THE SECOND VARIATION OF A DEFINITE INTEGRAL
WHEN ONE END-POINT IS VARIABLE*

BY

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The method applied in the following paper to the discussion of the second variation in the case in which one end-point is movable on a fixed curve, is closely analogous to that of Weierstrass † in his treatment of the problem for fixed end-points. The difference arises from the fact that in the present case terms outside of the integral sign must be taken into consideration. As a result of the discussion the analogue of Jacobi's criterion will be derived, defining apparently in a new way the critical point ‡ for the fixed curve along which the end-point varies. The relation between the critical and conjugate points is discussed in §4.

§1. The expression for the variation of the integral.

Consider a fixed curve $D$, 

$$x = f(u), \quad y = g(u),$$

and a fixed point $B (x_1, y_1)$. Let $C$ be a curve, 

$$x = \phi(t), \quad y = \psi(t),$$

cutting $D$ at $A (u = u_0, t = t_0)$, passing through $B (t = t_1)$, and making the integral

$$I = \int_{t_0}^{t_1} F(x, y, x', y') \, dt$$

a minimum with respect to values of the integral taken along other curves joining $D$ and $B$, and lying in a certain neighborhood of $C$. The following assumptions are made:

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† Lectures on the Calculus of Variations, 1879.
‡ The same as Kneser's "Brennpunkt." See his Variationsrechnung, p. 89.
§ Literal subscripts will be used to denote differentiation, partial when several variables are involved. The zero-subscript or $[\ ]_0$ means that in the function designated $t = t_0, u = u_0$. Unaccented letters refer to $D$; while accented letters refer to $C$. 

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1) The functions discussed are regular at the points considered;

2) \[ [x_0^2 + y_0^2]_0 = 0; \quad x^2 + y^2 = 0, \text{ for } t_0 \leq t \leq t_1; \]

3) \( F \) satisfies the usual homogeneity condition

\[
F(x, y, \kappa x', \kappa y') = \kappa F(x, y, x', y') \quad (\kappa > 0);
\]

4) \( F(x_0, y_0, x'_0, y'_0) = 0. \)

When the integral is taken along a curve,

\[
\bar{x} = \phi(t) + \xi(t), \quad \bar{y} = \psi(t) + \eta(t),
\]

the first variation can be put into the well-known form: *

\[
\delta I = \left[ F_x \xi + F_y \eta \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[ G_1 \xi + G_2 \eta \right] dt,
\]

where

\[
G_1 = F_x - \frac{d}{dt} F_{x'}, \quad G_2 = F_y - \frac{d}{dt} F_{y'}.
\]

According to Weierstrass † the second variation can be expressed in the form:

\[
\delta^2 I = \left[ R \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[ F_1 w'^2 + F_2 w^2 \right] dt,
\]

where

\[
R = L \xi^2 + 2M \xi \eta + N \eta^2, \quad w = y' \xi - x' \eta,
\]

the functions \( F_1, L, M, N, F_2 \) being defined by the following equations:

\[
F_1 = \frac{1}{y^2} F_{x'x'}, \quad F'_y = -\frac{1}{x'y}, \quad F_{xy}' = \frac{1}{x'y^2} F_{yy'},
\]

\[
L = F_{xx'} - y'y'' F_1, \quad N = F_{yy'} - x'x''' F_1,
\]

\[
M = F_{x'y'} + x''y' F_1 = F_{xy'} + x'y'' F_1,
\]

\[
F_2 = \frac{1}{y^2} \left( F_{xx'} - y'' F_1 - L' \right) = -\frac{1}{x'y} \left( F_{xy} + x'y'' F_1 - M' \right)
\]

\[
= \frac{1}{x'y^2} \left( F_{yy'} - x'' F_1 - N' \right).
\]

In the first place by considering variations of the curve which pass through the end-points \( A \) and \( B \) considered as fixed, the following two necessary conditions for a minimum are found:

*See Kneser, loc. cit., § 4. The arguments of \( F \) and its derivatives are always \( x, y, x', y' \).

