DEFINITIONS OF A LINEAR ASSOCIATIVE ALGEBRA BY INDEPENDENT POSTULATES*

BY

LEONARD EUGENE DICKSON

Introduction.

The term *linear associative algebra*, introduced by BENJAMIN PEIRCE, has the same significance as the term *system of (higher) complex numbers.* † In the usual theory of complex numbers, the coördinates are either real numbers or else ordinary complex quantities. To avoid the resulting double phraseology and to attain an evident generalization of the theory, I shall here consider systems of complex numbers whose coördinates belong to an arbitrary *field* F.

I first give the usual definition by means of a multiplication table for the n *units* of the system. It employs three postulates, shown to be independent, relating to n^3 elements of the field F.

The second definition is of abstract character. It employs four independent postulates which completely define a system of complex numbers.

The first definition may also be presented in the abstract form used for the second, namely, without the explicit use of units. The second definition may also be presented by means of units. Even aside from the difference in the form of their presentation, the two definitions are essentially different.

First Definition of a System of Complex Numbers.

Consider n quantities $e_1, e_2, \cdots, e_n$ linearly independent with respect to the field F and having a multiplication-table of the form

$$e_i e_k = \sum_{s=1}^{n} \gamma_{iks} e_s \qquad (i, k = 1, 2, \cdots, n), \tag{1}$$

where each γ_{iks} belongs to F. If $a_1, \cdots, a_n$ belong to F, the expression

$$a \equiv a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

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† A bibliography of the subject is given by STUDY, Encyklopädie der Mathematischen Wissenschaften, vol. 1, pp. 159–183. Cf. LIE-SCHEFFERS, *Continuierliche Gruppen*, ch. 21.

is called a *complex number* with the *coördinates* $a_1, a_2, \cdots, a_n$. Let

$$b \equiv b_1 e_1 + b_2 e_2 + \cdots + b_n e_n$$

be a second complex number. Addition and subtraction are defined thus:

$$a \pm b = (a_1 \pm b_1) e_1 + \cdots + (a_n \pm b_n) e_n.$$

In accordance with the distributive law, multiplication is defined thus:

$$ab = \sum_{i,k}^{1,\ldots,n} a_i b_k e_i e_k. \tag{2}$$

It follows from (1) that

$$ab = \sum_{s=1}^{n} u_s e_s, \qquad u_s \equiv \sum_{i,k}^{1,\ldots,n} \gamma_{iks} a_i b_k. \tag{3}$$

Whatever be the n^3 marks γ_{iks} of F, we have defined unambiguously certain operations called addition, subtraction and multiplication, which, when applied to any complex numbers with coördinates in F, lead uniquely to complex numbers with coördinates in F. It remains to impose certain conditions on the γ_{iks} such that there will result a *system of complex numbers*, viz., one for which the associative law for multiplication holds and for which division (as defined below) may in general be performed uniquely.

In view of (2), the associative law holds always if, and only if,

$$(e_i e_k) e_l = e_i (e_k e_l) \qquad (i, k, l = 1, 2, \cdots, n). \tag{4}$$

In view of (1) and the linear independence of $e_1, \cdots, e_n$, these relations give

$$\sum_{s=1}^{n} \gamma_{iks} \gamma_{slt} = \sum_{s=1}^{n} \gamma_{kls} \gamma_{ist}. \qquad (i, k, l, t = 1, \cdots, n). \tag{5}$$

In order that, for a general complex number a and an arbitrary complex number b, it shall be possible to determine uniquely a complex number x such that $ax = b$, the condition is that

$$\Delta_a \equiv \left| \sum_{i=1}^{n} \gamma_{iks} a_i \right| \text{ shall not vanish for every } a_1, \cdots, a_n. \tag{6}$$

The proof follows from formulæ analogous to (3).

Likewise, in order that it shall be possible to determine uniquely a complex number y such that $ya = b$, the condition is that

$$\Delta'_a \equiv \left| \sum_{k=1}^{n} \gamma_{iks} a_k \right| \text{ shall not vanish for every } a_1, \cdots, a_n. \tag{7}$$

Every system of complex numbers with respect to a field F defines n^3 marks γ_{iks} of F satisfying the conditions (5), (6), (7). *Inversely, such a set of marks γ_{iks} defines a system of complex numbers with respect to F.*

Independence of the Conditions (5), (6), (7).

