NEW PROOF OF A THEOREM OF OSGOOD'S IN THE
CALCULUS OF VARIATIONS*

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In the last number of the Transactions,† Professor Osgood has proved an important characteristic property of a strong minimum of an integral of the form

$$I = \int_{\tau_0}^{\tau_1} F(x, y, x', y') d\tau.$$  

His proof, however, is rather complicated, and the following note is intended to give a simpler proof of the theorem.

§1.

Introduction of curvilinear coordinates.

Suppose the integral (1) is taken along a continuous curve $C$ with continuously turning tangents:

$$C: \ x = \phi(\tau), \ y = \psi(\tau) \quad (\tau_0 \leq \tau \leq \tau_1)$$

joining two fixed points $A(\tau_0)$ and $B(\tau_1)$; further $\phi'^2 + \psi'^2 \neq 0$ in $(\tau_0, \tau_1)$.

Concerning the function $F(x, y, x', y')$ we make the same assumptions as Osgood on p. 277, l. c., except the assumption $F > 0$, which is not necessary for the present proof.

Now introduce instead of the rectangular coördinates $x, y$ any curvilinear coördinates

$$u = U(x, y), \ v = V(x, y),$$

where $U, V$ are single-valued functions with continuous first and second derivatives in a region $T$ of the $x,y$-plane containing the curve $C$; in the same region their Jacobian is supposed $\neq 0$. Interpret $u, v$ as the rectangular co-

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ordinates of a point in a \( u, v \)-plane and denote by \( T', C', A', B' \) the images of \( T, C, A, B \) respectively. Suppose further that the correspondence between \( T \) and \( T' \) is a one-to-one correspondence and that accordingly the inverse functions
\[
x = X(u, v), \quad y = Y(u, v),
\]
are single-valued functions with continuous first and second derivatives in \( T' \) and
\[
D = \frac{\partial (X, Y)}{\partial (u, v)} \neq 0
\]
in \( T' \).

Then the integral \( I \) is changed into
\[
I' = \int_{\tau_0}^{\tau_1} G(u, v, u', v') d\tau,
\]
the function \( G \) of the four arguments \( u, v, u', v' \) being defined by
\[
G(u, v, u', v') = F(X, Y, X_u u' + X_v v', Y_u u' + Y_v v'),
\]
where \( X_u = \frac{\partial X}{\partial u} \), etc.

The integral \( I' \) is taken, in the \( u, v \)-plane, along the image \( C' \) of \( C \).

From \( I = I' \) it follows that if the curve \( C \) minimize the integral \( I \), its image \( C' \) will minimize \( I' \), and vice versa; and if \( C \) be an extremal for \( I \), \( C' \) must be an extremal for \( I' \), and vice versa. Further WEIERSTRASS's function \( F_1 \) is an invariant for the above transformation, viz., if we denote the corresponding function derived from \( G \) by \( G_1 \), we obtain easily
\[
G_1 = D^2 F_1.
\]
Finally WEIERSTRASS's \( E \)-function is an absolute invariant, i.e., if we denote the new \( E \)-function by \( E' \) we have :
\[
E'(u, v; u', v'; \bar{u}', \bar{v}') = E(x, y; x', y'; \bar{x}', \bar{y}')
\]

where
\[
x' = X_u u' + X_v v', \quad \bar{x}' = X_u \bar{u}' + X_v \bar{v}',
\]
\[
y' = Y_u u' + Y_v v', \quad \bar{y}' = Y_u \bar{u}' + Y_v \bar{v}',
\]
as follows immediately from (5).

§2.

Proof of Osgood's theorem.

Now let
\[
x = \phi(t, \alpha), \quad y = \psi(t, \alpha)
\]


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be a set of extremals for the integral $I$, satisfying the following conditions:

1. The functions $\phi$, $\psi$ are single-valued functions of $t$, $a$ with continuous first and second derivatives in the region:

$$T_0 - \varepsilon \leq t \leq T_1 + \varepsilon, \quad |a - a_0| \leq \kappa \quad (\varepsilon > 0).$$

2. The extremal $C_0 : x = \phi(t, a_0), y = \psi(t, a_0)$ has no multiple point for $T - \varepsilon \leq t \leq T_1 + \varepsilon$, and passes through the two given points $A(t_0)$ and $B(t_1)$ where $T_0 < t_0 < t_1 < T_1$.