† Weierstrass's Lectures, 1879.
I. $C$ must be an extremal* satisfying $G_1 = 0$ and $G_2 = 0$;

II. $F_1$ must be $\equiv 0$ along the arc $AB$ of the curve $C$.†

In the second place consider variations which do not pass through $A$. Inasmuch as only a necessary condition is desired, $\xi$ and $\eta$ can be chosen in a special manner. Let $\xi_0$, $\eta_0$, $\xi$, $\eta$ be defined by the equations:

$$
\xi_0 = f(u_0 + \sigma) - f(u_0) = [x_u]_0 \sigma + [x_{uu}]_0 \frac{\sigma^2}{2} + \cdots,
$$

$$
\eta_0 = g(u_0 + \sigma) - g(u_0) = [y_u]_0 \sigma + [y_{uu}]_0 \frac{\sigma^2}{2} + \cdots,
$$

$$
\xi = \phi_1 \xi_0 + \phi_2 \eta_0,
$$

$$
\eta = \psi_1 \xi_0 + \psi_2 \eta_0,
$$

where $\phi_1$, $\phi_2$, $\psi_1$, $\psi_2$ are functions of $t$ satisfying the relations:

$$
\phi_1(t_0) = \psi_2(t_0) = 1, \quad \phi_2(t_0) = \psi_1(t_0) = 0,
$$

$$
\phi_1(t_1) = \phi_2(t_1) = \psi_1(t_1) = \psi_2(t_1) = 0.
$$

A curve $(2)$ constructed with $\xi$ and $\eta$ as in $(7)$ will be said to belong to the class $\bar{C}$. It is evident that each particular curve $\bar{C}$ cuts $D$ when $t = t_0$, and passes through $B$ when $t = t_1$.

For these special variations $\Delta I$ can be expressed as a power series in $\sigma$, say

$$
\Delta I = S_1 \sigma + S_2 \frac{\sigma^2}{2} + \cdots.
$$

$S_1$ and $S_2$ can be calculated from $\delta I$ and $\delta^2 I$. From (3) and (6), since $C$ passes through $B$,

$$
\delta I = - [F_x x_u + F_y y_u]_0 \sigma - [F_x x_{uu} + F_y y_{uu}]_0 \frac{\sigma^2}{2} + \cdots,
$$

and therefore from (4) and (9),

$$
S_1 = - [F_x x_u + F_y y_u]_0,
$$

$$
S_2 = - [F_x x_{uu} + F_y y_{uu} + L x_u^2 + 2 M x_y y_u + N y_u^2]_0 + \int_{t_0}^{t_1} \left[ F_1 \bar{w}'^2 + F_2 \bar{w}^2 \right] dt,
$$

where $\bar{w}$ and $\bar{w}'$ are the coefficients of $\sigma$ in $\bar{w}$ and its derivative.

From (8) it follows that a third necessary condition for the existence of a minimum is

$$
\Delta I = S_1 = 0.
$$

* E. g., see Kneser, loc. cit., § 8.
† Weierstrass's Lectures, 1879.
This is the well-known condition for transversality.* It follows also from (8) that if a minimum exists, $S_2$ must be $\geq 0$ for all curves of class $C$. The further discussion of $S_2$ is the principal object of this paper.

§2. A condition which prevents $S_2$ from becoming negative.

