CONCERNING THE EXISTENCE OF SURFACES CAPABLE OF CONFORMAL REPRESENTATION UPON THE PLANE IN SUCH A MANNER THAT GEODETIC LINES ARE REPRESENTED BY A PRESCRIBED SYSTEM OF CURVES*

BY

HENRY FREEMAN STECKER

Introduction.—This paper is in continuation of a previous paper † under nearly the same title. The notation given there is used in this paper with the exception that \( u, v \) are here used instead of \( \mu, \nu \).

We are concerned with a doubly infinite system of given curves:

\[
(1) \quad f_3(u, v) + Af_2(u, v) + Bf_1(u, v) = 0,
\]

of which the differential equation is †

\[
(2) \quad a_1du^3 + a_4dv^3 + a_2du^2dv + a_3du^2v + a_6(du^2v - dv^2u) = 0.
\]

The geodetic lines of any surface are given by:

\[
(3) \quad (EF_u - \frac{1}{2} EE_v - \frac{1}{2} FE_u) du^3 + (- GF_v + \frac{1}{2} GG_u + \frac{1}{2} FG_v) dv^3 \\
+ (EG_u - \frac{3}{2} FE_v - \frac{1}{2} GE_u + FF_v) du^2dv \\
+ (- GE_v + \frac{3}{2} FG_u + \frac{1}{2} EG_v - FF_v) du dv^2 \\
+ (EG - F^2)(du^2v - dv^2u) = 0 \quad (F_u = \partial F/\partial u, \text{ etc.}),
\]

where \( u \) and \( v \) are Gaussian coordinates on the surface. A comparison of (2) and (3) leads to the following system of partial differential equations:

---

* Presented to the Society at the Ithaca meeting under a slightly different title August 19, 1901. Received for publication November 6, 1901.

(a) \[ EF_u - \frac{1}{2} EE_v - \frac{1}{2} FE_u = \frac{a_1}{a_5} (EG - F^2), \]
(b) \[ -GF_v + \frac{1}{2} GG_u + \frac{1}{2} FG_v = \frac{a_4}{a_5} (EG - F^2), \]
(c) \[ EG_u - \frac{3}{2} FE_v - \frac{1}{2} GE_u + FF_u = \frac{a_2}{a_5} (EG - F^2), \]
(d) \[ -GE_v + \frac{3}{2} FG_u + \frac{1}{2} EG_v - FF_v = \frac{a_3}{a_5} (EG - F^2). \]

The solution of the problem depends upon that of this system.

Multiply equation (a) by \(-3F/E\) and add to equation (c). The result is:

\[ \begin{align*}
-2FF_u + EG_u - \frac{1}{2} GE_u + \frac{3}{2} F^2 E_u &= \left( \frac{a_2}{a_5} - \frac{3}{a_5} \frac{a_1F}{E} \right) (EG - F^2).
\end{align*} \]

Dividing this equation through by \(EG - F^2\), integrating with respect to \(u\), and representing by \(\psi(v)\) an arbitrary function of \(v\) only, we have:

\[ \begin{align*}
\frac{EG - F^2}{E^4} &= \psi(v) e^{\int \frac{a_2}{a_5} du} e^{-3\int \frac{a_1F}{a_5E} du}.
\end{align*} \]

In like manner, multiplying equation (b) by \(-3F/G\), and adding to equation (d), we find

\[ \begin{align*}
\frac{G^4}{EG - F^2} &= \phi(u) e^{\int \frac{a_2}{a_5} dv} e^{-3\int \frac{a_1F}{a_5G} dv},
\end{align*} \]

where \(\phi(u)\) is an arbitrary function of \(u\) only.

Beltrami’s investigations* were for the case, \(a_1 = a_2 = a_3 = a_4 = 0\), that is, where both of the exponentials in the right hand members of (6) and (7) become unity.

In my previous paper I considered the case \(F = 0\), that is, where (6) and (7) assume the forms:

\[ \begin{align*}
\frac{EG - F^2}{E^4} &= \psi(v) e^{\int \frac{a_2}{a_5} du},
\end{align*} \]

(8)

\[ \begin{align*}
\frac{G^4}{EG - F^2} &= \phi(u) e^{\int \frac{a_2}{a_5} dv}.
\end{align*} \]

(9)

It is proposed to investigate the case in which \(a_1 = a_4 = 0\), while \(a_2\) and \(a_3\) are unrestricted.

