

$$\int_a^X F(x)\,dx = \sum_{\nu=1}^{n} u_\nu + \theta u_{n+1}, \tag{24}$$

where* X lies on the interval $b_n b_{n+1}$ and θ is some number, $0 \leqq \theta \leqq 1$. Hence it follows easily that

$$\int_{a\ ((b))}^{b} F(x)\,dx = \sum_{\nu=1}^{\infty} u_\nu, \tag{25}$$

and similarly that $|F(x)|$ is not $((b))$-integrable from a to b, since the series $\sum_\nu |u_\nu|$ diverges.

Thus, the integral (20) is non-absolutely convergent, and so (§ 4 III) it is an essentially narrow (b)-integral.

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* We set $b_0 = a$ and $\sum_{\nu=1}^{0} u_\nu = 0$.

6°. *Exhibition for the* (b)*-integral* (20) *with respect to any two positive numbers* $\epsilon\,\delta$ *of an interval-set* J *not containing the point* b, *for which*

$$D_J < \delta, \quad \left| \int_J F(x)\,dx \right| > \epsilon. — \tag{26}$$

An example contravening HARNACK'S *theorem* 2.

In accordance with the footnote to note 1 of the first definition of § 2 this exhibition furnishes a more direct proof that the (b)-integral (20) is essentially narrow. And it evidently serves to disprove HARNACK'S theorem 2.

In view of the convergence for $\nu = \infty$ of the intervals h_ν to b there is an integer ν_0 such that

$$\sum_{\nu > \nu_0} |a_\nu - b_\nu| < \delta. \tag{27}$$

The series

$$\sum_{\nu > \nu_0} u_\nu$$

converges non-absolutely, and so contains a finite number (m) of positive terms $u_{\nu'}(\nu' = \nu_l > \nu_0;\ l = 1, 2, \cdots, m)$ such that

$$\sum_{\nu} u_{\nu'} > \epsilon. \tag{28}$$

For the interval-set J consisting of the m intervals $h_{\nu'}$ the inequalities (26) follow from (27) and (28).

7°. *Exhibition for the* (Ξ)*-integral* (20) $\big(\Xi = (b)\big)$ *of a set* H *closed and of content zero such that the corresponding* $(\Xi + \mathrm{H})$*-integral does not exist.*

The set H of all points $a_\nu b_\nu$ $(\nu = 1, 2, 3, \cdots)$ and the point b has the desired property. This follows directly from the exhibition of 6° in view of § 3 XIII′.

This example serves to show the presence of errors in certain statements of HARNACK* and BRODÉN* concerning the possibility of modifying the functional values of the integrand function at points of a set of content zero without altering the determination of the integral.

8°. *An example contravening* JORDAN'S *theorem* 3.

We have by 7° for the function $F(x)$ of (20) the (Ξ)-integral existent and the $(\Xi + \mathrm{H})$-integral non-existent, and furthermore $\Xi + \mathrm{H} = \mathrm{H}$.

Now we take by 4° a function $G(x)$ for which the $(\mathrm{H}) = (\Xi + \mathrm{H})$-integral exists. Then the $(\Xi + \mathrm{H})$-integral of $F(x) + G(x)$ is non-existent; for its existence would by § 3 II and XVII imply the existence of the $(\Xi + \mathrm{H})$-integral of $F(x)$.

9°. *An example contravening* HARNACK'S *theorem* 3.

The example of 8° is effective. For, if we set

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*Mathematische Annalen, vol. 24 (1884), p. 220, and vol. 52 (1899), p. 222.

$$J(x) = \int_{a(\Xi)}^{x} F(x)\,dx, \quad K(x) = \int_{a(\mathrm{H})}^{x} G(x)\,dx, \tag{29}$$

$$L(x) = J(x) + K(x), \tag{30}$$

the functions $J(x)$, $K(x)$, $L(x)$ are everywhere continuous on ab; further

$$L(x'') - L(x') = \int_{x'}^{x''} \big(F(x) + G(x)\big)\,dx$$

for every interval $x'x''$ of ab containing no singularity ζ of $F(x) + G(x)$; and the set of singularities $\mathrm{Z} = \Xi + \mathrm{H}$ is reducible; and yet by 8° the HARNACK integral

$$\int_{a(\mathrm{Z})}^{b} \big(F(x) + G(x)\big)\,dx \tag{31}$$

does not exist.

The HÖLDER integral (31) however exists and its value is $L(b)$.

The non-existent integral

$$\int_{a(\Xi+\mathrm{H})}^{b} F(x)\,dx \tag{32}$$

would afford a simpler example from the point of view of this paper. However from the original HARNACK point of view one would object that the set of singularities of $F(x)$ is Ξ and not $\Xi + \mathrm{H}$.

In the example given the integrand function $F(x) + G(x)$ is everywhere continuous on ab except at its singularities ζ.

THE UNIVERSITY OF CHICAGO,
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