COMPLETE SETS OF POSTULATES FOR THE THEORY OF REAL QUANTITIES*

BY

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The following paper presents two complete sets of postulates, or primitive propositions, either of which may be used as a basis for the ordinary algebra of real quantities. †

* Presented to the Society, under a slightly different title, December 29, 1902. Received for publication February 2, 1903. [For an abstract of an unpublished paper on the same subject, presented to the Society by the writer on April 26, 1902, see Bulletin of the American Mathematical Society, vol. 8 (1901–02), p. 371.]

† Cf. D. HILBERT, *Ueber den Zahlbegriff*, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. 8 (1900), pp. 180–184.—The axioms for real numbers enumerated by HILBERT in this note include many redundancies, and no attempt is made to prove the uniqueness of the system which they define. (Cf. Theorem II below.) The main interest of the paper lies in his new Axiom der Vollständigkeit, which, together with the axiom of Archimedes, replaces the usual axiom of continuity. The other axioms are given also in his *Grundlagen der Geometrie* (1899), § 13.

Complete sets of postulates for particular classes of real quantities (positive integral, all integral, positive real, positive rational) can be found in the following papers:

G. PEANO, *Sul concetto di numero*, Rivista di Matematica, vol. 1 (1891), pp. 87–102, 256–267; Formulaire de Mathématiques, vol. 3 (1901), pp. 39–44.—Here only the positive integers, or the positive integers with zero, are considered. An account of these postulates is given in the Bulletin of the American Mathematical Society, vol. 9 (1902–03), pp. 41–46. They were first published in a short Latin monograph by PEANO, entitled *Arithmetices principia nova methodo exposita*, Turin (1889).

A. PADOA: 1) *Essai d'une théorie algébrique des nombres entiers, précedé d'une introduction logique à une théorie déductive quelconque*, Bibliothèque du congrès international de philosophie, Paris, 1900, vol. 3 (published in 1901), pp. 309–365; 2) *Numeri interi relativi*, Rivista di Matematica, vol. 7 (1901), pp. 73–84; 3) *Un nouveau système irréductible de postulats pour l'algèbre*, Compte rendu du deuxième congrès international des mathématiciens, Paris, 1900 (published in 1902), pp. 249–256.—The second of these papers is an ideographical translation of the first; the third reproduces the principal results.

E. V. HUNTINGTON: 1) *A complete set of postulates for the theory of absolute continuous magnitude*; 2) *Complete sets of postulates for the theories of positive integral and positive rational numbers*; Transactions, vol. 3 (1902), pp. 264–279, 280–284.—The first of these papers will be cited below under the title: *Magnitudes*. [In the fifth line line of postulate 5, p. 267, the reader is requested to change "one and only one element A" to: *at least one element A* —a typographical correction which does not involve any further alteration in the paper.]

Among the other works which may be consulted in this connection are:

H. B. FINE, *The number system of algebra treated theoretically and historically*, Boston (1891).

O. STOLZ und J. A. GMEINER, *Theoretische Arithmetik*, Leipzig (1901–02). Two sections of this work have now appeared.

G. PEANO, *Aritmetica generale e algebra elementare*, Turin (pp. vii + 144, 1902).

The fundamental concept involved is that of an *assemblage*, or *class*, M, in which *a relation*, $⧀$, and *a rule of combination*, $\oplus$, and (in § 2) *another rule of combination*, $\odot$, are defined. Any object which belongs to the assemblage is called an *element.*

Thus, if a and b are elements of the assemblage, $a ⧀ b$ indicates that a stands in the given relation to b; while $a \oplus b$ and $a \odot b$ denote the objects uniquely determined by a and b (in their given order), according to the first and second rules of combination respectively.

(It will appear that the symbols

$$⧀, \oplus, \text{ and } \odot$$

obey the same formal laws as the symbols

$$<, +, \text{ and}$$

of elementary arithmetic, and they may therefore be called by the same names: *less than*, *plus*, and *times.*)

The relation $⧀$ and the rules of combination $\oplus$ and $\odot$ are wholly undetermined, except for the imposition of certain postulates (§ 1 and § 2), which are shown to be *consistent* (Theorems I and I′), and *independent* (Theorems III and III′).