Under these restrictions, the system (6) and (7) assumes the form (8) and (9), that is, the same form as for the case \(F = 0\). Moreover, these are evidently the only cases in which the right hand members of (6) and (7) are independent of \(E, G, F\).

*Annali di Matematica, ser. 1, vol. 7 (1866).
§ 1. Consideration of the form of (1) as restricted by the condition that $a_1 = a_4 = 0$.

Write

$$F'_1(u,v) = \frac{f'_1(u,v)}{f'_2(u,v)}, \quad F'_2(u,v) = \frac{f'_2(u,v)}{f'_3(u,v)}.$$  

Then we find:

$$a_1 = \frac{\partial F'_1}{\partial u} \frac{\partial^2 F'_2}{\partial u^2} - \frac{\partial F'_2}{\partial u} \frac{\partial^2 F'_1}{\partial u^2},$$  

$$a_4 = \frac{\partial F'_1}{\partial v} \frac{\partial^2 F'_2}{\partial v^2} - \frac{\partial F'_2}{\partial v} \frac{\partial^2 F'_1}{\partial v^2}.$$  

Equating each of these to zero, and integrating, we find:

$$F_2(u,v) = \psi_1(v) F_1(u,v) + \psi_2(v),$$  

$$F_2(u,v) = \Phi_1(u) F_1(u,v) + \Phi_2(u),$$  

and from these two equations:

$$F'_1(u,v) = \frac{\psi_2(v) - \phi_2(u)}{\phi_1(u) - \psi_1(v)},$$  

$$F'_2(u,v) = \frac{\phi_1(u) \psi_2(v) - \phi_2(u) \psi_1(v)}{\phi_1(u) - \psi_1(v)},$$  

where $\phi_1(u), \phi_2(u), \psi_1(v), \psi_2(v)$ are as yet arbitrary functions. Hence, when $a_1 = a_4 = 0$, equation (1) assumes the form:

$$(\phi_1(u) - \psi_1(v)) + A (\phi_1(u) \psi_2(v) - \phi_2(u) \psi_1(v)) + B (\psi_2(v) - \phi_2(u)) = 0.$$  

§ 2. Consideration of the values of the exponentials $e \int_a^s du$ and $e \int_a^s dv$ when $a_1 = a_4 = 0$.

We have:

$$a_2 = 2[F'_{1u} F'_{2uv} - F'_{2u} F'_{1uv}] + [F'_{1v} F'_{2uv} - F'_{2v} F'_{1uv}],$$  

$$a_3 = 2[F'_{1u} F'_{2uv} - F'_{2u} F'_{1uv}] + [F'_{1v} F'_{2uv} - F'_{2v} F'_{1uv}],$$  

$$a_5 = F'_{1u} F'_{2u} - F'_{1v} F'_{2v}$$  

(Differentiating (19) with respect to $u$ and comparing the result with (17) we find

$\downarrow$ Previous paper, loc. cit., p. 154.
(20) \[ a_2 = -a_5 + 3 \left[ F_{1u} F_{2u} - F_{2u} F_{1u} \right], \]
and in like manner, from (18) and (19), we also find

(21) \[ a_3 = a_5 - 3 \left[ F_{1u} F_{1u} - F_{1u} F_{2u} \right]. \]

From (12) and (13) we have

(22) \[ F_{2u} = \psi_1(v) F_{1u}, \]
(23) \[ F_{2u} = \phi_1(u) F_{1u}, \]
(24) \[ F_{2uv} = \psi_1(v) F_{1uv} + \psi_1(v) F_{1u} = \phi_1(u) F_{1uv} + \phi_1(u) F_{1u}. \]

These and (20) give

(25) \[ a_5 = F_{1u} F_{1u} \left[ \phi_1(u) - \psi_1(v) \right]. \]