Suppose now that $C$ satisfies the conditions I and III, and (instead of II) the condition that $F'_1$ is $> 0$ along the arc $AB$. Transform (4) by adding with Legendre,

$$0 = - [vw^2]_{t_0}^{t_1} + \int_{t_0}^{t_1} \frac{d(vw^2)}{dt} \, dt.$$ 

The integrand becomes a homogeneous quadratic expression in $w$ and $w'$. If for $t_0 \leq t \leq t_1$, a regular function $v$ exists satisfying the discriminant relation

$$(v) \quad v^2 - F'_1(F'_2 + v') = 0,$$

then $\delta^2 I$ becomes

$$(10) \quad \delta^2 I = [R - vw^2]_{t_0}^{t_1} + \int_{t_0}^{t_1} F'_1 \left[ w' + \frac{vw'}{F'_1} \right]^2 \, dt.$$

The integral of $(v)$ is expressible in terms of the integral of a linear equation. For when $v = - \frac{F'_1 U'}{U}$,

$$(U) \quad v^2 - F'_1(F'_2 + v') = \frac{F'_1}{U} (F''_1 U'' + F'_1 U' - F''_2 U) = 0.$$ 

Then

$$S_2 = - \left[ F'_2 x_{uu} + F'_2 y_{uu} + Lx^2 + 2Mx y + N y^2 + F'_1 \frac{\bar{w}^2}{U} \frac{U'}{U} \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} F'_1 \left[ \frac{U \bar{w}'}{U} - \frac{U' \bar{w}}{U} \right]^2 \, dt.$$ 

Assume the general integral of the differential equations $G'_1 = 0$ and $G'_2 = 0$, which are of the second order, to be

$$x = \phi(t, \alpha, \beta), \quad y = \psi(t, \alpha, \beta),$$

where $\alpha$ and $\beta$ are arbitrary constants. Suppose that these equations represent $C$ when $\alpha = \beta = 0$. Then two particular integrals of $(U)$ are†

$$\delta_1 = \begin{vmatrix} \phi_t & \phi_\alpha \\ \psi_t & \psi_\alpha \end{vmatrix}, \quad \delta_2 = \begin{vmatrix} \phi_t & \phi_\beta \\ \psi_t & \psi_\beta \end{vmatrix}.$$ 

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* Kneser, loc. cit., § 10.
† E. g., see Weierstrass's Lectures.
where \( \phi_t = \phi_t(t, 0, 0) \), etc. Suppose \( \partial_1 \) and \( \partial_2 \) to be linearly independent. Then the general integral of \( (U) \) is

\[
U = c_1 \partial_1 + c_2 \partial_2.
\]

Since \( \partial_1 \) and \( \partial_2 \) are linearly independent they satisfy the equation*

\[
\partial_1 \partial_2' - \partial_2 \partial_1' = \frac{c}{F_1} \quad (c \neq 0).
\]

A particular integral (12) can now be selected so that in \( S_2 \) the term outside of the integral vanishes. Put

\[
P = \left[ \frac{F_{xu}x_{uu} + F_{yu}y_{uu}}{x_u^2 + y_u^2} \right]_0 + L_0 \cos^2 \delta + 2M_0 \sin \delta \cos \delta + N_0 \sin^2 \delta,
\]

and

\[
Q = \left[ F_1 \left( \frac{y'x_u - x'y_u}{x_u^2 + y_u^2} \right) \right]_0 = \left[ F_1 \right]_0 \left( x_0^2 + y_0^2 \right) \sin^2(\gamma - \delta),
\]

where \( \gamma \) and \( \delta \) are the angles at \( A \) which \( C \) and \( D \) make with the \( x \)-axis. Then, from (11),

\[
S_2 = - \left[ P + Q \frac{U_0'}{U_0} \right] \left[ x_u^2 + y_u^2 \right]_0 + \int_{t_0}^t F_1 \left[ \frac{U'\dot{w} - U\dot{w}}{U} \right]^2 dt.
\]

Kneser has shown† that if \( F \neq 0 \) at \( A \), and \( D \) cuts \( C \) transversally, then \( D \) cannot be tangent to \( C \). Therefore \( Q \) is \( \neq 0 \). Since, furthermore, the equation (13) holds when \( t = t_0 \), \( c_1 \) and \( c_2 \) can be so determined that

\[
P + Q \frac{U_0'}{U_0} = 0.
\]