Any assemblage in which $⧀$ *and* $\oplus$ *are so defined as to satisfy the postulates* 1–10 *of* § 1, *or any assemblage in which* $⧀$, $\oplus$, *and* $\odot$ *are so defined as to satisfy the postulates* 1–14 *of* § 2, *shall be called a system of real quantities.* It is shown in Theorems II and II′ that all such systems are *equivalent;* in other words, the system of real quantities is uniquely determined by either of the two sets of postulates.

Each set of postulates is thus a *complete set.**

In the proofs of Theorems II and II′, use is made of certain theorems proved in the writer's paper on the postulates of magnitude.

In the proofs of Theorems I, III, and I′, III′, the existence is assumed of the ordinary system of real numbers — that is, the totality of finite and infinite decimal fractions (or the totality of CANTOR's *Fundamentalreihen* or DEDEKIND's *Schnitte*), built up from the system of positive integers in the usual way, by successive generalizations of the number concept.

The system of *real quantities* and the system of *real numbers* are of course mathematically equivalent concepts; the contrasted use of the terms in this paper is intended to correspond with HILBERT's distinction between the "axiomatic" and the "genetic" *methods of defining* that concept.

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* *Magnitudes*, p. 264.

§ 1. The first set of postulates for real quantities.

We consider in the first place an assemblage in which $\circledcirc$ and $\oplus$ are so defined that the following ten postulates are satisfied:

1. *If a and b belong to the assemblage, and $a \neq b$, then either $a \circledcirc b$ or $b \circledcirc a$.*

2. *The relation $\circledcirc$ is transitive; that is, if a, b, c belong to the assemblage, and $a \circledcirc b$ and $b \circledcirc c$, then $a \circledcirc c$.*

3. *The relation $c \circledcirc c$ is false for at least one element c.*

4. *If a and b belong to the assemblage, and $a \circledcirc b$, then there is an element x such that $a \circledcirc x$ and $x \circledcirc b$.*

5. *If S is any infinite sequence of elements (a_k) such that*

$$a_k \circledcirc a_{k+1}, \qquad a_k \circledcirc c \qquad\qquad (k = 1, 2, 3, \cdots),$$

(where c is some fixed element), then there is an element A having the following two properties:

> 1°) *$a_k \circledcirc A$ whenever a_k belongs to S;*
> 2°) *if $A' \circledcirc A$, then there is at least one element of S, say a_r, for which $A' \circledcirc a_r$.*

6. *If a, b and $b \oplus a$ belong to the assemblage, then $a \oplus b = b \oplus a$.*

7. *If a, b, c, $a \oplus b$, $b \oplus c$ and $a \oplus (b \oplus c)$ belong to the assemblage, then $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.*

8. *For every two elements a and b there is an element x such that $a \oplus x = b$.*

9. *If $x \circledcirc y$, then $a \oplus x \circledcirc a \oplus y$, whenever $a \oplus x$ and $a \oplus y$ belong to the assemblage.*

10. *The assemblage contains at least two distinct elements.**

It will be noticed that postulates 1–5 concern only the relation $\circledcirc$, and postulates 6–8 only the rule of combination $\oplus$.

Consistency of the postulates.

Theorem I.—*The postulates* 1–10 *are consistent with one another;* that is the postulates themselves, and therefore all their consequences, are free from contradiction.

* The necessity of adding this postulate was first pointed out to me by Mr. L. D. Ames. The addition and consequent modifications are made, May, 1903. It is to be noted that in *Magnitudes* the existence of at least one element in the assemblage might better have appeared (p. 267) as an explicit postulate than as a presupposition (p. 266, l. 6).

Proof.—To establish this theorem it is sufficient to prove the existence of some assemblage in which $\otimes$ and $\oplus$ are so defined that all the postulates are satisfied.*

One such assemblage (if we admit the axioms of positive integral numbers) is the system of all real numbers, built up from the system of positive integers by the "genetic" method with $\otimes$ and $\oplus$ defined as $<$ and $+$; another is the system of *positive* real numbers, with $\otimes$ and $\oplus$ defined as $<$ and $\times$.

Still another such system (if we admit the usual axioms of geometry of one dimension) is the system of points on a straight line through a fixed point O, with $\otimes$ defined as "on the left of," and $\oplus$ as the $+$ of vector analysis.

Deduction from the postulates.

The following propositions, 15–30, deduced from the fundamental propositions 1–10, will prepare the way for Theorem II.

In the first place, it follows from 6, 7, 8 that our assemblage is an *Abelian group with respect to* $\oplus$.† Hence:

15.‡ If a and b belong to the assemblage, then $a \oplus b$ does also.