Calculating next the bracketed expression in (21), we find it to be equal to

\[
F_{1u} \left\{ F_{1u} \left[ \phi_1(u) - \psi_1(v) \right] - F_{1u} \psi_1(v) \right\} = F_{1u} \frac{\partial}{\partial v} \left[ F_{1u} \left( \phi_1(u) - \psi_1(v) \right) \right].
\]

Hence we finally have

(26) \[ a_5 = a_5 - 3 F_{1u} \frac{\partial}{\partial v} \left[ F_{1u} \left( \phi_1(u) - \psi_1(v) \right) \right]. \]

Then from (25) and (26) we obtain :

\[
\int \frac{a_3}{a_5} dv = \int \frac{\partial a_5}{a_5} dv - 3 \int \frac{\partial}{\partial v} \frac{\partial}{\partial v} \left[ F_{1u} \left( \phi_1(u) - \psi_1(v) \right) \right] dv
\]

(27) \[ = \log \left( \frac{F_{1u}}{(\phi_1(u) - \psi_1(v))^2} \right). \]

From (14) we find :

\[
F_{1u} = \frac{\left[ \phi_1(u) - \psi_1(v) \right] \psi_1'(v) + \left[ \psi_2(v) - \phi_2(u) \right] \psi_1'(v)}{\phi_1(u) - \psi_1(v)^2},
\]

(28) \[
F_{1u} = \frac{\left[ \phi_1(u) - \psi_1(v) \right] \phi_1'(u) + \left[ \psi_2(v) - \phi_2(u) \right] \phi_1'(u)}{\phi_1(u) - \psi_1(v)^2},
\]
or say, \[ F_{1u} \equiv A/D^2, \] and \[ F_{1u} \equiv B/D^2. \]

Then \( \int (a_3/a_5) dv \) becomes \( \log (A/B^2) \). By similar reasoning we find that \( \int (a_3/a_5) du = \log (-A^2/B) \).

Hence the exponentials \( e^{\int a_3 du} \) and \( e^{\int a_3 dv} \) are equal, respectively, to \(-A^2/B\) and \(A/B^2\), where \(A\) and \(B\) have the meaning given in connection with (28).
§3. Reduction of the system of partial differential equations (4) under the restriction that \( a_1 = a_4 = 0 \), and conditions of integrability for the reduced system.

Representing the right hand members of (8) and (9) by \( R \) and \( S \) respectively, and placing \( p \) for \((RS)^t\), we have

\[
\begin{align*}
G &= pE, \\
F^2 &= E(p - RE^{-1}).
\end{align*}
\]

Also write \( t = p_uE^1 - R_u, \ h = p_vE^1 - R_v, \ l = 2pE^1 - \frac{3}{2}R \). Transforming system (4), we find for equation (a), after some reductions,

\[
Et + (l + R - pE^1) E_u - \sqrt{pE - RE^1} E_v = 0.
\]

Calculation shows that equation (a) reduces to (34), and, with a little more difficulty, we find that equations (b) and (d) reduce to the same equation which is:

\[
pE - RE - pE R_v + pp_uE \sqrt{pE - RE^1} E_u + p(pE^1 - \frac{1}{2}R) E_v = 0.
\]

Hence system (4) reduces, when \( a_1 = a_4 = 0 \), to equations (34) and (35). It remains to consider the conditions of integrability for this reduced system.

We find:

\[
E_u = \frac{2E}{R^2 l}, \quad E_v = \frac{2E}{R^2 k},
\]

where

\[
l = 2\Theta E^1 - RR_u + \frac{2\Pi}{p} \Delta, \quad k = 2\Pi E^1 - \frac{\Pi}{p} RR_u + 2\Theta \Delta,
\]

in which

\[
\Pi \equiv \begin{vmatrix} p & R \\ p_u & R_v \end{vmatrix}, \quad \Theta \equiv \begin{vmatrix} p & \frac{1}{2}R \\ p_u & R_v \end{vmatrix}, \quad \Delta \equiv \sqrt{pE - RE^1}.
\]

The condition of integrability, after some reduction, can be written:

\[
s^2(kl - lk) + s(l - k) + ls - ks_u = 0.
\]

