

THE APPROXIMATE DETERMINATION OF THE FORM OF MACLAURIN'S SPHEROID*

BY

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Preface.

Spherical harmonics render the approximate determination of the figure of a rotating mass of liquid a very simple problem. If ρ be the density, e the ellipticity, and ω the angular velocity of the spheroid, the solution is

$$\frac{\omega^2}{2\pi\rho} = \frac{8}{15} e.$$

This result is only correct as far as the first power of the ellipticity, but M. POINCARÉ has recently shown † how harmonic analysis may be so used as to give results which shall be correct as far as squares of small quantities; and I have myself used his method for the determination of the stability of the pear-shaped figure of equilibrium. ‡

Both these papers involved the use of ellipsoidal harmonic analysis, and it would be rather tiresome for a reader to extract the method from the complex analysis in which it is embedded. It therefore seems worth while to treat the well-worn subject of MACLAURIN'S spheroid as an example of the method in question. It will appear below that it would have been possible to obtain a more accurate result than that stated above, even if the rigorous solution of the problem had been beyond the powers of the mathematician.

My own personal reason for undertaking this task was that I desired a sort of collateral verification of the very complicated analysis needed in the case of my previous investigation.

§ 1. *Method of defining the spheroid.*

Let a sphere S be described concentrically with the spheroid, and let it be sufficiently large to enclose the whole of the spheroid. I call R the region

* Presented to the Society December 29, 1902. Received for publication December 13, 1902.

† Philosophical Transactions of the Royal Society, vol. 193 A (1902), pp. 333-373.

‡ *Ib.*, vol. 200 A, pp. 251-314.

between the sphere S and the spheroid; and suppose the density of the liquid in S to be $+\rho$, and that in R to be $-\rho$.

If S_i^s denotes any surface spherical harmonic of colatitude θ and longitude ϕ , it is usual to define the corresponding deformation of a sphere of radius a by the equation $r = a(1 + eS_i^s)$. But in the present investigation it will be found that there is a great saving of labor by defining it by the equation

$$r^3 = a^3(1 + 3eS_i^s).$$

The two forms give identical results as far as the first power of the ellipticity e , but not so when we are to consider the squares of small quantities.

In general I define S_i^s by one of the two alternative forms $P_i^s(\mu) \frac{\cos s\phi}{\sin s\phi}$, where $\mu = \cos \theta$. But in the case of the second zonal harmonic ($s = 0, i = 2$) it is convenient to write

$$S_2 = \frac{1}{3} - \mu^2.$$

The fourth zonal harmonic will occur explicitly below, and in accordance with the general definition to be adopted we have

$$S_4 = \frac{35}{8}\mu^4 - \frac{15}{4}\mu^2 + \frac{3}{8}.$$

The angular velocity is to be denoted by ω , and the colatitude θ or $\cos^{-1}\mu$ is measured from the axis of rotation.

We must now assume a general form for the equation to the spheroid, and shall subsequently determine the several ellipticities so that the surface may be a figure of equilibrium.

The radius of S being denoted by a , we may write the equation to the surface of the spheroid in the form

$$r^3 = a^3 [1 - 3c + 3eS_2 + 3fS_4 + 3 \sum f_i^s S_i^s].$$

In this expression e is the ellipticity corresponding to the second zonal harmonic, and it represents that term which exists alone in the ordinary approximate solution. Then I suppose that f and f_i^s are quantities of the order e^2 , and that there are f_i^s corresponding to all possible harmonics excepting the second and fourth zonal ones. Thus all the f_i^s are of order e^2 , excepting f_2 and f_4 which are zero. Lastly c is an arbitrary constant, and is only subject to the condition that it is greater than the greatest positive value of $eS_2 + fS_4 + \sum f_i^s S_i^s$. This condition ensures that S shall envelope the whole spheroid.

It is now convenient to replace the radius vector r by a new variable τ , defined by

$$(1) \quad \tau = \frac{a^3 - r^3}{3a^3}.$$

Thus the equation to the spheroid may be written

$$\tau = c - eS_2 - fS_4 - \sum f'_i S'_i.$$

The problem will be solved by making the energy of the system stationary. It will therefore be necessary to determine the energy lost in the concentration of the spheroid from a condition of infinite dispersion. This will involve the use of the formula for the gravity of S , and since the whole region R is inside S , we only require the formula for internal gravity.

If we were to continue the developments from this point all the formulæ would involve the constant c . But since it is merely needed for defining a sphere of reference of arbitrary size, it cannot finally appear in the formula for the energy. It is useless to encumber the analysis by the introduction of a constant which must disappear in the end, and it is legitimate and much shorter to treat c as zero from the first. It is however easier to maintain a clear conception of the processes if we continue to discuss the problem as though c were not zero, and as if S enveloped the whole spheroid. With this explanation we may write the equation to the spheroid in the form

$$(2) \quad \tau = -eS_2 - fS_4 - \sum f'_i S'_i.$$

§ 2. *The lost energy of the system.*

If the negative density in R were transported along conical tubes emanating from the center of S , it might be deposited as surface density on S ; I refer to such a condensation as $-C$. I do not, however, suppose the condensation actually effected, but I imagine the surface of S to be coated with equal and opposite condensations $+C$ and $-C$.

The system of masses forming the spheroid may then be considered as being as follows:

Density $+\rho$ throughout S , say $+S$.

Negative condensation on S , say $-C$.

Positive condensation on S and negative volume density $-\rho$ throughout R . This last forms a double system of zero mass, say D , and $D = C - R$.

The lost energy of the system clearly involves the lost energy of each of these three with itself, and the mutual lost energy of the three taken two and two together. Thus the lost energy may be written symbolically

$$\frac{1}{2}SS + \frac{1}{2}CC + \frac{1}{2}DD - SC + SD - CD.$$

Since D is $C - R$, the last three terms are equivalent to

$$-SC + (S - C)(C - R) = -SR + CR - CC.$$

Thus the gravitational lost energy is

$$\frac{1}{2}SS - SR + CR - \frac{1}{2}CC + \frac{1}{2}DD.$$

The lost energy of the system, as rendered statical by the imposition of a rotation potential, is clearly $\frac{1}{2}A\omega^2$, where A is the moment of inertia of the spheroid about the axis of rotation.

If A_s denotes the moment of inertia of the sphere S , and A_r the moment of inertia of the region R considered as being filled with positive density $+\rho$, we clearly have

$$A = A_s - A_r.$$

Thus if E denotes the lost energy of the system as rendered statical by the imposition of a rotation potential

$$(3) \quad E = \frac{1}{2}SS - SR + CR - \frac{1}{2}CC + \frac{1}{2}DD + \frac{1}{2}\omega^2(A_s - A_r).$$

§ 3. The energy $\frac{1}{2}SS - SR + CR - \frac{1}{2}CC$.