16. The element x in 8 is uniquely determined by a and b. $(x = b - a.)$

17. There is one and only one zero-element, 0, such that $a \oplus 0 = a$ for every element a.

Next, from 3, with 6, 8, and 9, we have:

18. The relation $a \otimes a$ is false for every element a.

* The question of the consistency (compatibility, Widerspruchslosigkeit) of any set of postulates is emphasized by HILBERT as the most important problem which can be raised concerning them, since the proof of the consistency of the postulates "is at the same time the proof of the mathematical existence" of the concept which they define. The method of proof here used for Theorem I (which is the only method now known) would probably not be satisfactory to HILBERT; for he writes: "While the proof of the compatibility of the geometrical axioms may be made to depend upon the theorem of the compatibility of the arithmetical axioms, the proof of the compatibility of the arithmetical axioms requires on the contrary a direct method"—that is, an absolute method depending only on the consideration of the axioms themselves, and not involving the introduction of an auxiliary system whose "mathematical existence" might be open to doubt. He has expressed his conviction "that it must be possible to find such a direct proof for the compatibility of the arithmetical axioms, by means of a careful study and suitable modification of the known methods of reasoning in the theory of irrational numbers," and has formulated the problem as the second in his list of unsolved mathematical problems. See Göttinger Nachrichten, 1900, p. 264; reprinted in the Archiv der Mathematik und Physik, 3d ser., vol. 1 (1901), p. 54; English translation by Dr. Mary W. NEWSON, in the Bulletin of the American Mathematical Society, 2d ser., vol. 8 (1901–02), p. 447.

† See the writer's definition of an Abelian group, p. 27 of the present volume of the Transactions.

‡ The paragraph numbers 11–14 are reserved for use in § 2.

For, take x so that $x \oplus a = c$, where c is an element for which $c \olessthan c$ is false. Then if $a \olessthan a$ were true, we should have $x \oplus a \olessthan x \oplus a$, or $c \olessthan c$.

From 1, 2, and 18 we have then the important theorem :

19. *Of the three relations:* $a = b$, $a \olessthan b$, $b \olessthan a$, *of any two elements* a *and* b, *at least one must be true, and not more than one can be true.*

Definitions.—When $a \olessthan b$ (a "less than" b), we write also: $b \ogreaterthan a$ (b "greater than" a). Every element a, not 0, is called *positive* or *negative* according as $a \ogreaterthan 0$ or $a \olessthan 0$. Thus (by 17 and 19) all the elements are divided into three classes: the positive elements, the negative elements, and zero.

20. If b is positive, $a \oplus b \ogreaterthan a$; if b is negative, $a \oplus b \olessthan a$.
For, by 9 and 15, if $0 \olessthan b$, then $a \olessthan a \oplus b$; and if $b \olessthan 0$, then $a \oplus b \olessthan a$.

21. If a and b are positive, $a \oplus b$ is positive; if a and b are negative, $a \oplus b$ is negative. (By 20 and 2.)

22. If $a \ogreaterthan b$, there is a positive x such that $a = b \oplus x$; if $a \olessthan b$, there is a positive y such that $a \oplus y = b$.
For, take x and y by 8; then both will be positive, since, by 20, if x were negative, we should have $a \olessthan b$, and if y were negative, $a \ogreaterthan b$.

23. If the sequence S in 5 is composed of positive elements, the element A (called the upper limit of the sequence) is also positive. (By 2.)

24. If a is positive, there is a positive x such that $x \olessthan a$. (By 4.)

25. We can now state the important result that *the positive elements form a system of absolute continuous magnitudes with respect to* $\oplus$. For (by 10, 19, and 22) there is at least one positive element; and (by 21, 20, 7, 22, 23 with 6, and 24) the postulates 1–6 in the writer's paper on magnitudes are satisfied.*

26. *Definition.*—By 16 and 17 every element a determines uniquely an element $-a$, such that $a \oplus (-a) = 0$; this element $-a$ is called the *opposite* of a. By 21, if a is positive, $-a$ is negative; if a is negative, $-a$ is positive; and if a is 0, $-a$ is also 0. Any two elements a and b form a pair of opposite elements if $a \oplus b = 0$.

27. From $a \oplus (-c) = b$ follows $a = b \oplus c$, and conversely. (By 7 and 26.)

28. $(-a) \oplus (-b) = -(a \oplus b)$. (By 6 and 7.)