Two such values are

\[
c_1 = P \partial_2(t_0) + Q \partial_2'(t_0), \quad c_2 = P \partial_1(t_0) + Q \partial_1'(t_0).
\]

If \( H(t, t_0) \) denotes the particular integral of \( (U) \) formed with these constants, then

\[
H(t, t_0) = P \Theta + Q \frac{\partial \Theta}{\partial t_0},
\]

where

\[
\Theta(t, t_0) = \begin{vmatrix} \partial_1(t) & \partial_2(t) \\ \partial_1(t_0) & \partial_2(t_0) \end{vmatrix}.
\]

The integral \( H \) is useful in forming a function \( v \) to satisfy condition \( (v) \), at least when \( B \) is near \( A \). For from (13) and (16), when \( t = t_0 \),

\[
H_0 = Q \frac{c}{[F_1]_0} \neq 0.
\]

* See Craig, Linear Differential Equations, vol. 1, p. 54.
† loc. cit., § 30.
These results lead to the following theorem:

If \( H(t, t_0) \neq 0 \) for \( t_0 \leq t \leq t_1 \), then for curves of class \( C \), \( S_2 \) can be expressed in the form

\[
S_2 = \int_{t_0}^{t_1} F_1 \left[ \frac{Uw' - U'w}{U} \right]^2 dt,
\]

which cannot become negative.

§ 3. The necessary condition.

By following still more closely the method of Weierstrass it can now be shown that the condition \( H(t, t_0) \neq 0 \) \( (t_0 \leq t < t_1) \) is necessary for the existence of a minimum. Suppose that this condition does not hold but that \( H \) has a zero \( t'_0 \) between \( t_0 \) and \( t_1 \). Then, as will be proved, variations of class \( C \) can be found which make \( S_2 \) and \( \Delta I \) negative.

Integrate by parts the first term in the integrand of (4). Then

\[
\delta^2 I = \left[ R + F_1ww' \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} w \left[ F'_1w'' + F_1'w' - F_1w \right] dt.
\]

Consider the equation,

\[
(U_e) \quad F_1 U'' + F_1' U' - (F_2 - \epsilon) U = 0,
\]

where \( \epsilon \) is a constant. From the theory of linear differential equations, an integral \( H_e \) of the equation \((U_e)\) exists, depending upon \( \epsilon \) for its value and having the following properties:

1) It is regular for \( t_0 \leq t \leq t_1 \);
2) \( [H_e]_0 = H_w \), \( [H_e']_0 = H'_w \);
3) If \( \eta > 0 \) is selected arbitrarily, \( \delta > 0 \) can be found such that \( |H_e - H| < \eta \) for \( t_0 \leq t \leq t_1 \), if \( |\epsilon| < \delta \).

\( H \) and \( H' \) can not both be zero at \( t'_0 \). For otherwise, since the functions involved are regular and \( F'_1 \neq 0 \), the expansion of the left member of \((U)\) could not be identically zero. From 3) therefore, \( \delta \) can be chosen so small that when \( |\epsilon| < \delta \), \( H_e \) also vanishes between \( t_0 \) and \( t_1 \), say at \( t_{\epsilon_0} \).

Curves can now be chosen of class \( C \), such that \( w \) satisfies the equation \((U_e)\). For example, let \( \xi \) and \( \eta \) be defined for \( t_0 \leq t \leq t_{\epsilon_0} \) by the equations

\[
(17) \quad w = y'\xi - x'\eta = (y'\xi_0 - x'\eta_0) \frac{H^*_w}{H'_w} H_w,
\]

\[
(18) \quad \delta^2 I = \left[ R - F_1ww' \right]_{t_0}^{t_1} + \epsilon \int_{t_0}^{t_1} w^2 dt.
\]
From (9) and (18) by calculation as before, and since \( H \) satisfies (15), it follows that

\[
S_2 = \epsilon \int_{t_0}^{t_0+\delta} \overline{w}^2 \, dt.
\]

The function \( \overline{w} \) can not be identically zero unless \( H_\epsilon \) is so; and by 3) \( H_\epsilon \) can not vanish identically if \( \delta \) is taken small enough, since \( H \) does not. Hence for certain functions \( \xi, \eta \) as in (7), \( S_2 \neq 0 \) and can be made positive or negative by taking values of \( \epsilon \) opposite in sign. From \$1 \$ therefore the arc \( AB \) can not make \( I \) a minimum.