After a somewhat long calculation we are able to write this in the form:

\[
AE^1\Delta_E + BE\Delta_E + CE^1\Delta + DE^1 + ME + N + 2\frac{\Pi}{p} \Delta_v - 2\Theta \Delta_u = 0,
\]

where

\[
A = \frac{8}{R^2} \left( \frac{\Pi^2}{p} - \Theta^2 \right), \quad B = 4 \frac{R^u}{R} \left( \Theta - \frac{\Pi^2}{p^2} \right), \quad C = 4 \frac{R^e}{R^2} \left( \Theta^2 - \frac{\Pi^2}{p} \right),
\]

\[
D = 2 \frac{R^u}{R} \left( 1 - \frac{\Theta}{p} \right) + 2 \left( \Theta - \Pi_u \right) + \frac{4}{R} (\Pi R_e - \Theta R_u),
\]
Calculating the values of $\Delta_x$, $\Delta_u$, $\Delta_v$ we finally have (38) in the form

\begin{equation}
D_1 E - N_1 E^4 + (D E^4 + N) \sqrt{p E - R E^4} = 0,
\end{equation}

where

\begin{align*}
D_1 &\equiv \frac{1}{2} p B - \frac{1}{4} R A - R C + M p + \frac{1}{p} \rho - \Theta p, \\
N_1 &\equiv \frac{1}{4} R B + R E - \frac{1}{p} R_v + \Theta R_u.
\end{align*}

Our conclusion is that the necessary and sufficient conditions that the system of partial differential equations (34) and (35) shall be integrable are:

\begin{equation}
D = 0, \quad D_1 = 0, \quad N = 0, \quad N_1 = 0.
\end{equation}

§ 4. Consideration of the form which the functions $\phi(u)$, $\phi_1(u)$, $\phi_2(u)$, $\psi(v)$, $\psi_1(v)$, $\psi_2(v)$ assume under the conditions of integrability.

It is desirable to make $\phi(u)$ and $\psi(v)$, which are arbitrary, the means, as far as possible, of satisfying the conditions of integrability. If the third condition, $N = 0$, of (40) were as complicated as the other three, the problem would seem almost beyond our power. Fortunately it is somewhat simpler than the others and by means of the conclusion which we are able to reach from the condition $N = 0$, we are able so to reduce the others that they can be controlled.

Putting $a = -A^2/B$ where $A$ and $B$ have the meaning given in connection with (28), so that

\begin{align*}
P &\equiv \frac{a_u A_v - a_u A_{uv} - a_{uu} A_v}{a_{uu} A^2}, \\
Q &\equiv \frac{a a_v - a_a a_v}{a^2 a_{uu}},
\end{align*}

we can, after a long reduction, put $N = 0$ in the form:

\begin{equation}
\frac{d\psi(v)}{dv} + 2P \psi(v) - 2Q \psi(v)^{-1} = 0,
\end{equation}

so that

\begin{equation}
\psi(v)^\delta = e^{-\int_P dv} \int e^{\int_P dv} Q dv + C.
\end{equation}

Calculation shows that the right hand member of (42) can be a function of $N$ alone, only when $\partial Q/\partial u = 0$ and $P$ is of the form $f_1(u)f_2(v)$. When $Q$ is reduced it assumes the final form:

\[ Q \equiv \frac{B(2hB^3 + kA^3)}{A^2 N}, \]

where
\[ h = \phi_1'(u)\phi_2''(u) - \phi_2'(u)\phi_1''(u), \quad k = \psi_1'(v)\psi_2''(v) - \psi_2'(v)\psi_1''(v), \]
and
\[ N = A^2 B_{uu} - 2AB^2 A_{uu} - 2B^2 A_u^2 + 4ABA_u B_u - 2A^2 B_u^2. \]

Since \( A \) is not a factor of \( B \), and each is a function of both \( u \) and \( v \) (unless we consider a trivial case), it follows that \( A \) must be a factor of \( 2hB^3 + kA^3 \). This condition is satisfied, with least restriction upon the form of the curve, by putting \( h = 0 \). Then as a further condition, either the product of \( AB \) and a function of \( v \) only must be equal to \( N \), or else \( k \) must vanish, and we find that the latter includes the former. Hence \( Q \) must vanish.