29. If $a \olessthan b$, then $-a \ogreaterthan -b$. (By 22, 27, and 20.)

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* *Magnitudes*, p. 267.

30. From 26 and 28 we see that *the negative elements will also form a system of absolute continuous magnitudes with respect to* $\oplus$ since the negative elements can be put into one-to-one correspondence with the positive elements, in such a way that if a and b correspond to $-a$ and $-b$, then $a \oplus b$ will correspond to $(-a) \oplus (-b)$.

Sufficiency of the postulates to define an assemblage.

THEOREM II.—*Any two assemblages, $M(\otimes, \oplus)$ and $M'(\otimes, \oplus)$, which satisfy the postulates* 1–10 *are equivalent;* that is, they can be brought into one-to-one correspondence in such a way that when a and b in M correspond to a' and b' in M', we shall have:

1°) $a' \otimes b'$ whenever $a \otimes b$; and

2°) $a \oplus b$ will correspond to $a' \oplus b'$.

Proof.—First make the positive elements of M correspond to the positive elements of M' as in Theorem II of the writer's paper on magnitudes.* Next, make the negative element of M correspond to the negative elements of M', by making a correspond to a' whenever $-a$ corresponds to $-a'$.

Finally, make the zeros of the two systems correspond to each other.

The relation 2°) then clearly holds when a and b are both positive, or both negative, or when either is 0.

To show that it holds also when one is positive and the other negative, let a be positive, b negative, and $a \oplus b = c$; and let a', b', c' be the elements of M' corresponding to a, b, c in M. Then if c is positive, we have $a = c \oplus (-b)$, where a, c, and $-b$ are all positive; whence $a' = c' \oplus (-b')$, or $a' \oplus b' = c'$. If c is negative, we have $b = c \oplus (-a)$, where b, c, and $-a$ are all negative; whence, $a' \oplus b' = c'$, as before. And if $c = 0$, we have $a = -b$, whence $a' = -b'$, or $a' \oplus b' = 0 = c'$.

Thus relation 2°) is always satisfied.

To prove relation 1°), notice that if $a \otimes b$ there is a positive x such that $a \oplus x = b$. Then $a' \oplus x' = b'$, where x' is also positive. Hence $a' \otimes b'$.

From Theorems I *and* II *we see that the postulates* 1–10 *define essentially a single assemblage*, which we may call *the system of real quantities with respect to* $\otimes$ *and* $\oplus$. The relation $\otimes$ may then be called the relation of (algebraically) *less than*, and the rule of combination $\oplus$, the *addition of real quantities.*

For example, each of the systems mentioned in the proof of Theorem I is a system of real quantities in the sense here defined.

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* *Magnitudes*, p. 277.

Independence of the postulates.

THEOREM III.—*The postulates* 1–10 *are independent;* that is, no one is a consequence of the other eight. This is shown by the following systems, each of which satisfies all the other postulates, but not the one for which it is numbered.*

(1) The system of all real numbers, with $\circledcirc$ so defined that $a \circledcirc b$ is true when and only when a and b are positive or zero, and $a < b$; and $a \oplus b$ defined as $a + b$.

(2) The system of all real numbers, with $a \circledcirc b$ defined as $a \neq b$, and $a \oplus b$ as $a + b$.

Here 2 is not satisfied, for from $a \circledcirc b$ and $b \circledcirc a$ it does not follow that $a \circledcirc a$. In 5, take $A = c$.

(3) The system of all real numbers, with $a \circledcirc b$ defined as $a \leqq b$, and $a \oplus b$ as $a + b$.

(4) The system of all integral numbers, with $\circledcirc$ defined as $<$, and $\oplus$ as $+$.

Here 5 is satisfied vacuously (to use a term of Professor MOORE's), since no infinite sequence S of the kind described can occur.

(5) The system of all rational numbers, with $\circledcirc$ defined as $<$, and $\oplus$ as $+$.

To show that 5 fails, consider any infinite sequence of increasing rational fractions used to define $\sqrt{2}$.

(6) The system of all positive real numbers, with $\circledcirc$ defined as $<$, and $\oplus$ defined by the relation $a \oplus b = b$.

(7) The system of all positive real numbers, with $\circledcirc$ defined as $<$, and $\oplus$ defined by the relation $a \oplus b = \sqrt{ab}$.

