COMPLETE SETS OF POSTULATES FOR THE THEORIES OF POSITIVE INTEGRAL AND POSITIVE RATIONAL NUMBERS*

BY

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By properly modifying the set of postulates considered in the preceding paper, we can construct two different sets of postulates such that every assemblage which satisfies either of these new sets will be equivalent to the system of *positive integers*, when $a \circ b = a + b$.*

In the first set (§ 1), postulates 1–5 are left unchanged, while 6 is replaced by a new postulate 6′. In the second set (§ 2), postulates 1–3 are retained, while postulates 4, 5 and 6 are replaced by a single postulate, 4″. Both of these sets are complete sets of postulates in the sense defined on p. 264, although one contains six postulates and the other only four. A problem is therefore at once suggested, to which no satisfactory answer has as yet been given, viz., "when several complete sets of postulates define the same system, which shall be regarded as the best?"

By a further modification of the postulates, in which 1–3 are still retained, while 4, 5 and 6 are replaced by 4‴ and 5‴, we obtain (§ 3) a complete set of postulates for the theory of *positive rational numbers.*

§ 1. *First set of postulates for positive integers.*

1, 2, 3, 4, 5. *The postulates* 1, 2, 3, 4, 5 *on p.* 267.

6′. *There is an element E such that $x \circ y \neq E$ whenever $y \neq E$.*

THEOREM I′.—*The postulates* 1, 2, 3, 4, 5 *and* 6′ *are consistent.* (Cf. Th. I.)

For, one system which satisfies them all is the system of positive integers, with $a \circ b$ defined as $a + b$. (In 6′ take $E = 1$.)

Another such system is the assemblage of all positive integral powers of 2: $2^1, 2^2, 2^3, \cdots$, with $a \circ b$ defined as $a \times b$. (In 6′ take $E = 2$.)

* Presented to the Society February 22, 1902. Received for publication March 8, 1902.

* Of course this work throws no new light on the fundamental nature of the numbers, since the whole system of ordinal numbers is assumed in defining the sequence of multiples of any element (12). For discussion of the more fundamental problems, see DEDEKIND, *Was sind und was sollen die Zahlen?* and the papers of PEANO and PADOA already cited.

THEOREM II′.—*All the assemblages which satisfy postulates* 1, 2, 3, 4, 5 *and* 6′ *are equivalent.* (Cf. Th. II.)

To establish this theorem, notice first that propositions 7–26 in the preceding paper, being deduced from 1, 2, 3, 4, 5 alone, will hold here also.

Hence, the element E in 6′ is a minimum element; that is, whatever element a may be, $a \geqq E$. (For, if we suppose $a < E$ then, by 7, $a \circ y = E$, which is impossible by 16 if $y = E$, and by 6′ if $y \neq E$; hence, by 8, $a \geqq E$.)

Therefore, as we saw in 22, Case I, every element must be some multiple of E; all the multiples of E belong to the assemblage, by 1, and by 13*f* no two of them are the same.

Then any two assemblages which satisfy 1–5, 6′ can be brought into one-to-one correspondence by assigning the multiples of E in M to the same multiples of E' in M'; and when this is done, $a \circ b$ will correspond with $a' \circ b'$ by 13*b*.

Thus the postulates 1, 2, 3, 4, 5 and 6′ are *sufficient.*

THEOREM III′.—*The postulates* 1, 2, 3, 4, 5 *and* 6′ *are independent.* (Cf. Th. III.)

To establish the independence of these postulates we employ the following assemblages:

M'_1) Let M'_1 be the assemblage of all positive integers less than 10, while $a \circ b = a + b$. Then M'_1 satisfies all the other postulates, but not 1.

M'_2) Let M'_2 be the assemblage of all positive integers and 0, while $a \circ b = a + b$. Then M'_2 satisfies all the other postulates, but not 2.

In 5 notice that whenever $a \leqq b$ numerically, $a < b$ in the sense in which this notation is used in 5. Hence the sequence S will have a maximum element which can be taken as A.