The first restriction is that \( \phi_1'(u)\phi_2''(u) - \phi_2'(u)\phi_1''(u) \) and \( \psi_1'(v)\psi_2''(v) - \psi_2'(v)\psi_1''(v) \) each vanish. Integrating, we have
\[ \phi_2(u) = \phi_1(u) + k_1u + k_2, \]
\[ \psi_2(v) = \psi_1(v) + h_1v + h_2, \]
where \( k_1 \) and \( k_2 \) are to be determined. Integrating the equation \( Q = 0 \) with respect to \( v \), we find, after some reduction, that the following expression must be a function of \( u \) only:
\[ \frac{m_1 + m_2\psi(v)\psi'(v) + m_3\psi(v) + m_4\psi_1(v) + m_5\psi_1'(v) + m_6\psi_2(v) + m_7\psi_2'(v)}{n_1 + n_2\psi(v)\psi'(v) + n_3\psi(v)\psi_1(v) + n_4\psi(v) + h_1\psi_1(u)v^2\psi_1''(v)}, \]
where \( m_i \) and \( n_i \) do not contain \( v \). Calculation shows that \( h_1 \) and \( h_2 \) must vanish, after which the above expression assumes the form \(-2\phi_1''(u)/\phi_1'(u)\), where \( l = -k_2 + h_2 \).

Calculating now the value of \( P \), we find it to be \(-2\psi_1''(v)/\psi_1'(v)\), so that
\[ \frac{d\psi(v)}{dv} - 2\frac{\psi''(v)}{\psi_1'(v)}\psi(v) = 0, \]
or
\[ \psi(v) = k [\psi'_1(v)]^2. \]

Collecting our results thus far, we have
\[ \phi_2(u) = \phi_1(u) + k_2, \]
\[ \psi_2(v) = \psi_1(v) + h_2, \]
(45)
\[ \psi(v) = k [\psi'_1(v)]^2. \]

It remains to consider the remaining three conditions of integrability. Of these, \( N_1 = 0 \) is the most simple. A little consideration shows that it must
be of the form $\phi''(u) + \beta_2 \phi'(u)/\beta_1 + \beta_3/\beta_1 = 0$, where $\beta_i$ is independent of $\phi(u)$. It turns out that $\beta_2 \equiv 0$, and that $\beta_3/\beta_1$ is equal to

$$-(\gamma k^4 \frac{[\psi_1''(v)]^2}{\psi_1(v)} \phi_1''(u)) - \left\{ \frac{17}{k} \left[ \frac{\psi_1''(v)}{\psi_1'(v)} \right]^2 - \frac{2}{k} \left[ \frac{\psi_1''(v)}{\psi_1'(v)} \right]^3 \right\} [\phi'(u)]^2. \tag{46}$$

This must be a function of $u$ only, which requires that $\psi_1(v)$ satisfy the system:

$$[\psi_1''(v)]^2 = \lambda_1 \psi_1'(v), \tag{47}$$

$$17 [\psi_1''(v)]^2 - 2 \psi_1'''(v) \psi_1'(v) = \lambda_2 [\psi_1'(v)]^3,$$

where $\lambda_i$ is constant. The system (47) admits of but one solution, $\lambda_i = 0$, $\psi_1''(v) = 0$. That is, $N_1 = 0$ adds to (45) the two results:

$$\phi(u) = b_1 u + b_2, \tag{48}$$

$$\psi_1(v) = a_1 v + a_2.$$

The conditions, $D = 0$, and $D_1 = 0$, remain for consideration. The equations (45) and (48) reduce $D = 0$ to the form:

$$p R_u - p (R_u)^2 + \frac{1}{2} p R_p p_u = 0,$$

which is satisfied by either $R_u = 0$, or $p - p R_u + \frac{1}{2} p R_p p = 0$. A little calculation shows that $D_1 = 0$ is also satisfied if $R_u = 0$. It remains to determine whether $D_1 = 0$ is also satisfied when $\Theta = p$. We find that $D_1 = 0$ assumes the form:

$$p R_u - R_p p = \frac{2}{3} p,$$

which we are to consider simultaneously with $p R_u - \frac{1}{2} R p p = p$. This requires either that $p = 0$, or that $R_u = \frac{1}{3} p$.