Here 7 fails, since $(2 \oplus 2) \oplus 8 = 4$, while $2 \oplus (2 \oplus 8) = \sqrt{8}$. In 8, take $x = b^2/a$.

(8) The system of all positive real numbers, with $\circledcirc$ defined as $<$, and $\oplus$ as $+$.

(9) The system of all real numbers, with $\circledcirc$ so defined that $a \circledcirc b$ is always true when $b \neq 0$ and false when $b = 0$; and $a \oplus b$ defined as $a + b$.

Here 2 holds, for since $b \circledcirc c$, c cannot be 0. In 3, take $c = 0$. In 4 and 5, x and A may be any elements not 0. To show that 9 fails, take $y = -a$.

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*On this method, used by PEANO and HILBERT, see especially PADOA, *loc. cit.*

(10) A system comprising only one element a, with $\otimes$ and $\oplus$ so defined that $a \otimes a$ is false, and $a \oplus a = a$.

Remarks on the further development of the theory.

In the present paper we have drawn only such deductions from the postulates 1–10 as were needed to establish Theorem II. The first step in the further development of the theory would be naturally the definition of multiplication. Any positive quantity being chosen as a "unit," and denoted by 1, every positive element can be expressed as the limit of an infinite sequence of rational fractions of the form $m1/n$, where m and n are positive integers.*

The product $a \cdot b$, or ab, of any two real quantities a and b is then defined as follows:

1°) when a and b are positive,

$$ab = \lim \left(\frac{mp}{nq} 1 \right),$$

where $\lim (m1/n) = a$ and $\lim (p1/q) = b$;

2°) when a or b is negative, $ab = -[(-a)b] = -[a(-b)]$;

3°) when a or b is zero, $a0 = 0b = 0$.

The commutative, associative and distributive laws for the multiplication of real quantities then follow readily from the corresponding properties of positive integers; and the quotient b/a, where $a \neq 0$, is then uniquely defined as that quantity x which satisfies the relation $ax = b$.

§ 2. The second set of postulates for real quantities.

We consider in the second place an assemblage in which $\otimes$, $\oplus$, and $\odot$ are so defined that the following fourteen postulates are satisfied:

1–10. *The same as postulates* 1–10 *in* § 1.

11. *If a and b belong to the assemblage, then $a \odot b$ does also.*

12. *First distributive law: $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$, whenever a, b, c, $a \odot b$, etc., all belong to the assemblage.*

13. *Second distributive law: $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$, whenever a, b, c, $a \oplus b$, etc., all belong to the assemblage.*

14. *If* 0 *is an element such that $c \oplus 0 = c$ for every element c, and if $0 \otimes a$ and $0 \otimes b$, then $0 \otimes a \odot b$, whenever $a \odot b$ belongs to the assemblage.*

Consistency of the postulates of the second set.

THEOREM I′.—*The postulates* 1–14 *are consistent with one another.*

For, one assemblage which satisfies them all is the system of all real numbers, with $\otimes = <$, $\oplus = +$, and $\odot = \times$. Another such assemblage is the system

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*Cf. 39, below.

of all positive real numbers, with $\otimes = <$, $\oplus = \times$, and $\odot$ defined as follows: $a \odot b = 2^{\alpha\beta}$, where α and β are real numbers such that $a = 2^{\alpha}$ and $b = 2^{\beta}$.

Deductions from the postulates of the second set.

As consequences of the postulates 1–14 we have in the first place all the propositions 15–30 in § 1, and further, the propositions 31–44, below. The object of the work is to establish the commutative and associative laws for the operation $\odot$, and thus prepare the way for Theorem II′.

31. If a and b are positive, $a \odot b$ will be positive. (By 14, with 11, 17, 19.)

32. For every element a, $a \odot 0 = 0 = 0 \odot a$.

For, $a \odot b = a \odot (b \oplus 0) = (a \odot b) \oplus (a \odot 0)$, by 11 and 12; hence $a \odot 0 = 0$, by 17. Similarly, $0 \odot a = 0$, using 13 in place of 12.

33. $a \odot (-b) = -(a \odot b)$ and $(-a) \odot b = -(a \odot b)$.

For $(a \odot - b) \oplus (a \odot b) = a \odot (-b \oplus b) = a \odot 0 = 0$, by 12, 26, and 32. Hence the first part of the theorem is true by 26. The second part follows in a similar way, if 13 is used in place of 12.