M'_3) Let M'_3 be the assemblage of all positive integers, while $a \circ b$ is defined as follows: when $a \neq b$, $a \circ b = b$; when $a = b$, $a \circ a = a + 1$. Then M'_3 satisfies all the other postulates, but not 3.

Postulates 1 and 2 clearly hold. Postulate 3 fails, for $a \circ (a \circ a) = a + 1$ while $(a \circ a) \circ a = a$. In 4 take $y = b$; then, since $a \neq y$, $a \circ y = b$. To satisfy 6′ take $E = 1$. In 5 notice that whenever $a \neq b$ we shall have $a < b$ in the sense in which this notation is used in 5. Hence S may be any infinite sequence in which no two successive elements are the same; and any element not belonging to S will answer for c. Now take the element c as A; thus 5 is satisfied.

M'_4) Let M'_4 be the assemblage of all positive integers greater than 1, while $a \circ b = a + b$. Then M'_4 satisfies all the other postulates, but not 4.

To show that 4 fails, take $a = 6$ and $b = 7$. In 6′ take $E = 2$. In 5 notice that whenever $b - a > 1$ numerically, $a < b$ in the sense in which this

notation is used in 5. Hence 5 is satisfied, since no infinite sequence which satisfies the conditions stated can occur.

M'_5) * Let M'_5 be the assemblage of all couples of the form (α, β), where α is any integer (positive negative or 0), and β any positive integer or 0, provided that β shall not be 0 unless α is positive. Let the rule of combination be the following:

$$(\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, \beta_1 + \beta_2).$$

Then M'_5 satisfies all the other postulates, but not 5.

(These couples (α, β) may be represented by those points in the upper half of the complex plane whose coördinates are both integral, together with the points $(1, 0), (2, 0), (3, 0), \cdots$ on the α-axis.) Postulates 1, 2, 3 are clearly satisfied. To see that 4 is satisfied, let $a = (\alpha_1, \beta_1)$ and $b = (\alpha_2, \beta_2)$. Then if $\beta_1 > \beta_2$ take $x = (\alpha_1 - \alpha_2, \beta_1 - \beta_2)$; if $\beta_1 < \beta_2$ take $y = (\alpha_2 - \alpha_1, \beta_2 - \beta_1)$; and if $\beta_1 = \beta_2$ take $x = (\alpha_1 - \alpha_2, 0)$ or $y = (\alpha_2 - \alpha_1, 0)$ according as $\alpha_1 > \alpha_2$ or $\alpha_1 < \alpha_2$. To see that 5 fails, consider the sequence $(1, 0), (2, 0), (3, 0), \cdots$ with $c = (1, 1)$. In 6′ take $E = (1, 0)$.

M'_6) Let M'_6 be the assemblage of all positive real numbers, while $a \circ b = a + b$. Then M'_6 satisfies all the other postulates, but not 6′

§ 2. *Second set of postulates for positive integers.*

1, 2, 3. *The postulates* 1, 2, 3 *on p.* 267.

4″. *There is an element E such that every element is some multiple of E; that is, given an element a there is always some ordinal number m such that $mE = a$.* (See 13*a*.)

THEOREM I″.—*The postulates* 1, 2, 3 *and* 4″ *are consistent.*

For both the assemblages mentioned under Th. I′ satisfy them all.

THEOREM II″.—*All the assemblages which satisfy the postulates* 1, 2, 3 *and* 4″ *are equivalent.*

For, propositions 13*a* to 13*h* in the preceding paper, being deduced from 1, 2, 3 alone, will hold here also. Therefore by 13*f* no two multiples of E will be the same.—But by 1 all the multiples of E are elements of the assemblage, and by 4″ there are no elements which are not multiples of E.

Hence the theorem follows precisely as under Th. II′; that is, the postulates 1, 2, 3 and 4″ are *sufficient.*

THEOREM III″.—*The postulates* 1, 2, 3 *and* 4″ *are independent.*

The assemblages which we use in the proof of independence here are the following:

* The system M'_5 was suggested to me by Professor BÔCHER.

M''_1) Let M''_1 be the assemblage of all positive integers less than 10, while $a \circ b = a + b$. Then M''_1 satisfies all the other postulates, but not 1.