If now the point \( A' \) defined on \( G \) by \( t_0' \) is said to be the critical point for the curve \( D \), a fourth necessary condition can be stated as follows:

IV. If the extremal \( C \), which passes through the fixed point \( B \) and cuts the fixed curve \( D \) transversally, is to make the integral

\[
I = \int_{t_0}^{t_1} F(x, y, x', y') \, dt
\]
a minimum, then \( B \) must not lie beyond the critical point defined by \( D \) on \( C \); or analytically,

\[
H(t, t_0) \neq 0 \quad \text{for} \quad t_0 \leq t < t_1.
\]

§ 4. Relation between the conjugate and critical points.

The point conjugate to \( A \) is defined * by the zero \( t_0'' \) of \( \Theta(t, t_0) \), which is nearest to \( t_0 \). The functions \( \Theta \) and \( H \) are both integrals of \( U \) of the form (12), and are linearly independent since \( \Theta_0 = 0 \) and \( H_0 \neq 0 \). By a theorem concerning linear differential equations of the second order \( \dagger \) their zeros must separate each other, and \( H = 0 \) has therefore one root between \( t_0 \) and \( t_0'' \).

The expression for \( H \) involves the curvature of \( D \) at \( A \) linearly. The curvature is

\[
(19) \quad \frac{1}{r} = \frac{x_uy_{uu} - x_{uu}y_u}{[x_u^2 + y_u^2]^{3/2}}.
\]

By differentiating (1) for \( \kappa \) it is found that

\[
x' F_x + y' F_y = F.
\]

From this equation and III the values of $F_x$ and $F_y$ at $A$ can be determined, and by substitution in (16) $H$ becomes

$$H(t, t_0) = \left(\frac{P_1}{r} + P_2\right)\Theta + Q\frac{\partial \Theta}{\partial t_0},$$

where

$$P_1 = \frac{F_0}{\sqrt{x_0^2 + y_0^2 \sin (\gamma - \delta)}},$$

$$P_2 = L_0 \cos^2 \delta + 2M_0 \sin \delta \cos \delta + N_0 \sin^2 \delta.$$ 

Suppose $C$ and $A$ fixed, and $D$ changeable but always transversal to $C$ at $A$. Then if the expression (20) for $H$ is put equal to zero and solved for $r$, the resulting function of $t$ will express the value which the radius of curvature of $D$ at $A$ must have in order that $t$ may determine the critical point for $D$. By the use of (18), (14) and (21) the function and its derivative are found to be

$$r = \frac{-P_1 \Theta}{P_2 \Theta + Q\frac{\partial \Theta}{\partial t_0}},$$

$$\frac{dr}{dt} = -\frac{c^2 \sqrt{x_0^2 + y_0^2}}{\left[\frac{P_2 \Theta + Q\frac{\partial \Theta}{\partial t_0}}{P_1}\right]^2} F_0 \sin (\gamma - \delta).$$

The denominator of $r$ vanishes once between $t_0$ and $t_0''$ for the same reason that $H$ does. From 4) of § 1 the derivative $dr/dt$ is $\pm 0$ and has the sign of

$$\frac{F_0}{F_1} \sin (\gamma - \delta).$$

The radius of curvature (19) is positive when its direction is related to that of the curve $D$ for increasing $u$ as the $+y$-axis is to the $+x$-axis; otherwise it is negative.

From these results the following theorems can be stated if it is supposed that $F_0 > 0$:

1) The critical point for a curve $D$ which cuts the extremal $C$ transversally at $A$, always lies between $A$ and its conjugate $A''$.