It is in the first place necessary to obtain certain preliminary analytical and numerical results.

If we write $d\sigma$ for $d\mu d\phi$, it is clear that an element dv of volume is given by

$$dv = a^3 d\tau d\sigma = \frac{3M}{4\pi\rho} d\tau d\sigma,$$

where M is the mass of the sphere S .

When we integrate throughout the region R the limits of τ are $-eS_2 - fS_4 - \sum f'_i S'_i$ to zero.

I now define certain integrals, viz:

$$\phi_i^s = \frac{3}{4\pi} \int (S'_i)^2 d\sigma, \quad \omega_i^s = \frac{3}{4\pi} \int (S_2)^2 S'_i d\sigma.$$

It is well known that

$$\int_{-1}^{+1} [P_i^s(\mu)]^2 d\mu = \frac{2}{2i+1} \frac{(i+s)!}{(i-s)!},$$

and since in every case, excepting that of S_2 , $S'_i = P_i^s(\mu) \frac{\cos s\phi}{\sin s\phi}$, we have

$$\phi_i^s = \frac{3}{2i+1} \frac{(i+s)!}{(i-s)!}.$$

Since $S_2 = -\frac{2}{3}P_2(\mu)$, the value of ϕ_2 is derivable from the same general formula. Hence we have

$$(4) \quad \phi_0 = 3, \quad \phi_2 = \frac{4}{15}, \quad \phi_4 = \frac{1}{3}.$$

Since S_i' involves either $\cos s\phi$ or $\sin s\phi$, ω_i' vanishes unless $s = 0$; hence we need only consider ω_i .

The function $(S_2)^2$ may be expanded in terms of zonal harmonics. Assume then

$$(S_2)^2 = \sum \eta_i S_i.$$

Multiplying both sides by $3S_i/4\pi$, and integrating throughout angular space, we find

$$\omega_i = \eta_i \phi_i, \quad \text{and} \quad (S_2)^2 = \sum \frac{\omega_i}{\phi_i} S_i.$$

But, by actual substitution,

$$(S_2)^2 = \frac{4}{45} S_0 - \frac{4}{21} S_2 + \frac{8}{35} S_4.$$

Therefore

$$(4) \quad \frac{\omega_0}{\phi_0} = \frac{4}{45}, \quad \frac{\omega_2}{\phi_2} = -\frac{4}{21}, \quad \frac{\omega_4}{\phi_4} = \frac{8}{35},$$

and all the higher ω 's vanish.

It will be noticed that $\omega_0 = \phi_2$, hence we have $\omega_0/\phi_0 = \frac{1}{3}\phi_2$. Also

$$(4) \quad \omega_0 = \frac{4}{15}, \quad \omega_2 = -\frac{16}{3^2 \cdot 5 \cdot 7}, \quad \omega_4 = \frac{8}{3 \cdot 5 \cdot 7}.$$

The equation to the spheroid possesses a certain property which permits us to effect a great saving in the subsequent work. If k be any arbitrary number it is clear that to the order of squares of small quantities we may write the equation to the spheroid in the form

$$\tau = -(e + ke^3)S_2 - fS_4 - \sum f_i' S_i',$$

since we have only imported a new term of the order e^3 . Now when k is zero the energy will be found to involve terms in $e^2, e^3, e^4, e^2f, f^2, (f_i')^2$. If then we write $e + ke^3$ for e , as is clearly permissible, the term in e^2 will give rise to a term in $2ke^4$. Hence it follows that the term in e^4 in the energy is really indeterminate, and that the retention of it would give rise to a fictitious accuracy. It is therefore permissible to omit the term in e^4 , while retaining other terms of the fourth order. Again the moment of inertia will involve terms in e, e^2, e^3, ef , and the same argument shows that the retention of the term in e^3 would give rise to a fictitious accuracy.*

It is now necessary to determine the volume of the region R ; it is

$$\alpha^3 \iint d\tau d\sigma = -\alpha^3 \int [eS_2 + fS_4 + \sum f_i' S_i'] d\sigma = 0.$$

* If the coefficient of the terms in e^2 and e were zero, the term in e^4 would not be fictitious. This was the case in discussing the pear-shaped figure of equilibrium.

From this it follows that the mass of S is equal to that of the spheroid, and therefore M is the mass of the spheroid. It is this result which makes the choice of τ as independent variable so convenient.

We are now in a position to determine the several contributions to the lost energy.

The lost energy of the sphere, denoted by $\frac{1}{2}SS$, is known to be $3M^2/5a$. This is a constant and may be omitted as being of no further interest. The internal potential of S is given by

$$V = \frac{2}{3}\pi\rho(3a^2 - r^2).$$

But $r^3 = a^3(1 - 3\tau)$, and $r^2 = a^2(1 - 2\tau - \tau^2 \dots)$. Therefore as far as squares of small quantities,

$$V = \frac{M}{a} + \frac{M}{a}(\tau + \frac{1}{2}\tau^2).$$

Since the volume of R is zero the first term of V contributes nothing to the lost energy SR , and the second term of V will give the whole. Therefore to the fourth order

$$\begin{aligned} SR &= \frac{3M^2}{4\pi\rho a} \iint (\tau + \frac{1}{2}\tau^2) d\tau d\sigma \\ &= \frac{3M^2}{8\pi\rho a} \int [e^2(S_2)^2 + 2efS_2S_4 + 2\sum ef_i^2 S_2S_4^i + f^2(S_4)^2 + 2\sum ff_i^2 S_4S_4^i \\ &\quad + (\sum f_i^2 S_4^i)^2 - \frac{1}{3}e^3(S_2)^3 - e^2f(S_2)^2S_4 - \sum e^2f_i^2(S_2)^2S_4^i] d\sigma \\ &= \frac{M^2}{2a} [e^2\phi_2 + f^2\phi_4 + \sum (f_i^2)\phi_4^i - \frac{1}{3}e^3\omega_2 - e^2f\omega_4]. \end{aligned}$$

Thus on rearranging the terms we have

$$(5) \quad \frac{1}{2}SS - SR = \frac{M^2}{a} [-\frac{1}{2}e^2\phi_2 + \frac{1}{3}e^3\omega_2 + \frac{1}{2}e^2f\omega_4 - \frac{1}{2}f^2\phi_4 - \frac{1}{2}\sum (f_i^2)\phi_4^i].$$

We have next to consider the terms depending on the condensation C . Since $dv/d\tau = a^3d\sigma$ the amount of matter in the region R , if filled with density $+\rho$, which stands on an element of unit area is

$$\rho a \int d\tau = -\rho a(eS_2 + fS_4 + \sum f_i^2 S_4^i).$$

This expression gives the surface density of the condensation $+C$, and it is expressed in surface harmonics.