If $p = 0$, then, either $G = 0$, or $E G - F^2 = 0$, the latter of which is excluded. If $G = 0$ we find that $E = 1/R^2$ and $F = \pm 1$, and the corresponding surface is an imaginary ruled surface.

The assumption $R_u = 0$ would lead us to the relation

$$(b_1^2 u + b_2^2)^2(\frac{4}{3} u + \delta)^3 = \lambda k^2 a_1^2 c^2,$$

which would require that $\phi(u)$ vanish, and hence lead us to the same imaginary surface as for $p = 0$.

Hence: In order that a real surface exist, it is necessary and sufficient that we have:

$$\phi(u) = b_1 u + b_2, \quad \psi_1(v) = k a_1^2, \tag{49}$$

$$\psi_2(v) = \psi_1(v) + h_2, \quad \phi_2(u) = \phi_1(u) + k_2,$$
where the constants must not cause \( \phi(u) \) or \( \psi(v) \) to vanish.

If now we calculate \( a_2 \) and \( a_3 \) we find that they also vanish, so that we are led to the conclusion that there is for our proposed problem no new solution.

§ 5. Integration of the system of partial differential equations.

The conditions of integrability (40) being satisfied, we proceed to integrate the corresponding systems, (34) and (35), of partial equations.

They may now be written:

\[
E_u = -2nE^3, \quad E_v = -2nE^3\sqrt{p - RE^{-1}},
\]

where \( n \) is the constant \( b_1/(k_2 - h_2)k^4a,c_1 \).

The integrals of these are, respectively,

\[
E^{-1} = nu + f_1(v), \quad \sqrt{p - RE^{-1}} = -\frac{1}{2}nv + f_2(u),
\]

where \( f_1(v) \) and \( f_2(u) \) are yet to be determined.

We have the identical relation:

\[
p - [f_2(u) - \frac{1}{2}nv]^2 \equiv R [nu + f_1(v)].
\]

Writing \( u = 0 \), we have at once:

\[
f_1(v) \equiv p_0 - \frac{[f_2(0) - \frac{1}{2}nv]^2}{R},
\]

and further consideration shows that \( f_2(u) \equiv f_2(0) \), a constant which we denote by \( \delta \). Writing \( D \equiv m(b_1u + b_2) - [\delta - \frac{1}{2}nv]^2 \), we finally have:

\[
E = \frac{R^2}{D^2}, \quad F = \frac{R(\delta - \frac{1}{2}nv)}{D}, \quad G = \frac{m(b_1u + b_2)R^2}{D^2}.
\]

§ 6. Curvature of the surface corresponding to the problem considered.

The Gaussian curvature of any surface is given by:

\[
K = \frac{1}{\sqrt{EG - F^2}} \left[ \frac{\partial}{\partial v} \left( \frac{\sqrt{EG - F^2}}{E} \right) \begin{bmatrix} 11 \\ 2 \end{bmatrix} - \frac{\partial}{\partial u} \left( \frac{\sqrt{EG - F^2}}{G} \right) \begin{bmatrix} 12 \\ 2 \end{bmatrix} \right],
\]

where,

\[
\begin{bmatrix} 11 \\ 2 \end{bmatrix} = -\frac{FE_u + 2EF_u - EE_v}{2(EG - F^2)}, \quad \begin{bmatrix} 12 \\ 2 \end{bmatrix} = \frac{EG_u - FE_v}{2(EG - F^2)}.
\]

Referring to system (4) we find that \( \begin{bmatrix} 11 \\ 2 \end{bmatrix} = \frac{a_1}{a_5} \). Writing \( EG - F^2 = \Delta \), we have, from the same system,

\[
\frac{EG_u - FE_v}{\Delta} = \frac{a_2}{a_5} + \frac{1}{2} \frac{GE_u}{\Delta} + \frac{FE_v - FF_u}{\Delta},
\]
\[- \frac{F}{E} \frac{a_1}{a_5} = \frac{1}{2} \frac{F E_x - F F_u}{\Delta} + \frac{1}{2} \frac{F^2 E_u}{E \Delta}, \]

and from these,
\[
\{12\} = \frac{1}{2} \frac{a_2}{a_5} - \frac{1}{2} \frac{F}{E} \frac{a_1}{a_5} + \frac{1}{4} \frac{E_u}{E}. \]