2) The position of the critical point is determined by the curvature of $D$ at $A$.

3) If the radius of curvature of $D$ at $A$ is supposed to vary continuously from 0 to $\infty$ on the same side of $D$ as the arc $AB$, and from $\infty$ to 0 on the

*See Kneser, loc. cit., p. 111.
opposite side, then the critical point moves continuously from $A$ to $A''$ when there is a minimum, and from $A''$ to $A$ when there is a maximum.

§ 5. Relation between the preceding results and those of Kneser.

Sufficient conditions.

Kneser has derived a necessary condition which corresponds to IV. He shows that it is possible to find a set of extremals,*

\[(22) \quad x = \xi(t, a), \quad y = \eta(t, a),\]

each cutting $D$ transversally when $t = t_0$, and giving $C$ for $a = 0$. The curve $D$ is then represented in the vicinity of $C$ by the equations

\[x = \xi(t_0, a), \quad y = \eta(t_0, a),\]

where $a$ is the parameter. The condition III of transversality requires that

\[\left[ F_x' \xi_a + F_y' \eta_a \right]_0 = 0\]

for every $a$ near zero, since each curve of the system (22) is transversal to $D$. This equation can be differentiated for $a$ and the derivatives of $F$ expressed in terms of $F_1$, $L$, $M$, $N$, from equations (5). From (14) and (20), $P$ and $Q$ depend only upon the curvature and direction of $D$, and are independent therefore of the parameter representation. It follows that for $a = 0$,

\[
\frac{\partial}{\partial a} \left[ F_x' \xi_a + F_y' \eta_a \right]_0 = \left[ P + Q \frac{\Delta'(0,0)}{\Delta(0,0)} \right] \left[ \xi_a + \eta_a \xi_a \right]_0 = 0,
\]

where

\[\Delta(t, a) = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix}.\]

$\Delta(t, 0)$ must therefore satisfy (15). It can be proved, as for $\partial_1$ and $\partial_2$, that $\Delta(t, 0)$ is also an integral of $(U)$. Since both $H$ and $\Delta(t, 0)$ are integrals of $(U)$ satisfying (15) they must be linearly dependent. That is,

\[H(t, t_0) = C\Delta(t, 0), \quad C \neq 0.\]

The condition IV can therefore be restated in Kneser's form:

IV'. If $C$ as in IV is to make $I$ a minimum, then a necessary condition is

\[\Delta(t, 0) \neq 0 \text{ for } t_0 \leq t < t_1.\]

Kneser proves this condition † by discussing the case in which $B$ coincides

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* Kneser, loc. cit., §30.
† loc. cit., §25.
with the critical point $A'$. He shows that unless the envelope has a singular point of a particular kind at $A'$, there is no minimum, and so none when $B$ lies beyond $A'$. The result is a stronger condition than IV', namely,

\[(23) \quad \Delta(t, 0) \neq 0, \quad \text{for} \quad t_0 \leq t \leq t_1.\]

But his proof does not hold if the envelope has the exceptional form mentioned.

The method given in § 3 applies when $B$ lies beyond $A'$, and then includes Kneser's exceptional case. It cannot be used when $B$ and $A'$ coincide. For then it is not certain that the integral $H_\varepsilon$ can be made to vanish between $t_0$ and $t_1$, and since $w$ must vanish for $t = t_1$, the functions $\xi, \eta$ cannot be constructed as in (17).

If the conditions II and IV are amended to read:

II'. $F_1 > 0$ for points $(x, y)$ on $AB$, and for any $(x', y') \neq (0, 0)$,

IV'. $H(t, t_0) \neq 0$ for $t_0 \leq t \leq t_1$,

then $\Delta(t, 0)$ satisfies (23). According to Kneser a field can be constructed about $AB$, and the four conditions I, II', III', IV' are sufficient conditions for the arc $AB$ to make the integral a minimum.

The University of Minnesota,
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