Now by the usual formula of spherical harmonic analysis the internal potential of surface density $\rho a\epsilon S_i^i$ is

$$\frac{4\pi\rho}{2i+1} \frac{r^i}{a^{i-2}} S_i^i = \frac{3M}{(2i+1)a} \frac{r^i}{a^i} S_i^i.$$

As far as the first power of τ ,

$$r^i = a^i(1 - i\tau),$$

but in the case $i = 2$, as far as squares of small quantities,

$$r^2 = a^2(1 - 2\tau - \tau^2).$$

Hence it follows that the internal potential, say V_c , of the condensation + C is given by

$$(6) \quad V_c = -\frac{3M}{a} \left[\frac{1}{5}e(1 - 2\tau - \tau^2) S_2 + \frac{1}{9}f(1 - 4\tau) S_4 + \sum \frac{(1 - i\tau)}{2i+1} f_i^i S_i^i \right].$$

On multiplying this by $3M\bar{d}\tau d\sigma/4\pi$ and integrating throughout R we shall obtain the lost energy CR .

Thus

$$\begin{aligned} CR &= \frac{3M}{4\pi} \iint V_c \bar{d}\tau d\sigma \\ &= -\frac{9M^2}{4\pi a} \iint \left[\frac{1}{5}e S_2 + \frac{1}{9}f S_4 + \sum \frac{1}{2i+1} f_i^i S_i^i \right. \\ &\quad \left. - \tau \left(\frac{2}{5}e S_2 + \frac{4}{9}f S_4 + \sum \frac{i}{2i+1} f_i^i S_i^i \right) - \frac{1}{5}\tau^2 e S_2 \right] d\tau d\sigma \\ &= \frac{9M^2}{4\pi a} \int \left\{ \left[\frac{1}{5}e S_2 + \frac{1}{9}f S_4 + \sum \frac{1}{2i+1} f_i^i S_i^i \right] [e S_2 + f S_4 + \sum f_i^i S_i^i] \right. \\ &\quad \left. + \frac{1}{2} \left[\frac{2}{5}e S_2 + \frac{4}{9}f S_4 + \sum \frac{i}{2i+1} f_i^i S_i^i \right] [e^2 (S_2)^2 + 2ef S_2 S_4 \right. \right. \\ &\quad \left. \left. + 2 \sum ef_i^i S_2 S_i^i] - \frac{1}{3.5} e^3 (S_2)^3 \cdot e S_2 \right\} d\sigma. \end{aligned}$$

The term in e^4 may be now omitted in accordance with the principle explained above, and therefore

$$(7) \quad \begin{aligned} CR &= \frac{3M^2}{a} \left\{ \frac{1}{5}e^2 \phi_2 + \frac{1}{9}f^2 \phi_4 + \sum \frac{1}{2i+1} (f_i^i)^2 \phi_i^i \right. \\ &\quad \left. + \frac{1}{2} \left[\frac{2}{5}e^3 \omega_2 + \frac{4}{9}e^2 f \omega_4 + \frac{4}{9}e^2 f \omega_i \right] \right\} \\ &= \frac{3M^2}{a} \left[\frac{1}{5}e^2 \phi_2 + \frac{1}{5}e^3 \omega_2 + \frac{2}{3} \frac{8}{9}e^2 f \omega_4 + \frac{1}{9}f^2 \phi_4 + \sum \frac{1}{2i+1} (f_i^i)^2 \phi_i^i \right] \end{aligned}$$

In order to find $\frac{1}{2}CC$ we have only to deal with surface density. Then the value of V_c at the surface is given by (6) with $\tau = 0$; therefore

$$\frac{1}{2}V_c = -\frac{3M}{2a} \left[\frac{1}{5}eS_2 + \frac{1}{9}fS_4 + \sum \frac{1}{2i+1} f'_i S'_i \right].$$

An element of mass of the surface density $+C$ is

$$-\frac{3M}{4\pi} [eS_2 + fS_4 + \sum f'_i S'_i] d\sigma.$$

Multiplying these two together and integrating, we find

$$\frac{1}{2}CC = \frac{3M^2}{2a} \left[\frac{1}{5}e^2\phi_2 + \frac{1}{9}f^2\phi_4 + \sum \frac{1}{2i+1} (f'_i)^2 \phi'_i \right].$$

And subtracting this from (7)

$$(8) \quad CR - \frac{1}{2}CC = \frac{3M^2}{a} \left[\frac{1}{10}e^2\phi_2 + \frac{1}{5}e^3\omega_2 + \frac{2}{5}e^2f\omega_4 + \frac{1}{8}f^2\phi_4 + \frac{1}{2} \sum \frac{1}{2i+1} (f'_i)^2 \phi'_i \right].$$

Again adding this to (5) we have

$$(9) \quad \frac{1}{2}SS - SR + CR - \frac{1}{2}CC = \frac{M^2}{a} \left[-\frac{1}{5}e^2\phi_2 + \frac{2}{3}e^3\omega_2 + \frac{7}{3}e^2f\omega_4 - \frac{1}{3}f^2\phi_4 - \sum \frac{i-1}{2i+1} (f'_i)^2 \phi'_i \right].$$

It remains to determine the value of the term $\frac{1}{2}DD$, and for this end we must investigate the theory of double layers, according to the ingenious method devised by M. POINCARÉ.

§ 4. *Double layers.**

Let a closed surface S be intersected at every point by a member of a family of curves, and let α be the angle between the curve and the outward normal at any point. At every point of S measure along the curve an infinitesimal arc τ , and let τ be a function of the two coördinates which determine position on S . The extremities of these arcs define a second surface S' , and every element of area $d\sigma$ of S has its corresponding element $d\sigma'$ on S' . Suppose that S is coated with surface density δ , and that S' is coated with surface density $-\delta'$, where $\delta d\sigma = \delta' d\sigma'$. The system SS' may then be called a double layer, and its total mass is zero. We are to discuss the potential of such a system.

Let U_+ and U_- be the external and internal potentials of density δ on S , and U_0 their common value at a point P of S . At P take a system of rec-

* This section is contained in my paper on the *Stability of the Pear-shaped Figure of Equilibrium*, Philosophical Transactions of the Royal Society, vol. 200A (1903), pp. 251-314 but is reproduced here in order that the present investigation may be complete in itself.

tangular axes, n being along the outward normal, and s and t mutually at right angles in the tangent plane.

In the neighborhood of P

$$U_+ = U_0 + n \frac{dU_+}{dn} + s \frac{dU_+}{ds} + t \frac{dU_+}{dt} \dots,$$

$$U_- = U_0 + n \frac{dU_-}{dn} + s \frac{dU_-}{ds} + t \frac{dU_-}{dt} \dots.$$

In the first of these n is necessarily positive, in the second negative.