Following the customary method, we exhibit for $j = 5, 6, 7$, a set S_j of n^3 marks γ_{iks} of F for which the jth condition fails while the remaining two conditions are satisfied. It suffices to take $n = 2$, whence

$$\Delta_a \equiv \begin{vmatrix} \gamma_{111}a_1 + \gamma_{211}a_2 & \gamma_{112}a_1 + \gamma_{212}a_2 \\ \gamma_{121}a_1 + \gamma_{221}a_2 & \gamma_{122}a_1 + \gamma_{222}a_2 \end{vmatrix}, \quad \Delta'_a \equiv \begin{vmatrix} \gamma_{111}a_1 + \gamma_{121}a_2 & \gamma_{112}a_1 + \gamma_{122}a_2 \\ \gamma_{211}a_1 + \gamma_{221}a_2 & \gamma_{212}a_1 + \gamma_{222}a_2 \end{vmatrix}.$$

$$S_5. \qquad \begin{cases} \gamma_{111} = 1, & \gamma_{112} = 0, & \gamma_{121} = 0, & \gamma_{122} = 1, \\ \gamma_{211} = 1, & \gamma_{212} = 0, & \gamma_{221} = 1, & \gamma_{222} = 0.* \end{cases}$$

Then (5) fails for $i = 2$, $k = 1$, $l = 2$, $t = 1$, since

$$\sum_{s=1,2} \gamma_{21s}\gamma_{s21} = 0, \qquad \sum_{s=1,2} \gamma_{12s}\gamma_{2s1} = 1.$$

But (6) and (7) hold, since

$$\Delta_a \equiv \begin{vmatrix} a_1 + a_2 & 0 \\ a_2 & a_1 \end{vmatrix} = a_1(a_1 + a_2), \qquad \Delta'_a \equiv \begin{vmatrix} a_1 & a_2 \\ a_1 + a_2 & 0 \end{vmatrix} = -a_2(a_1 + a_2).$$

$$S_6. \qquad \begin{cases} \gamma_{111} = 1, & \gamma_{112} = 0, & \gamma_{121} = -1, & \gamma_{122} = 0, \\ \gamma_{211} = 0, & \gamma_{212} = 1, & \gamma_{221} = 0, & \gamma_{222} = -1. \end{cases}$$

$$\Delta_a \equiv \begin{vmatrix} a_1 & a_2 \\ -a_1 & -a_2 \end{vmatrix} = 0, \qquad \Delta'_a \equiv \begin{vmatrix} a_1 - a_2 & 0 \\ 0 & a_1 - a_2 \end{vmatrix} = (a_1 - a_2)^2.$$

We may verify directly that conditions (5) are satisfied; or we may verify relations (4), employing relations (1) which here become

$$e_1e_1 = e_1, \qquad e_1e_2 = -e_1, \qquad e_2e_1 = e_2, \qquad e_2e_2 = -e_2.$$

$$S_7. \qquad \begin{cases} \gamma_{111} = 1, & \gamma_{112} = 0, & \gamma_{121} = 0, & \gamma_{122} = 1, \\ \gamma_{211} = -1, & \gamma_{212} = 0, & \gamma_{221} = 0, & \gamma_{222} = -1. \end{cases}$$

* When F does not have modulus 2, we may take for S_5 the set
$\gamma_{111} = 1$, $\gamma_{112} = 0$, $\gamma_{121} = 0$, $\gamma_{122} = -1$, $\gamma_{211} = 0$, $\lambda_{212} = -1$, $\gamma_{221} = -1$, $\gamma_{222} = 0$.
Then $\Delta_a = \Delta'_a = -(a_1^2 + a_2^2)$; while (5) fails for $i = 2$, $k = 2$, $l = 1$, $t = 1$, since $-1 \neq +1$.

$$\Delta_a \equiv \begin{vmatrix} a_1 - a_2 & 0 \\ 0 & a_1 - a_2 \end{vmatrix} = (a_1 - a_2)^2, \qquad \Delta'_a \equiv \begin{vmatrix} a_1 & a_2 \\ -a_1 & -a_2 \end{vmatrix} = 0.$$

That conditions (5) are satisfied follows from the fact that the set S_7 can be derived from the set S_6 by interchanging γ_{iks} with γ_{kis}.

Second Definition of a System of Complex Numbers.

We consider a system of elements $A = (a_1, a_2, \cdots, a_n)$ each uniquely defined by n marks of the field F together with their sequence. The marks $a_1, \cdots a_n$ are called the *coördinates* of A. The element $(0, 0, \cdots, 0)$ is called zero and designated 0.

Addition of elements is defined thus:

$$A + B = (a_1 + b_1, a_2 + b_2, \cdots, a_n + b_n). \tag{8}$$

It follows that there is an element $D = (a_1 - b_1, \cdots, a_n - b_n)$ such that $D + B = A$.

Consider a second rule of combination of the elements having the properties:*

1. For any two elements A and B of the system, $A \cdot B$ is an element of the system whose coördinates are bilinear functions of the coördinates of A and B, with fixed coefficients belonging to F.

2. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, if $A \cdot B$, $B \cdot C$, $(A \cdot B) \cdot C$, $A \cdot (B \cdot C)$ belong to the system.

3. There exists in the system an element I such that $A \cdot I = A$ for every element A of the system.

4. There exists in the system at least one element A such that $A \cdot Z \neq 0$ for any element $Z \neq 0$.

That any system of elements given by the second definition is a system of complex numbers according to the usual (first) definition is next shown.† From 1 and (8) follows the distributive law:

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C). \tag{9}$$

For any element I satisfying 3, $I \cdot B = B$ *for every element* B.

In proof, let A be one of the elements satisfying 4, and let B' be such that $B + B' = 0$ (see above). Then by 2

* Note that 3 assumes the existence of a right-hand identity element. Postulate 4 is milder than the assumption of a left-hand identity I', while from the existence of I' would follow 4, A being taken as I'.

† The inverse is true. That a system of complex numbers contains an identity element (so that 3 and 4 follow) is shown in LIE-SCHEFFERS, p. 614.

$$A\cdot(I\cdot B)=(A\cdot I)\cdot B=A\cdot B.$$

Hence, by (9) applied twice,

$$A\cdot[(I\cdot B)+B']=(A\cdot B)+(A\cdot B')=A\cdot(B+B')=A\cdot 0.$$

By 1, $A\cdot 0=0$. Hence by 4, $(I\cdot B)+B'=0$, so that $I\cdot B=B$.

There is an unique element I satisfying 3.

For, let I be the given element and I' a second element, each satisfying 3. Then $I'\cdot I=I'$ by 3. By the preceding theorem, $I'\cdot B=B$ for every B, whence $I'\cdot I=I$. It follows that $I'=I$.

There is an unique element I such that $I\cdot B=B$ for every B.

For, let I_1 be one such element and let I be the unique element satisfying 3. Then $I_1\cdot B=B$ gives $I_1\cdot I=I$, while 3 gives $I_1\cdot I=I_1$. Hence $I_1=I$.

From the three preceding results it follows that there is an unique element I such that $A\cdot I=I\cdot A=A$ for every A.

To pass to the form of representation used in the first definition, we make $A\equiv(a_1, a_2, \cdots, a_n)$ correspond to $a\equiv a_1e_1+a_2e_2+\cdots+a_ne_n$. In view of 1, there exist constant marks γ_{iks} of F such that relation (3) holds and, as special cases, relations (1). Condition 2 thus leads to relations (5). Since there is an unique solution* $X=I$ of $A\cdot X=A$, where A satisfies 4, the determinant Δ_a does not vanish for every $a_1, \cdots, a_n$. Since there is an unique solution $X=I$ of $X\cdot A=A$, where A is such that $ZA\neq 0$ if $Z\neq 0$, Δ'_a does not vanish for every $a_1, \cdots, a_n$. Since conditions (5), (6), and (7) are satisfied, the system of elements forms a system of complex numbers.

Independence of the Postulates 1, 2, 3, 4.

For $i=1, 2, 3, 4$, we exhibit a system Σ_i of elements for which the ith postulate fails, while the remaining three postulates hold.

Σ_1. Take $A\cdot B=A$. Or take $A\cdot B=A+B$ with $I=0$ and $A=0$ in 4.

Σ_2. Take $n=2$, and for $A\cdot B$ take the law of combination

$$(a_1, a_2)(\alpha_1, \alpha_2)=(a_1\alpha_1+a_1\alpha_2+a_2\alpha_2, a_2\alpha_1).$$

Then 3 is satisfied for $I=(1, 0)$, and 4 for $A=(0, 1)$ since

$$(0, 1)(z_1, z_2)=(z_2, z_1).$$

But 2 fails for $A=(0, 1)$, $B=(1, 0)$, $C=(0, 1)$.

Σ_3. We employ the system S_7. Hence $A\cdot B$ is given by

* For, by the proof of the first theorem, $X\cdot B=B$ for every B, whence $X=I$.

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 - a_2 b_1, a_1 b_2 - a_2 b_2).$$

Hence 1 and 2 hold. Since $(1, 0)(z_1, z_2) = (z_1, z_2)$, 4 holds. To show that 3 fails, we note that $A \cdot B = A$ requires

$$b_1(a_1 - a_2) = a_1, \quad b_2(a_1 - a_2) = a_2,$$

so that b_1 and b_2 are not independent of a_1 and a_2.

Σ_4. We employ the system S_6. Hence $A \cdot B$ is given by

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1 - a_1 b_2, a_2 b_1 - a_2 b_2).$$

Hence 1 and 2 hold. Also 3 holds for $I = (1, 0)$. But 4 fails since

$$(a_1, a_2)(1, 1) = (0, 0).$$

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