3. If we denote by $\Delta(t, a)$ the Jacobian

$$\Delta(t, a) = \frac{\partial(\phi, \psi)}{\partial(t, a)},$$

then

$$\Delta(t, a_0) \neq 0 \quad \text{in} \quad (T_0 - \varepsilon, T_1 + \varepsilon).$$

4. The inequality

$$F_1' (\phi(t, a_0), \psi(t, a_0), \cos \lambda, \sin \lambda) > 0$$

holds for every $t$ of the interval $T_0 - \varepsilon \leq t \leq T_1 + \varepsilon$ and for every $\lambda$.

Under these circumstances if we denote by $R_\kappa$ the region:

$$R_\kappa : \quad T_0 \leq t \leq T_1, \quad |a - a_0| \leq \kappa,$$

and denote by $S_\kappa$ the image of $R_\kappa$ in the $x,y$-plane, then $\kappa$ can be taken so small that

1. $\Delta(t, a) \neq 0$ in $R_\kappa$.
2. $F_1(x, y, \cos \lambda, \sin \lambda) > 0$ for every $x, y$ in $S_\kappa$ and for every $\lambda$.
3. The correspondence between $R_\kappa$ and $S_\kappa$ is a one-to-one correspondence.*

The inverse functions

$$t = t(x, y), \quad a = a(x, y)$$

will then be single-valued with continuous first and second derivatives in $S_\kappa$ and their Jacobian will be $\neq 0$ in $S_\kappa$.

Then $S_\kappa$ is a "field" surrounding the arc $AB$ of the extremal$\ dagger$ $C_0$, and if

$$C : \quad x = \tilde{\phi}(\tau), \quad y = \tilde{\psi}(\tau), \quad (\tau_0 \equiv \tau \equiv \tau_1),$$

be any other curve with continuously turning tangents drawn from $A$ to $B$ in $S_\kappa$, for which $\tilde{\phi}'^2 + \tilde{\psi}'^2 \neq 0$, then Weierstrass's theorem holds, according to which

$$\Delta I = I_c(AB) - I_c(AB) = \int_{\tau_0}^{\tau_1} E(x, y; x', y'; \tilde{x}', \tilde{y}') \, d\tau,$$

* See Kneser, Lehrbuch der Variationsrechnung, § 14, and Osgood, l. c., p. 278. Both proofs have to be supplemented by the following preliminary lemma: If for every $\kappa$, however small, there existed points $(x, y)$ in $S_\kappa$ to which correspond in $R_\kappa$ at least two distinct points $(t', a')$ and $(t'', a'')$, then there must exist, in $R_\kappa$, a point $(t_0, a_0)$ such that in every vicinity of it pairs of distinct points can be found whose images in the $x, y$-plane coincide.
where \(x', y'\) refer to the extremal of the set (8) passing through \(x, y\), and \(\bar{x}', \bar{y}'\) to the curve \(C\). Hence Weierstrass infers that \(\Delta I > 0\) by making use of the following theorem connecting the functions \(E\) and \(F_1\):

\[(12) \quad E(x, y; \cos \bar{\theta}, \sin \bar{\theta}; \cos \bar{\theta}, \sin \bar{\theta}) = (1 - \cos \omega) F_1(x, y, \cos \bar{\theta}^*, \sin \bar{\theta}^*),\]

where \(\omega = \bar{\theta} - \bar{\theta}\); \(\theta^*\) is an angle between \(\theta\) and \(\bar{\theta}\); and the angles are so measured that \(|\omega| \leq \pi\).

Now let \(0 < h < \kappa\) and use \(S_h\) in the analogous signification as \(S_\kappa\); then Osgood's theorem may be stated as follows: There exists a positive quantity depending upon \(h, \epsilon_h\), such that for every curve \(C\) joining \(A\) and \(B\) drawn in the interior of \(S_\kappa\) but not wholly contained in \(S_h\),

\[\Delta I > \epsilon_h.\]

To prove this theorem it is now only necessary to introduce instead of \(x, y\) the curvilinear coordinates

\[(13) \quad u = t(x, y), \quad v = a(x, y)\]

which satisfy for the regions \(S_\kappa\) and \(R_\kappa\) the conditions of §1, and to make use of the remarks made there.