Now $dU_+/ds = dU_-/ds = dU/ds$; and the like holds for the differentials with respect to t .

Also by POISSON'S equation,

$$\frac{dU_-}{dn} - \frac{dU_+}{dn} = 4\pi\delta.$$

Let PP' be one of the family of curves whereby the double layer is defined, and let P' lie on S' , so that PP' is τ . By the definition of α the normal elevation of S' above S is $\tau \cos \alpha$.

Let v, v' be the potentials of the double layer at P and at P' .

The potential of S' at P' differs infinitely little in magnitude, but is of the opposite sign from that of S at P ; it is therefore $-U_0$. The point P' lies on the positive side of S at a point whose coördinates may be taken to be

$$n = \tau \cos \alpha, \quad s = \tau \sin \alpha, \quad t = 0.$$

Therefore the potential of S at P' is

$$U_0 + \tau \cos \alpha \frac{dU_+}{dn} + \tau \sin \alpha \frac{dU_+}{ds}.$$

Therefore

$$v' = \tau \cos \alpha \frac{dU_+}{dn} + \tau \sin \alpha \frac{dU_+}{ds}.$$

Again the potential of S at P is U_0 , and since P lies on the negative side of S' and has coördinates relatively to the n, s, t axes at P' given by

$$n = -\tau \cos \alpha, \quad s = -\tau \sin \alpha, \quad t = 0;$$

since further the density on S' is negative, we have

$$v = \tau \cos \alpha \frac{dU_-}{dn} + \tau \sin \alpha \frac{dU_-}{ds}.$$

Therefore

$$v - v' = \tau \cos \alpha \left[\frac{dU_-}{dn} - \frac{dU_+}{dn} \right] = 4\pi\tau\delta \cos \alpha.$$

The differential with respect to n of the potential of S falls abruptly by $4\pi\delta$ as we cross S normally from the negative to the positive side; and the differential of the potential of S' rises abruptly by the same amount as we pass on across S' . It follows that dv/dn on the inside of S is continuous with its value on the outside of S' .

The surface S to which this theorem is to be applied is a slightly deformed sphere, and the curves are radii drawn from the center of the sphere which is deformed. The radii are normal to the sphere, and where they meet S the angle α will be proportional to the deformation whereby S is derived from the sphere. It follows that $\cos \alpha$ will only differ from unity by a term proportional to the square of the deformation, and as it is only necessary to retain terms of the order of the first power of the deformation, we may treat $\cos \alpha$ as unity.

We thus have the result

$$v - v' = 4\pi\tau\delta.$$

Suppose the curve PP' produced both ways, and that M_0, M_1 are two points on it, either both on the same side or on opposite sides of the double layer.

Let M_0M_1 be equal to ζ , let ζ be measured in the same direction as n , and let ζ be a small quantity whose first power is to be retained in the results.

Let v_0, v_1 be the potential of the double layer at M_0 and M_1 respectively.

When ζ does not cut the layer we have

$$v_0 - v_1 = -\zeta \frac{dv}{dn},$$

and when it does cut the layer

$$v_0 - v_1 = 4\pi\tau\delta - \zeta \frac{dv}{dn}.$$

In the application which I shall make of this result the surface S' will actually be inside S . Then v_0 will denote the potential at any point not lying in the infinitely small space between S and S' , and v_1 is the potential at a point more towards the inside of the sphere by a distance ζ ; δ is the surface density on the external surface S and τ is measured inwards. If then we still choose to measure n outwards, as I shall do, our formula becomes

$$v_0 - v_1 - \zeta \frac{dv}{dn} = 4\pi\tau\delta \text{ or } 0,$$

according as ζ does or does not cut the double layer.

It may be well to remark that v being proportional to $\tau\delta$, $\zeta dv/dn$ is small compared with $4\pi\tau\delta$. It is also important to notice that the term $4\pi\tau\delta$ is independent of the form of the surface, and that dv/dn will be the same to the first

order of small quantities for a slightly deformed sphere as for the sphere itself.

We have now to apply these results to our problem.

The position of a point in the region R may be defined by the distance measured inwards from the sphere S along one of the radii orthogonal to S . The surface of the spheroid as defined in this way is given by ϵ , a function of θ and ϕ . Any point on a radius may then be defined by $s\epsilon$, where s is a proper fraction. If s is the same at every point the surface s is a deformed sphere; $s = 1$ gives the spheroid and $s = 0$ the sphere S .

If $d\sigma$ is an element of area of S , the corresponding element on the surface s will be $(1 - \lambda\epsilon s)d\sigma$. The value of λ will be determined hereafter, and it is only necessary to remark that it is positive because the areas must decrease as we travel inwards.

Let s and $s + ds$ be two adjacent surfaces; then the mass of negative density enclosed between them in the tube of which

$$(1 - \lambda\epsilon s)d\sigma \quad \text{and} \quad [1 - \lambda\epsilon(s + ds)]d\sigma$$

are the ends is $-\rho\epsilon(1 - \lambda\epsilon s)d\sigma ds$. If this element of mass be regarded as surface density on s , that surface density is clearly $-\rho\epsilon ds$. If the same element of mass were carried along the orthogonal tube and deposited as surface density on S that surface density would be $-\rho\epsilon(1 - \lambda\epsilon s)$. The sum for all values of s of all such transportals would constitute the condensation $-C$ already considered.

The double system D consists of the volume density $-\rho$ in R , and the positive condensation $+C$ on S , the total mass being zero.

Let z , a proper fraction, define a surface between the sphere S and the spheroid. Consider one of the orthogonal lines, and let V_0 be the potential of D at the point P where the line leaves S and V_z the potential at the point Q where it cuts z . Then I require to find $V_0 - V_z$.

Since s denotes a surface intermediate between S and the spheroid, $dsd(V_0 - V_z)/ds$ is the excess of the potential at P above that at Q of surface density $-\rho\epsilon ds$ on s and surface density $+\rho\epsilon(1 - \lambda\epsilon s)ds$ on S . Such a system is a double layer, but there is a finite distance between the two surfaces, and the form of $d(V_0 - V_z)/ds$ will clearly be different according as z is greater or less than s .

The arc ϵs may be equally divided by a large number of surfaces, and we may take t to define any one of them. Now we may clothe each intermediate surface t with equal and opposite surface densities $\pm\rho\epsilon[1 - \lambda\epsilon(s - t)]dt$.

