ON THE ENVELOPE OF THE AXES OF A SYSTEM OF CONICS
PASSING THROUGH THREE FIXED POINTS*

BY

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In a recent number of the Annals of Mathematics † I have shown that the envelope of the asymptotes of a system of conics passing through three fixed points consists of two three-cusped hypocycloids, touching the three straight lines that join the three fixed points in pairs. I propose now to show that the envelope of the axes of the same system of conics consists of two three-cusped hypocycloids touching three concurrent straight lines.

The foci of a conic may be regarded as four of the vertices of a complete four-side circumscribing the conic, the other two vertices being the circular points at infinity; then the straight line at infinity is one diagonal line of this four-side, and the axes are the other two diagonal lines.

The coordinates of the circular points at infinity are \((1, -e^{\alpha}, -e^{-\beta})\) and \((1, -e^{-\alpha}, -e^{\beta})\); let us denote these points for the present by \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\).

Let the equation of the conic be

\[ U = \lambda_1 y z + \lambda_2 x z + \lambda_3 x y = 0, \]

the three fixed points through which the conic is to pass being the vertices of the triangle of reference; and put

\[ U_1 = \lambda_1 y z_1 + \lambda_2 z_1 x_1 + \lambda_3 x_1 y_1; \quad U_2 = \lambda_1 y z_2 + \lambda_2 z_2 x_2 + \lambda_3 x_2 y_2; \]

\[ U_1' = x_1 \frac{\partial U}{\partial x} + y_1 \frac{\partial U}{\partial y} + z_1 \frac{\partial U}{\partial z}; \quad U_2' = x_2 \frac{\partial U}{\partial x} + y_2 \frac{\partial U}{\partial y} + z_2 \frac{\partial U}{\partial z}. \]

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The equations of the tangents from the circular points at infinity are
\[ U_1'^2 = 4UU_1 \]
and
\[ U_2'^2 = 4UU_2; \]
and the foci are the intersections of these two pairs of straight lines.
The equation \( U_2U_1'^2 = U_1U_2'^2 \) evidently represents a third pair of straight lines passing through the foci, and must therefore represent the axes.
Now the condition for similarity may be expressed in the form *
\[ \sum (\lambda_i^2 \sin^2 A - 2\lambda_2 \lambda_3 \sin B \sin C) = t^2(\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2, \]
where \( t \) is the tangent of the angle between the asymptotes; or,
\[ \sum (\lambda_i^2 - 2\lambda_2 \lambda_3 \cos A) = s^2(\lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C)^2, \]
that is,
\[ U_1U_2 = s^2P^2, \]
where \( s \) is the secant of the angle between the asymptotes, and
\[ P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C. \]
Hence the equations of the axes may be expressed in the form
\[ U_1U'_2 = sPU_1', \quad \text{or} \quad U_2U'_1 = sPU'_2; \]
and
\[ U_1U'_2 = -sPU'_1', \quad \text{or} \quad U_2U'_1 = -sPU'_2. \]
Using the first of these equations, we may write the tangential coordinates of the corresponding axis in the form
\[ u = U_1(\lambda_2x_2 + \lambda_3y_2) - sP(\lambda_2z_1 + \lambda_3y_1), \]
\[ v = U_1(\lambda_3x_2 + \lambda_1z_2) - sP(\lambda_3x_1 + \lambda_1z_1), \]
\[ w = U_1(\lambda_1x_2 + \lambda_2z_2) - sP(\lambda_1y_1 + \lambda_2x_1). \]
Noticing that
\[ \lambda_1(y_2z_1 + y_2x_1) + \lambda_2(z_2x_1 + z_2y_1) + \lambda_3(x_2y_1 + x_3y_1) = -2P, \]
we have, on multiplying these equations first by \( x_1, y_1, z_1 \) and adding, and then by \( x_2, y_2, z_2 \), and adding,
\[ V = x_1u + y_1v + z_1w = -2PU_1 - 2sPU_1, \]
\[ W = x_2u + y_2v + z_2w = 2U_1U_2 + 2sP^2. \]
*See the paper entitled, *On some curves, etc.*, referred to above.
Hence, taking account of the relation \( U_1 U_2 = s^2 P^2 \), we find,
\[
V = \frac{\sqrt{W}}{\sqrt{2}(s+1)}, \quad U_1 = -\frac{V \sqrt{s}}{\sqrt{2}(s+1) W}.
\]

Now writing the coordinates in the form
\[
u = (z_2 U_1 - z_1 s P) \lambda_2 + (y_2 U_1 - y_1 s P) \lambda_3,
\]
\[
v = (z_2 U_1 - z_1 s P) \lambda_1 + (x_2 U_1 - x_1 s P) \lambda_3,
\]
\[
w = (y_2 U_1 - y_1 s P) \lambda_1 + (x_2 U_1 - x_1 s P) \lambda_2,
\]
and using the equation
\[
P = \lambda_1 \cos A + \lambda_2 \cos B + \lambda_3 \cos C,
\]
we have, on eliminating \( \lambda_1, \lambda_2, \lambda_3 \) the relation
\[
\begin{vmatrix}
u & 0 & z_2 U_1 - z_1 s P & y_2 U_1 - y_1 s P \\
v & z_2 U_1 - z_1 s P & 0 & x_2 U_1 - x_1 s P \\
w & y_2 U_1 - y_1 s P & x_2 U_1 - x_1 s P & 0 \\
1/(s+1) & \cos A & \cos B & \cos C
\end{vmatrix} = 0.
\]

On substituting the values of \( P \) and \( U_1 \) found above, and reducing by means of the relations
\[
x_2 V + x_1 W = 2(u - v \cos C - w \cos B), \text{ etc.,}
\]
we finally obtain the equation of the envelope in the form
\[
\begin{vmatrix}
u & 0 & u \cos B + v \cos A - w & w \cos A + u \cos C - v \\
v & u \cos B + v \cos A - w & 0 & v \cos C + w \cos B - u \\
w & w \cos A + u \cos C - v & v \cos C + w \cos B - u & 0 \\
1/(s+1) & \cos A & \cos B & \cos C
\end{vmatrix} = 0,
\]
or
\[
\sum [u(v \cos C + w \cos B - u)\{u \cos(B-C) - v \cos B - w \cos C\}] - \frac{2}{s+1}(v \cos C + w \cos B - u)(w \cos A + u \cos C - v)(u \cos B + v \cos A - w) = 0.
\]

It may be shown that this curve has the straight line at infinity for a double tangent, the circular points at infinity being the points of contact.
It must therefore be of the fourth order and have three cusps; and hence for all values of \( s \) (except \( s = -1 \)) it is a three-cusped hypocycloid.

It may easily be shown that it always touches the perpendicular bisectors of the sides of the triangle of reference; in the special case, \( s = -1 \), the curve degenerates into the points at infinity on these three lines.

The two axes envelope the same curve only in the case of the equilateral hyperbola, for which \( s = \infty \).

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