Accordingly the extremals for the integral \(I'\) are the lines \(v = \text{const.}\), and, therefore, in the \(u,v\)-plane Weierstrass's theorem takes the form:

\[\Delta I = \Delta I' = \int_0^{s_1} E'(u, v; \cos 0, \sin 0; \cos \omega, \sin \omega) \, ds \]

\[= \int_0^{s_1} (1 - \cos \omega) G_1(u, v, \cos \theta^*, \sin \theta^*) \, ds.\]

These integrals are taken along the image \(\bar{C}'\) in the \(u,v\)-plane (i.e., \(t,a\)-plane) of the curve \(\bar{C}\); \(\omega\) denotes the angle between the positive tangent to \(\bar{C}'\) in the point \(u, v\) and the positive \(u\)-axis; and \(s\) is the arc of the curve \(\bar{C}'\).

But from (9), (10), and (6) it follows that we can assign a positive quantity \(m\) such that

\[G_1(u, v, \cos \lambda, \sin \lambda) \equiv m > 0\]

for every \(u, v\) in \(R_\kappa\) and for every \(\lambda\). Therefore

\[\Delta I \equiv m \int_0^{s_1} (1 - \cos \omega) \, ds,\]

that is

\[\Delta I \equiv m \left(l - (t_1 - t_0)\right),\]

if \(l\) denotes the length of the curve \(\bar{C}'\).

† Lectures on the Calculus of Variations, 1882.
Now suppose that $\bar{C}$ is not wholly contained in the interior of $S_\lambda$ and therefore passes through a point $P$ of one of the two extremals $a = a_0 \pm \delta$, then $\bar{C}'$ passes through a point $P'$ whose ordinate is $v = a_0 \pm \delta$. Hence $l$ is greater than or equal to the length of the broken line $A'P'B'$. But if we choose $Q'$ on the same line $v = \text{const.}$ as $P'$ so that $A'Q' = B'Q'$, then

$$A'P'B' \geq A'Q'B',$$

and therefore

$$\Delta I \geq 2m \left[ \sqrt{\delta^2 + \left( \frac{t_1 - t_0}{2} \right)^2} - \left( \frac{t_1 - t_0}{2} \right) \right],$$

which proves Osgood's theorem.*

Osgood's theorem can easily be extended to the case where one endpoint, say $B$, is fixed, the other, $A$, movable on a given curve. For this purpose, it is only necessary to choose for the new coördinates Kneser's curvilinear coördinates $u, v$ (Kneser, §16); a slight modification of the above reasoning leads then to the inequality

$$\Delta I \geq m \left[ \sqrt{\delta^2 + (u_1 - u_0)^2 - (u_1 - u_0)} \right].$$

* Osgood bases his proof upon the following lemma: "Let $f(x)$ be a single valued continuous function of $x$ in the interval $a \leq x \leq b$, and let $f(x)$ have a continuous derivative $f'(x)$ at all points of this interval. Let $a < l \leq b$ and $|f(l) - f(a)| = L > 0$. Then

$$\int_a^l f'(x)^2 \, dx \leq \frac{L^2}{54(b - a)^2},$$

provided that $L < 3\sqrt{2}(b - a)^2$.

This lemma may be proved as follows: Since the value of the integral does not change if $f(x)$ is replaced by $M \pm f(x), M$ being a constant, we may confine ourselves to functions $f(x)$ for which $f(a) = 0, f(l) = L$. This remark reduces the question to the problem of minimizing the above integral with these given initial values. The solution is $y = L(x - a)/(b - a);$ it satisfies Weierstrass' sufficient conditions for a minimum, and furnishes for the integral the minimum value: $(L^2)/(l - a) \leq (L^2)/(b - a)$, which under the above inequality for $L$ is greater than $\frac{1}{2}(L^2)/(b - a)^2$. 

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It is, however, to be remembered that the introduction of Kneser's coördinates presupposes that

\[ F(\phi(t, \alpha), \psi(t, \alpha), \phi'(t, \alpha), \psi'(t, \alpha)) > 0 \]

in the region \( R_\kappa \).

University of Chicago,
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