The density

$$+\rho\epsilon[1 - \lambda\epsilon(s - t)]dt \quad \text{on} \quad t,$$

together with

$$- \rho\epsilon [1 - \lambda\epsilon(s - t - dt)] dt \quad \text{on} \quad t + dt,$$

constitutes an infinitesimal double layer; and since the positive density on each t surface may be coupled with the negative density on the next interior surface, the finite double layer may be built up from a number of infinitesimal double layers. Hence $dt dt d^2(V_0 - V_z)/ds dt$ is the excess of the potential at P above that at Q of an infinitesimal double layer of thickness ϵdt , and with surface density $\rho\epsilon [1 - \lambda\epsilon(s - t)] dt$ on its exterior surface.

We may now apply the result $v_0 - v_1 - \zeta dv/dn = 4\pi\delta\tau$ or 0 , according as ζ does or does not cut the double layer, and it is clear that

$$\frac{d^2}{ds dt} \left[V_0 - V_z - \epsilon z \frac{dV}{dn} \right] = 4\pi\rho\epsilon^2 [1 - \lambda\epsilon(s - t)] \quad \text{or} \quad 0,$$

according as z is greater or less than t .

In the next place, we must integrate this from $t = s$ to $t = 0$, and the result will have two forms.

First, suppose $z > s$; then for all the values of t , $z > t$, and the first alternative holds good. Therefore

$$\frac{d}{ds} \left(V_0 - V_z - \epsilon z \frac{dV}{dn} \right) = 4\pi\rho\epsilon^2 [s - \frac{1}{2}\lambda\epsilon s^2].$$

Secondly, suppose $z < s$; then from $t = s$ to $t = z$, $z < t$ and the second alternative holds, while from $t = z$ to $t = 0$, $z > t$ and the first holds. Therefore

$$\frac{d}{ds} \left(V_0 - V_z - \epsilon z \frac{dV}{dn} \right) = 4\pi\rho\epsilon^2 [z - \lambda\epsilon(sz - \frac{1}{2}z^2)].$$

We have now to integrate again from $s = 1$ to $s = 0$.

From $s = 1$ to $s = z$, $z < s$ and the second form is applicable; from $s = z$ to $s = 0$, $z > s$ and the first form applies.

Therefore

$$\begin{aligned} V_0 - V_z - \epsilon z \frac{dV}{dn} &= 4\pi\rho\epsilon^2 \int_z^1 [z - \lambda\epsilon(sz - \frac{1}{2}z^2)] ds + 4\pi\rho\epsilon^2 \int_0^z [s - \frac{1}{2}\lambda\epsilon s^2] ds \\ &= 4\pi\rho\epsilon^2 \{ z(1 - z) - \lambda\epsilon [\frac{1}{2}z(1 - z^2) - \frac{1}{2}z^2(1 - z)] \\ &\quad + \frac{1}{2}z^2 - \frac{1}{6}\lambda\epsilon z^3 \} \\ &= 2\pi\rho\epsilon^2 \{ 2z - z^2 - \lambda\epsilon(z - z^2 + \frac{1}{3}z^3) \}. \end{aligned}$$

Finally, we have to multiply $-\frac{1}{2}(V_0 - V_z)$ by an element of negative mass at the point defined by z and integrate throughout R . The physical meaning of this integral will be considered subsequently.

We have already seen that such an element of mass is given by

$$-\rho dv = -\rho\epsilon(1 - \lambda\epsilon z) d\sigma dz$$

and the limits of integration are $z = 1$ to $z = 0$.

Therefore

$$\begin{aligned} \frac{1}{2} \int (V_0 - V_s) \rho dv &= \pi\rho^2 \int \int \epsilon^3 (1 - \lambda\epsilon z) \{2z - z^2 - \lambda\epsilon(z - z^2 + \frac{1}{3}z^3)\} dz d\sigma \\ &\quad + \frac{1}{2}\rho \int \int \epsilon^2 z (1 - \lambda\epsilon z) \frac{dV}{dn} dz d\sigma. \end{aligned}$$

In this expression we neglect terms of the order ϵ^5 and note that $\epsilon^3 z^2 dV/dn$ is of that order.

Thus

$$\begin{aligned} \frac{1}{2} \int (V_0 - V_s) \rho dv &= \pi\rho^2 \int \int \epsilon^3 [2z - z^2 - \lambda\epsilon(z + z^2 - \frac{2}{3}z^3)] dz d\sigma \\ &\quad + \frac{1}{2}\rho \int \int \epsilon^2 z \frac{dV}{dn} dz d\sigma \quad (z = 1 \text{ to } 0), \\ &= \frac{2}{3}\pi\rho^2 \int \epsilon^3 (1 - \lambda\epsilon) d\sigma + \frac{1}{2}\rho \int \epsilon^2 \frac{dV}{dn} d\sigma, \end{aligned}$$

the integrals being taken all over the surface of the sphere.

We must now consider the meaning of the integral

$$\frac{1}{2} \int (V_0 - V_s) \rho dv.$$

Let P be a point on S and Q a point on R on the same orthogonal line.

Let $-U$ be the potential at Q of the density $-\rho$ throughout R , and $-U_0$ its value at P .

Let δ be the surface density of the positive concentration on S , W its potential at Q , and W_0 its value at P .

The lost energy of the double system consisting of $-\rho$ throughout R , and δ on S is

$$\frac{1}{2} \int U \rho dv + \frac{1}{2} \int W_0 \delta d\sigma - \frac{1}{2} \int U_0 \delta d\sigma - \frac{1}{2} \int W \rho dv.$$

This is equal to

$$\frac{1}{2} \int (U - W) \rho dv - \frac{1}{2} \int (U_0 - W_0) \delta d\sigma.$$

Consider the triple integral

$$\int \int \int (U_0 - W_0) \rho dv.$$

Here $dv = \epsilon(1 - \lambda\epsilon s) d\sigma ds$, $U_0 - W_0$ is not a function of s , and the limits of s are 1 to zero. Therefore

$$\int \int \int (U_0 - W_0) \rho dv = \int \int (U_0 - W_0) \left[\int_0^1 \epsilon(1 - \lambda\epsilon s) \rho ds \right] d\sigma.$$

But

$$\int_0^1 \epsilon(1 - \lambda\epsilon s) \rho ds$$

is equal to δ the surface density of concentration. Therefore

$$\int \int \int [U_0 - W_0] \delta d\sigma = \int \int \int (U_0 - W_0) \rho dv.$$

We may now revert to the Gaussian notation with single integral sign, and we see that the lost energy of the system is

$$\frac{1}{2} \int [(W_0 - U_0) - (W - U)] \rho dv.$$

But $W - U$ is the potential of the double system at Q , and is therefore V_s ; and $W_0 - U_0$ is the potential of the double system at P , and is therefore V_0 .