34. $(-a) \odot (-b) = a \odot b$. (By 33 and 26.)

35. If a and b are both positive, or both negative, $a \odot b$ is positive; if one is positive and the other negative, $a \odot b$ is negative. (By 31, 34, and 33.) Hence, if $a \odot b = 0$, then either $a = 0$ or $b = 0$.

36. If c is positive, and $x \otimes y$, then $c \odot x \otimes c \odot y$, and $x \odot c \otimes y \odot c$.

For, by 22 take a positive z so that $x \oplus z = y$; then $c \odot z$ is positive, by 31. Now $(c \odot x) \oplus (c \odot z) = c \odot (x \oplus z) = c \odot y$, by 12. Hence $c \odot x \otimes c \odot y$, by 20. The second part of the theorem follows in a similar way, if 13 is used in place of 12.

37. If a and b are positive, and $a \otimes a'$, $b \otimes b'$, then $a \odot b \otimes a' \odot b'$. (By 36.)

Since (by 25) the positive elements form a system of absolute continuous magnitudes, we may use the theorems concerning multiples and sub-multiples for such elements.* Hence:

* *Magnitudes*, 13-34.

38. If a and b are positive,

$$\left(\frac{m}{n}a\right)\odot\left(\frac{p}{q}b\right)=\frac{mp}{nq}(a\odot b).$$

For, by mathematical induction from 12 and 13 we have $a\odot(mb)=m(a\odot b)$ and $(ma)\odot b=m(a\odot b)$, whence, replacing b by b/m in the first equation, and a by a/m in the second, we obtain:

$$a\odot(b/m)=(a\odot b)/m \qquad \text{and} \qquad (a/m)\odot b=(a\odot b)/m.$$

From these four equations the theorem is readily inferred.

39. If c is any fixed positive element, any positive element a may be expressed in the form: *

$$a=\lim_{n=\infty}\left(\frac{m}{n}c\right),$$

where the positive integer m corresponding to any positive integer n is determined by the following relation:

$$\frac{m}{n}c \circledcirc\!\!\!\!\!\raisebox{-0.6ex}{=}\; a \circledcirc \frac{m+1}{n}c.$$

40. If a, b, c are positive, and

$$a=\lim\left(\frac{p}{n}c\right) \qquad \text{and} \qquad b=\lim\left(\frac{q}{n}c\right),$$

then

$$a\odot b=\lim\frac{pq}{nn}(c\odot c).$$

For, by 37 and 39 we have:

$$\left(\frac{p}{n}c\right)\odot\left(\frac{q}{n}c\right) \underline{\circledcirc}\; a\odot b \circledcirc \left(\frac{p+1}{n}c\right)\odot\left(\frac{q+1}{n}c\right),$$

whence, by 38,

$$\frac{pq}{nn}(c\odot c) \underline{\circledcirc}\; a\odot b \circledcirc \frac{pq}{nn}(c\odot c)\oplus\frac{1}{n}\left(\frac{p+q+1}{n}\right)(c\odot c),$$

in which the last term can be made as small as we please by increasing n, since p/n and q/n remain finite.

* A detailed elementary discussion of this familiar application of the theory of limits is given in the writer's dissertation, *Ueber die Grund-Operationen an absoluten und complexen Grössen*, Braunschweig (1901), chapters II and III.

In view of 40, the commutative and associative laws for $\odot$ follow at once from the corresponding properties of positive integers—first when all the elements are positive, and then, by 32, 33, and 34, when any of the elements concerned are negative or zero. Hence, in all cases:

41. $a \odot b = b \odot a$, and:

42. $(a \odot b) \odot c = a \odot (b \odot c)$.

From 38 we derive also the following theorem:

43. For every two elements a and b, provided $a \neq 0$, there is an element x such that $a \odot x = b$. $(x = b/a.)$

Proof.—If a and b are positive, and c is any fixed positive element, there is, by 38, a positive integer n_1 such that $a \odot (c/n) \otimes b$ for every positive integer $n > n_1$; and for every such n there is a positive integer m such that

$$a \odot \frac{m}{n} c \underline{\otimes} b \otimes a \odot \frac{m+1}{n} c.$$

The sequence $(m/n)c$ will then have a limit, x, which will be the required element.

If a [or b] is negative, take $x = -z$, where z is the positive element such that $(-a) \odot z = b$ [or $a \odot z = (-b)$]. If $b = 0$, take $x = 0$.