M''_2)* Let M''_2 be an assemblage containing only a single element, a, the rule of combination being, of course, $a \circ a = a$. Then M''_2 satisfies all the other postulates, but not 2.

Postulate 2 clearly fails, and 1 and 3 clearly hold. In 4″ take $E = a$ and $m = 1$; then $a = 1E$, so that 4″ also holds.

M''_3) Let M''_3 be an assemblage containing only two elements, a and b, the rule of combination being taken as follows:

$$a \circ a = b; \quad a \circ b = b; \quad b \circ a = a; \quad b \circ b = a.$$

Then M''_3 satisfies all the other postulates, but not 3.

Postulate 3 fails, since, for example, $(a \circ a) \circ a = b \circ a = a$, while $a \circ (a \circ a) = a \circ b = b$. Postulates 1 and 2 clearly hold. In 4″ take $E = a$, then $a = 1E$ and $b = 2E$, so that 4″ also holds.

M''_4) Let M''_4 be the assemblages of all positive real numbers, while $a \circ b = a + b$. Then M''_4 satisfies all the other postulates, but not 4″.

Since there is no element E of which 2 and $\sqrt{2}$ are multiples, 4″ is not satisfied.

§ 3. *The set of postulates for positive rational numbers.*

1, 2, 3. *The postulates* 1, 2, 3 *on p.* 267.

4‴. *Given any elements a and b, there is an element z of which both a and b are multiples: $a = pz$, $b = qz$.* (See 13*a*.)

5‴. *Given any element a and any ordinal number m, there is an element x such that $mx = a$.* (See 13*a*.)

THEOREM I‴.—*The postulates* 1, 2, 3, 4‴ *and* 5‴ *are consistent.*

For, one assemblage which satisfies them all is the system of positive rational fractions, with $a \circ b = a + b$.

Another such assemblage is the system of all positive rational powers of 2, with $a \circ b = a \times b$.

THEOREM II‴.—*All the assemblages which satisfy postulates* 1, 2, 3, 4‴ *and* 5‴ *are equivalent.*

In the first place, propositions 13 will hold, by 1, 2, 3.

Next, 20 will hold. For, by 4‴ let $a = pz$ and $a' = p'z$; then if $a > a'$ we have $pz > p'z$, whence, by 13*g*, p comes later than p', and $p(mz) > p'(mz)$; therefore by 13*e*, $m(pz) > m(p'z)$, or, $ma > ma'$.

* The system M''_2 might have been used in place of M_2 on p. 278.

From 20 follows the truth of 21.

Further, the element x in $5'''$ is uniquely determined. For, suppose x and x' are both such that $mx = mx' = a$, and by $4'''$ let $x = pz$ and $x' = p'z$. Then $m(pz) = m(p'z)$, or, by 13*e*, $p(mz) = p'(mz)$, whence, by 13*f*, $p = p'$, that is, $x = x'$.

Therefore 28, and hence 31 and 32, will hold.

If now a "unit element" E be chosen at pleasure, *every element*, a, *will be some rational fraction of* E. For, by $4'''$, $a = pz$ and $E = qz$; therefore, by 13*e*, $qa = q(pz) = p(qz) = pE$, whence, by 28, $a = pE/q$.

Then if M and M' are any two assemblages satisfying all these postulates, we can assign to every element pE/q in M a definite element pE'/q in M'; the correspondence thus established will be a one-to-one correspondence, by 32*c*; and $a \circ b$ will correspond with $a' \circ b'$ by 32*b*.

Thus the postulates 1, 2, 3, $4'''$ and $5'''$ are *sufficient*.

THEOREM III'''.—*The postulates* 1, 2, 3, $4'''$ *and* $5'''$ *are independent.*

This theorem is proved by the use of the following assemblages:

M_1''') Let M_1''' be the assemblage of all positive rational fractions less than 10, while $a \circ b = a + b$.

M_2''', M_3''', M_4''') The assemblages M_2'', M_3'', M_4'' in § 2.

M_5''') Let M_5''' be the assemblage of all positive integers, while $a \circ b = a + b$.

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