Accordingly the lost energy

$$\frac{1}{2} DD = \frac{1}{2} \int (V_0 - V_s) \rho dv = \frac{2}{3} \pi \rho^2 \int (\epsilon^3 - \lambda\epsilon^4) d\sigma + \frac{1}{2} \rho \int \epsilon^2 \frac{dV}{dn} d\sigma.$$

§ 5. The energy $\frac{1}{2} DD$.

The element of surface of the sphere was written $d\sigma$ in § 4, but in order to accord with the notation used elsewhere we must now write it $a^2 d\sigma$.

The first term in $\frac{1}{2} DD$ was

$$\frac{2}{3} \pi \rho^2 \int (\epsilon^3 - \lambda\epsilon^4) d\sigma,$$

and when the notation for the element of surface is changed we may write it

$$\frac{1}{2} \frac{M\rho}{a} \int (\epsilon^3 - \lambda\epsilon^4) d\sigma.$$

In this expression ϵ is the length measured along a radius from the sphere S to the spheroid. We have denoted the outward normal by n , and therefore to the second order of small quantities,

$$-dn = -dr = a(1 + 2\tau)d\tau.$$

The distance measured inward from the sphere to the point defined by τ in the region R is, to the first order of small quantities, $-n = a\tau$. Again

$$(10) \quad \begin{aligned} \epsilon &= - \int dn = a \int (1 + 2\tau) d\tau \\ &= -a [eS_2 + fS_4 + \sum f_i^2 S_i^2 - e^2(S_2)^2]. \end{aligned}$$

Since $-n$ is what was denoted ϵ s in the general investigation of § 4, we have

$$\begin{aligned} dv &= -a^2(1 + \lambda n)dn d\sigma \\ &= a^3(1 - \lambda a\tau)(1 + 2\tau)d\tau d\sigma \\ &= a^3[1 + (2 - \lambda a)\tau]d\tau d\sigma. \end{aligned}$$

But since $dv = a^3 d\tau d\sigma$, we have $\lambda = 2/a$.

Therefore

$$\begin{aligned} \epsilon^3 &= -a^3 [e^3(S_2)^3 + 3e^2f(S_2)^2 S_4 + 3 \sum e^2 f_i^2 (S_2)^2 S_i^2 - 3e^4(S_2)^4], \\ \lambda \epsilon^4 &= 2a^3 e^4 (S_2)^4. \end{aligned}$$

Whence

$$\epsilon^3 - \lambda \epsilon^4 = -a^3 [e^3(S_2)^3 + 3e^2f(S_2)^2 S_4 + 3 \sum e^2 f_i^2 (S_2)^2 S_i^2 - e^4(S_2)^4].$$

This must be multiplied by $\frac{1}{2}M\rho/a$ and integrated throughout angular space. Then since, as before, we may omit the term in e^4 , this contribution to the energy becomes

$$(11) \quad \frac{M^2}{a} \left[-\frac{1}{2}e^3\omega_2 - \frac{3}{2}e^2f\omega_4 \right].$$

The second term

$$\frac{1}{4}\rho \int \epsilon^2 \frac{dV}{dn} d\sigma$$

in $\frac{1}{2}DD$ remains for consideration. It will clearly be a term in e^4 and as such might be omitted, but it is of some interest to see how it may be computed, and I therefore proceed.

In order to evaluate dV/dn it suffices to imagine the volume density $-\rho$ in the region R concentrated on a surface bisecting the space between S and the

spheroid. We may then treat the system D as an infinitesimal double layer of thickness $\frac{1}{2}\epsilon$ and with density $+C$ or $-\rho a(eS_2 + fS_4 + \Sigma f_i S_i')$ on its outer surface. In the present instance it suffices to consider only the leading terms in the density and thickness. Hence by (10) the product denoted $\tau\delta$ in the general investigation becomes

$$\left(-\frac{1}{2}aeS_2\right)\left(-\rho aeS_2\right) = \frac{1}{2}\rho a^2 e^2 \left[\frac{1}{3}\phi_2 S_0 + \frac{\omega_2}{\phi_2} S_2 + \frac{\omega_4}{\phi_4} S_4 \right].$$

We thus have $\tau\delta$ expanded in surface harmonics.

Now consider two functions

$$V_e = \sum A_i \frac{ia^{i-1}}{r^{i+1}} S_i', \quad \text{for space external to } S,$$

$$V_i = -\sum A_i \frac{(i+1)r^i}{a^{i+2}} S_i', \quad \text{for internal space.}$$

They are solid harmonics and as such satisfy LAPLACE'S equation throughout space. Hence they are the external and internal potentials of a distribution of matter on S , but since they are not continuous, while their differentials are continuous, that matter constitutes a double layer.

At the surface $r = a$,

$$V_e - V_i = \sum (2i+1) \frac{A_i}{a^2} S_i'.$$

But this must be equal to $4\pi\tau\delta$. Hence

$$2\pi\rho a^2 e^2 \left[\frac{1}{3}\phi_2 S_0 + \frac{\omega_2}{\phi_2} S_2 + \frac{\omega_4}{\phi_4} S_4 \right] = \frac{A_0}{a^2} S_0 + \frac{5A_2}{a^2} S_2 + \frac{9A_4}{a^2} S_4.$$

Therefore

$$A_0 = \frac{3}{2}Ma e^2 \cdot \frac{1}{3}\phi_2; \quad A_2 = \frac{3}{2}Ma e^2 \cdot \frac{1}{5}\frac{\omega_2}{\phi_2}; \quad A_4 = \frac{3}{2}Ma e^2 \cdot \frac{1}{9}\frac{\omega_4}{\phi_4}.$$

Now

$$\frac{dV}{dn} = \frac{dV_e}{dr} = \frac{dV_i}{dr} \quad (r = a).$$

$$= -\sum i(i+1) \frac{A_i}{a^3} S_i'$$

$$= -\frac{3}{2} \frac{Me^2}{a^2} \left[\frac{6}{5} \frac{\omega_2}{\phi_2} S_2 + \frac{2.0}{9} \frac{\omega_4}{\phi_4} S_4 \right].$$

Then since

$$\rho\epsilon^2 a^2 = \rho a^4 e^2 (S_2)^2 = \frac{3Ma}{4\pi} e^2 \left[\frac{1}{3}\phi_2 + \frac{\omega_2}{\phi_2} S_2 + \frac{\omega_4}{\phi_4} S_4 \right],$$

we have (on writing $a^2 d\sigma$ for the $d\sigma$ of the general investigation of § 4)

$$(12) \quad \begin{aligned} \frac{1}{4}\rho a^2 \int \epsilon^2 \frac{dV}{dn} d\sigma &= -\frac{9M^2}{32\pi a} e^4 \int \left[\frac{6}{5} \left(\frac{\omega_2}{\phi_2} \right)^2 (S_2)^2 + \frac{2.0}{9} \left(\frac{\omega_4}{\phi_4} \right)^2 (S_4)^2 \right] d\sigma \\ &= -\frac{3}{8} \frac{M^2}{a} e^4 \left[\frac{6}{5} \frac{(\omega_2)^2}{\phi_2} + \frac{2.0}{9} \frac{(\omega_4)^2}{\phi_4} \right]. \end{aligned}$$

This term involves e^4 , and may be omitted, but, as stated above, it seemed worth while to show how it may be computed.