Put
\[
M = \frac{G^i}{\phi(u)^i E}, \quad N = \frac{\psi^i}{E^i},
\]
\[
a = - \frac{1}{2} \frac{a_3}{a_5} + \frac{3}{2} \frac{a_4}{a_5} \frac{F}{G}, \quad \beta = \frac{1}{2} \frac{a_2}{a_5} - \frac{3}{2} \frac{a_1}{E} \frac{F}{E},
\]
\[
H = \frac{\sqrt{\Delta}}{E}, \quad \{11\} = W, \quad \{12\} = U.
\]

Then from (6) and (7), we have
\[
H = Me^{-adu} = Ne^{-bdv},
\]
and some calculation gives:
\[
K = \frac{1}{E} \left\{ \frac{\partial}{\partial v} \left( \frac{a_1}{a_5} \right) - U_u + \begin{vmatrix} \frac{a_1}{a_5} & \beta + \frac{\partial \log N}{\partial u} \\ \beta + \frac{\partial \log M}{\partial N} & a \end{vmatrix} \right\}.
\]

Then calculating \(U_u, \partial \log N/\partial u, \) and \(\partial \log M/\partial v\) we find as a final expression for the total curvature of a surface corresponding to the system of partial differential equations (4):
\[
K = \frac{1}{E} \left\{ \frac{\partial}{\partial v} \left( \frac{a_1}{a_5} \right) - \frac{1}{2} \frac{\partial^2 \log E}{\partial u^2} + \frac{F}{E} \frac{\partial}{\partial u} \left( \frac{a_1}{a_5} \right) + \frac{1}{2} \frac{a_4}{a_5} \frac{E F_u - F E_u}{E^2} \right\} + \frac{1}{E} \begin{vmatrix} \frac{a_1}{a_5} & \frac{1}{2} \frac{a_3}{a_5} - \frac{3}{2} \frac{a_4}{a_5} \frac{F}{E} - \frac{1}{4} \frac{E_u}{E} \\ \frac{1}{2} \frac{a_2}{a_5} - \frac{1}{2} \frac{a_4}{a_5} \frac{F}{E} + \frac{1}{4} \frac{E_u}{E} & - \frac{1}{2} \frac{a_3}{a_5} + \frac{3}{2} \frac{a_4}{a_5} \frac{F}{G} + \frac{3}{4} \frac{G_u}{G} - \frac{E_u}{E} \end{vmatrix}.
\]

If \(a_1 = a_4 = 0,\) this reduces to:
\[
K = \frac{1}{E} \left\{ - \frac{1}{2} \frac{\partial}{\partial u} \left( \frac{a_2}{a_5} \right) - \frac{1}{2} \frac{\partial^2 \log E}{\partial u^2} - \frac{1}{4} \left( \frac{a_2}{a_5} \right)^2 + \frac{1}{16} \left( \frac{E_u}{E} \right)^2 \right\}.
\]

But then, \(E = R^2/(A - B^2)^2,\) where \(A = mb_1 u + mb^2\) and \(B = \delta - \frac{1}{2} nv.\)
Hence:
\[
\frac{1}{16} \left( \frac{E_u}{E} \right)^2 - \frac{1}{4} \frac{\partial^2 \log E}{\partial u^2} = -\frac{1}{4} \frac{m^2 b^2_1}{(A - B^2)^2}.
\]

Making use of this relation we have:
\[
K = \frac{1}{E} \left[ -\frac{1}{2} \frac{\partial}{\partial u} \left( \frac{a_2}{a_5} \right) - \frac{1}{4} \left( \frac{a_2}{a_5} \right)^2 \right] - \frac{1}{4} \frac{m^2 b^2_1}{R^2}.
\]

Here we notice that if \( a_2 \) were zero the curvature would be constant—which agrees with known results.

Cornell University.