In view of 41, 42, and 43 we see that *the elements of our assemblage, excluding the element* 0, *form an Abelian group with respect to* $\odot$.* Hence (using also 32):

44. There is one and only one unit-element, 1, such that $a \odot 1 = a$ for every element a. In particular, $1 \odot 1 = 1$.

We notice also, in passing, that every assemblage which satisfies postulates 1–14 will form a *field* † *with respect to* $\oplus$ *and* $\odot$.

Sufficiency of the postulates to define an assemblage.

THEOREM II′.—*Any two assemblages,* $M(\otimes, \oplus, \odot)$ *and* $M'(\otimes, \oplus, \odot)$, *which satisfy the postulates* 1–14 *are equivalent;* that is, they can be brought into one-to-one correspondence in such a way that when a and b in M correspond to a' and b' in M' we shall have:

* Cf. 15–17, above.

† See the definitions of a field by L. E. DICKSON and by E. V. HUNTINGTON, p. 13 and p. 31 of the present volume of the Transactions.

1°) $a' \otimes b'$ whenever $a \otimes b$;

2°) $a \oplus b$ will correspond to $a' \oplus b'$; and

3°) $a \odot b$ will correspond to $a' \odot b'$.

Proof.—By § 1, Theorem II, place M and M' in one-to-one correspondence in such a way that relations 1°) and 2°) are satisfied, the elements $0, 1$ in M being made to correspond to the elements $0', 1'$ in M'. Then 3°) will also be satisfied.

For, consider first any two positive elements, a and b, in M, and the two corresponding positive elements, a' and b', in M'; and let $a = \lim (p/n) 1$ and $b = \lim (q/n) 1$. Then $a' = \lim (p/n) 1'$ and $b' = \lim (q/n) 1'$. Hence, by 40 and 44, $a \odot b = \lim (pq/nn) 1$ and $a' \odot b' = \lim (pq/nn) 1'$, which are corresponding elements.

When either a or b is negative or zero, 3°) is then clearly satisfied in view of 32, 33, and 34.

From Theorems I′ *and* II′ *we see that the postulates* 1–14 *define essentially a single assemblage*, which we may call *the system of real quantities with respect to* $\otimes$, $\oplus$, *and* $\odot$. The relation $\otimes$ may then be called *less than*, and the rules of combination $\oplus$ and $\odot$, *addition* and *multiplication* of real quantities.

The definition of the system of real quantities in § 1 is clearly consistent with that in § 2; in the first case multiplication is treated as a derived concept, in the second, as a fundamental concept.

Independence of the postulates of the second set.

THEOREM III′.—*The postulates* 1–14 *are independent.*

The systems [1]–[9] which we use to prove the independence of the first nine of these fourteen postulates are the same as the systems (1)–(9) above, with $\odot$ defined by the relation $a \odot b = ab$.

Here it should be especially noticed that the systems [6] and [7] satisfy postulates 12, 13, and 14 (the last vacuously); and that the systems [2], [3], and [9] satisfy postulate 14.

To prove the independence of postulate 10 we use a system [10] comprising only one element a, with $\otimes$, $\oplus$, and $\odot$ so defined that $a \otimes a$ is false, $a \oplus a = a$, and $a \odot a = a$.

To prove the independence of postulates 11–14, we use the system of all real numbers, with $\otimes = <$, $\oplus = +$, and $\odot$ defined as follows:

[11] $a \odot b = ab$ when a and b are rational; $a \odot b$ not in the assemblage when either a or b is irrational.

[12] $a \odot b = ab$ when $b \neq 0$; and $a \odot 0 = a$.

Here 12 fails, since we have, for example, $2 \odot (3 \oplus -3) = 2 \odot 0 = 2$, while $(2 \odot 3) \oplus (2 \odot -3) = 6 \oplus (-6) = 0$. To show that 13 is satisfied, consider the two cases: $c = 0$ and $c \neq 0$.

[13] $a \odot b = ab$ when $a \neq 0$; and $0 \odot b = b$.

Here 13 fails, since we have, for example, $(3 \oplus -3) \odot 2 = 0 \odot 2 = 2$, while $(3 \odot 2) \oplus (-3 \odot 2) = 6 \oplus -6 = 0$. To show that 12 is satisfied, consider the two cases $a = 0$ and $a \neq 0$.

[14] $a \odot b = -ab$.

Here 12 and 13 are clearly satisfied, while 14 is not.

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