Then adding (11) to (9) and omitting (12), we have for the whole gravitational lost energy

$$(13) \quad \frac{M^2}{a} \left[-\frac{1}{5}e^2\phi_2 + \frac{4}{15}e^3\omega_2 + \frac{1}{15}e^2f\omega_4 - \frac{1}{3}f^2\phi_4 - \sum \frac{i-1}{2i+1} (f'_i)^2\phi'_i \right].$$

§ 6. *Moment of inertia.*

Since ω^2 is of order e , the moment of inertia must be determined to the cubes of small quantities.

We have, to the square of τ ,

$$x^2 + y^2 = r^2 \sin^2 \theta = \frac{3M}{4\pi\rho a} \left(\frac{2}{3} + S_2 \right) (1 - 2\tau - \tau^2).$$

The region R is to be considered as filled with density $+\rho$, and the element of mass is $3M d\tau d\sigma / 4\pi$.

Hence the moment of inertia of the region R is

$$\begin{aligned} C_r &= \left(\frac{3M}{4\pi} \right)^2 \frac{1}{\rho a} \int \int \left(\frac{2}{3} + S_2 \right) (1 - 2\tau - \tau^2) d\tau d\sigma, \\ &= - \left(\frac{3M}{4\pi} \right)^2 \frac{1}{\rho a} \int \int \left[\frac{2}{3} + S_2 \right] [eS_2 + fS_4 + \sum f'_i S'_i + e^2 (S_2)^2 \\ &\quad + 2efS_2 S_4 + 2 \sum ef'_i S_2 S'_i - \frac{1}{3}e^3 (S_2)^3] d\tau d\sigma. \end{aligned}$$

The term in e^3 may be omitted for the reason assigned above, and we have

$$\begin{aligned} C_r &= - \left(\frac{3M}{4\pi} \right)^2 \frac{1}{\rho a} \int [e(S_2)^2 + e^2 (S_2)^3 + 2ef (S_2)^2 S_4 \\ &\quad + 2 \sum ef'_i (S_2)^2 S'_i + \frac{2}{3}e^2 (S_2)^2] d\sigma \\ &= - \frac{3M^2}{4\pi\rho a} [(e + \frac{2}{3}e^2)\phi_2 + e^2\omega_2 + 2ef\omega_4]. \end{aligned}$$

The moment of inertia of S is $\frac{2}{3}Ma^2$. Therefore

$$C_s = \frac{3M^2}{4\pi\rho a} \left(\frac{2}{3} \right).$$

Then

$$C = C_s - C_r = \frac{3M^2}{4\pi\rho a} \left[\frac{2}{5} + (e + \frac{2}{3}e^2)\phi_2 + e^2\omega_2 + 2ef\omega_4 \right].$$

Whence

$$(14) \quad \frac{1}{2}C\omega^2 = \frac{M^2}{a} \frac{\omega^2}{4\pi\rho} \left[\frac{3}{5} + (\frac{3}{2}e + e^2)\phi_2 + \frac{3}{2}e^2\omega_2 + 3ef\omega_4 \right].$$

§ 7. *Solution of the problem.*

From (13) and (14) we have

$$E \div \frac{M^2}{a} = -\frac{1}{5}e^2\phi_2 + \frac{4}{15}e^3\omega_2 + \frac{1}{15}\frac{3}{2}e^2f\omega_4 - \frac{1}{3}f^2\phi_4 - \sum \frac{i-1}{2i+1} (f_i^*)^2 \phi_i^* + \frac{\omega^2}{4\pi\rho} \left[\frac{3}{5} + (\frac{3}{2}e + e^2)\phi_2 + \frac{3}{2}e^2\omega_2 + 3ef\omega_4 \right].$$

The conditions for a figure of equilibrium are

$$\frac{dE}{de} = 0, \quad \frac{dE}{df} = 0, \quad \frac{dE}{df_i^*} = 0,$$

with ω^2 constant.

The last of these gives at once $f_i^* = 0$, and the two others give

$$(15) \quad -\frac{2}{5}e\phi_2 + \frac{4}{5}e^2\omega_2 + \frac{2}{15}\frac{3}{2}ef\omega_4 + \frac{\omega^2}{4\pi\rho} \left[(\frac{3}{2} + 2e)\phi_2 + 3e\omega_2 + 3f\omega_4 \right] = 0,$$

$$\frac{1}{15}\frac{3}{2}e^2\omega_4 - \frac{2}{3}f\phi_4 + \frac{\omega^2}{4\pi\rho} \cdot 3e\omega_4 = 0.$$

From the former of these as a first approximation

$$\frac{\omega^2}{4\pi\rho} = \frac{4}{15}e.$$

The second equation then gives

$$\frac{2}{3}f\phi_4 = \frac{1}{15}\frac{3}{2}e^2\omega_4 + \frac{4}{5}e^2\omega_4 = \frac{5}{8}e^2\omega_4.$$

Therefore

$$(16) \quad f = \frac{5}{2}e^2 \frac{\omega_4}{\phi_4} = \frac{4}{7}e^2.$$

We see that f is of order e^2 , as was assumed to be the case. Now it is of no use to retain terms in e^3 in the first of (15), because we have neglected the term in e^4 in E . Thus the first of (15) reduces to

$$-\frac{2}{5}e\phi_2 + \frac{4}{5}e^2\omega_2 + \frac{\omega^2}{4\pi\rho} \left[(\frac{3}{2} + 2e)\phi_2 + 3e\omega_2 \right] = 0.$$

Whence

$$\frac{\omega^2}{4\pi\rho} \left[1 + \frac{4}{3}e + 2e \frac{\omega_2}{\phi_2} \right] = \frac{4}{15}e \left(1 - 2e \frac{\omega_2}{\phi_2} \right);$$

or

$$(17) \quad \begin{aligned} \frac{\omega^2}{4\pi\rho} &= \frac{4}{15}e \left(1 - \frac{4}{3}e - 4e \frac{\omega_2}{\phi_2} \right) = \frac{4}{15}e \left[1 - \frac{4}{3}e + \frac{1}{2} \frac{e}{1} e \right], \\ &= \frac{4}{15}e \left(1 - \frac{4}{3}e \right). \end{aligned}$$

It follows from (16) and (17) that the equation to the surface of equilibrium is

$$(18) \quad r^3 = a^3 \left[1 + 3eS_2 + \frac{1}{7}e^2S_4 \right],$$

and

$$\frac{\omega^2}{2\pi\rho} = \frac{8}{15}e \left(1 - \frac{4}{3}e \right).$$

It remains to verify that the solution (18) is correct.

The equation to an ellipsoid of revolution, whose equatorial and polar radii are a_1 and $a_1(1 - e_1)$, is

$$r^2 = \frac{a_1^2}{\frac{\cos^2 \theta}{(1 - e_1)^2} + \sin^2 \theta}.$$

If we determine r^3 by developing this expression as far as e_1^2 , it appears that the equation to the ellipsoid may be written in the form

$$r^3 = a_1^3(1 - e_1) \left[1 + 3 \left(e_1 + \frac{5}{14}e_1^2 \right) S_2 + \frac{1}{7}e_1^2 S_4 \right].$$

Since the volume of this ellipsoid is $\frac{4}{3}\pi a_1^3(1 - e_1)$ and that of our spheroid of equilibrium was $\frac{4}{3}\pi a^3$ it follows that

$$a^3 = a_1^3(1 - e_1).$$

If then we write

$$e = e_1 + \frac{5}{14}e_1^2, \quad e^2 = e_1^2,$$

the equation to the ellipsoid of revolution becomes

$$r^3 = a^3 \left[1 + 3eS_2 + \frac{1}{7}e^2S_4 \right],$$

and this is the form determined above in (18).

If the eccentricity of the ellipsoid be denoted by $\sin \gamma$, we have

$$\cos^2 \gamma = (1 - e_1)^2 = \left(1 - e + \frac{5}{14}e^2 \right)^2 = 1 - 2e + \frac{1}{7}e^2.$$

Therefore

$$\cos \gamma = 1 - e + \frac{5}{14}e^2,$$

$$\sin^2 \gamma = 2e \left(1 - \frac{6}{7}e \right);$$

whence

$$\cos \gamma \sin^2 \gamma = 2e \left(1 - \frac{1}{7} 3e\right),$$

$$\cos \gamma \sin^4 \gamma = 4e^2.$$

Now it is known that the rigorous solution for the angular velocity of MACLAURIN'S ellipsoid may be written in the form

$$\frac{\omega^2}{4\pi\rho} = \cos \gamma \sum_1^{\infty} \frac{(2n-1)! \sin^{2n} \gamma}{(2n+1)(2n+3)[(n-1)!]^2 2^{2n-3}}.$$

Taking the first two terms

$$\begin{aligned} \frac{\omega^2}{4\pi\rho} &= \frac{2}{15} \sin^2 \gamma \cos \gamma + \frac{3}{5} \sin^4 \gamma \cos \gamma \\ &= \frac{2}{15} e \left(1 - \frac{4}{7} e\right). \end{aligned}$$

This agrees with the second of (18), and the solution is found to be correct as far as squares of small quantities.

The approximate solution found above is insufficient to enable us to discuss the stability of the Maclaurin figure, but it may be well to indicate how a more accurate approximate solution would give the required result.

Let us suppose that V is the gravitational lost energy corresponding to the equilibrium ellipticities e, f, f'_i and the angular velocity ω .

Then we are to regard V and the moment of inertia C as functions of e, f, f'_i , so that e, f, f'_i are the solutions of the equations

$$(19) \quad \frac{\partial V}{\partial e} + \frac{1}{2} \omega^2 \frac{\partial C}{\partial e} = 0, \quad \frac{\partial V}{\partial f} + \frac{1}{2} \omega^2 \frac{\partial C}{\partial f} = 0, \quad \frac{\partial V}{\partial f'_i} + \frac{1}{2} \omega^2 \frac{\partial C}{\partial f'_i} = 0.$$

Now suppose that the ellipticities corresponding to any neighboring form are $e + \delta e, f + \delta f, f'_i + \delta f'_i$, and that $V + \delta V, C + \delta C$ are the corresponding values of V and C . Conceive also that these variations from the equilibrium figure are made subject to constancy of the angular momentum, then it is clear that U , the sum of the potential and kinetic energies, is given by

$$U = - (V + \delta V) + \frac{1}{2} \frac{(C\omega)^2}{C + \delta C}.$$

If we omit constant terms and only retain squares of small quantities we have

$$U = - \delta V - \frac{1}{2} \omega^2 \delta C + \frac{1}{2} \omega^2 \frac{(\delta C)^2}{C}.$$

For the sake of brevity I will only retain the two ellipticities e, f , since this will suffice to indicate the general law. Now if V and C be expanded by

TAYLOR's theorem in powers of δe , δf , and if we bear in mind the conditions (19) for the figure of equilibrium, it is easy to show that

$$\delta V + \frac{1}{2}\omega^2\delta C = \frac{1}{2}\left(\frac{\partial^2 V}{\partial e^2} + \frac{1}{2}\omega^2\frac{\partial^2 C}{\partial e^2}\right)(\delta e)^2 + \left(\frac{\partial^2 V}{\partial e\partial f} + \frac{1}{2}\omega^2\frac{\partial^2 C}{\partial e\partial f}\right)\delta e\delta f \\ + \frac{1}{2}\left(\frac{\partial^2 V}{\partial f^2} + \frac{1}{2}\omega^2\frac{\partial^2 C}{\partial f^2}\right)(\delta f)^2,$$

$$\frac{1}{2}\omega^2\frac{(\delta C)^2}{C} = \frac{1}{2}\frac{\omega^2}{C}\left(\frac{\partial C}{\partial e}\right)^2(\delta e)^2 + \frac{\omega^2}{C}\frac{\partial C}{\partial e}\frac{\partial C}{\partial f}\delta e\delta f + \frac{1}{2}\frac{\omega^2}{C}\left(\frac{\partial C}{\partial f}\right)^2(\delta f)^2.$$

Hence

$$U = -\frac{1}{2}\left\{\frac{\partial^2 V}{\partial e^2} + \frac{1}{2}\omega^2\left[\frac{\partial^2 C}{\partial e^2} - \frac{2}{C}\left(\frac{\partial C}{\partial e}\right)^2\right]\right\}(\delta e)^2 \\ - \left\{\frac{\partial^2 V}{\partial e\partial f} + \frac{1}{2}\omega^2\left[\frac{\partial^2 C}{\partial e\partial f} - \frac{2}{C}\frac{\partial C}{\partial e}\frac{\partial C}{\partial f}\right]\right\}\delta e\delta f - \text{etc.}$$

This is a quadratic function of δe , δf and the vanishing of the Hessian would give the condition for the change from stability to instability. However, we need just one term more than that found above to obtain even a first approximation to the limit of the stability of the Maclaurin spheroid. It is accordingly useless to pursue the topic further.

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