ON THE REDUCIBILITY OF LINEAR GROUPS*

BY

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The object of this note is a two-fold generalization of Loewy's theorem proved in these Transactions, vol. 4, pp. 171-177. His theorem may be conveniently stated as follows: If \( R \) is the domain of all real numbers and \( C \) the domain of all complex numbers, any group of linear homogeneous transformations with coefficients in \( R \) which is irreducible in \( R \), but reducible in \( C \), can be transformed linearly into a decomposable group \( (G \overline{G}) \), where \( G \) and \( \overline{G} \) are two groups irreducible in \( C \), with coefficients not all in \( R \), such that the coefficients in every transformation of \( \overline{G} \) are the conjugate imaginaries of the corresponding coefficients for \( G \).

In seeking a generalization, we note that the domain \( C \) may be considered as derived from \( R \) by the adjunction of a root \( i \) of the quadratic equation \( x^2 + 1 = 0 \) belonging to and irreducible in \( R \). For the generalization, \( R \) is replaced by a general domain \( F \) (or field not having a modulus) and \( R(i) \) is replaced by the domain \( F(\rho_0) \) given by the extension of \( F \) by the adjunction of a root \( \rho_0 \) of an equation \( f(x) = 0 \) of degree \( r \) belonging to and irreducible in \( F \). The generalization will therefore be two-fold. Let the roots of \( f(x) = 0 \) be \( \rho_0, \rho_1, \ldots, \rho_{r-1} \). If \( G_{11} \) is a group of transformations with coefficients \( C_{\psi}(\rho_0) \) in the domain \( F(\rho_0) \), let \( G^{(s)}_{11} \) denote the group of transformations with the coefficients \( C_{\psi}(\rho_s) \); in particular, \( G^{(0)}_{11} = G_{11} \). The coefficients of \( G_{11}, G'_{11}, \ldots, G^{(r-1)}_{11} \) are thus conjugate with respect to \( F \). The generalized theorem is as follows:

Let \( G \) be a group of linear homogeneous transformations with coefficients in a domain \( F \), such that \( G \) is irreducible in \( F \) but is reducible in the domain \( F(\rho_0) \) given by the extension of \( F \) by the adjunction of a root \( \rho_0 \) of an equation belonging to and irreducible in \( F \) and having as its roots \( \rho_0, \rho_1, \ldots, \rho_{r-1} \). Then \( G \) can be transformed linearly into a decomposable group \(*\)

* Present to the Society at the Boston summer meeting, August 31-September 1, 1903. Received for publication April 27, 1903.

† When the irreducible equation is a normal equation, the groups \( G^{(s)}_{11} \) \( (s = 0, 1, \ldots, r - 1) \) are all irreducible in the same (normal) domain. Loewy's case furnishes an example.
where $G^{(i)}_{11}$ is a group irreducible in $F(\rho_i)$ with coefficients not all in $F'$, and $G_{11}, G'_{11}, \ldots, G^{(r-1)}_{11}$ are conjugate with respect to $F$.

The proof starts as in Loewy, § 1. The first variation * occurs at the bottom of p. 173; we now take $r$-fold decomposable matrices

$$H = \begin{bmatrix} G & 0 & \cdots & 0 \\ 0 & G' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G \end{bmatrix}, \quad Q = \begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & P' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P^{(r-1)} \end{bmatrix}.$$

Corresponding changes are to be made in the first two statements on p. 174. Thus, the diagonal groups in (6) are to be replaced by

$$G_{11}, G_{22}, G'_{11}, G'_{22}, G''_{11}, G''_{22}, \ldots, G^{(r-1)}_{11}, G^{(r-1)}_{22}.$$  

In place of the transformation † (7), we have

$$(7') \quad y_{jk} = \sum_{i=1}^{n} C_{ki}^{(i)} y_{ji}^* \quad (k = 1, \ldots, n; j = 0, \ldots, r-1),$$

where $C_{ki}^{(i)}$ is a rational function of $\rho_i$ with coefficients in $F$, and

$$(7'') \quad C_{ki}^{(i)} = 0 \quad (k = 1, \ldots, m; i = m + 1, \ldots, n; j = 0, \ldots, r-1).$$

Introduce two pairs each of $rn$ new variables defined by

$$(8') \quad y_{sk} = Y_{0k} + \rho_s Y_{1k} + \rho_s^2 Y_{2k} + \cdots + \rho_s^{r-1} Y_{r-1,k} \quad (s = 0, \ldots, r-1),$$

$$(8'') \quad y_{sk} = Y_{0k}^* + \rho_s Y_{1k}^* + \rho_s^2 Y_{2k}^* + \cdots + \rho_s^{r-1} Y_{r-1,k}^* \quad (k = 1, \ldots, n).$$

This may be done since the determinant

$$\Delta = \begin{vmatrix} 1 & \rho_0 & \rho_0^2 & \cdots & \rho_0^{n-1} \\ 1 & \rho_1 & \rho_1^2 & \cdots & \rho_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \rho_{r-1} & \rho_{r-1}^2 & \cdots & \rho_{r-1}^{n-1} \end{vmatrix} = \Pi (\rho_i - \rho_j) \neq 0.$$

* The statement on p. 173, lines 7–8, is apparently not used later; a proof follows readily from the main theorem under consideration.

† Loewy's notation is unwieldy even in his simple case. I write $y_{sk}, y_{sk}$ for his $y_s, z_s$. The transformed variables are marked * instead of being primed.
Solving (8') for fixed \( k \), while \( s = 0, \ldots, r - 1 \), we get

\[
\Delta Y_{rk} = \sum_{s=0}^{r-1} (-1)^s D_s y_{sk} \quad (i=0, 1, \ldots, r-1),
\]

where

\[
D_s = \begin{vmatrix}
1 & \rho_0 & \ldots & \rho_{0}^{s-1} & \rho_0^{s+1} & \ldots & \rho_0^{r-1} \\
1 & \rho_1 & \ldots & \rho_{1}^{s-1} & \rho_1^{s+1} & \ldots & \rho_1^{r-1} \\
& \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \rho_{s-1} & \ldots & \rho_{s-1}^{s-1} & \rho_{s-1}^{s+1} & \ldots & \rho_{s-1}^{r-1} \\
1 & \rho_{s+1} & \ldots & \rho_{s+1}^{s-1} & \rho_{s+1}^{s+1} & \ldots & \rho_{s+1}^{r-1} \\
& \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & \rho_{r-1} & \ldots & \rho_{r-1}^{s-1} & \rho_{r-1}^{s+1} & \ldots & \rho_{r-1}^{r-1}
\end{vmatrix}.
\]

Substituting for \( y_{sk} \) in (e) its value from (7') and then eliminating \( y_{kj} \) by (8'), we obtain

\[
Y_{ik} = \sum_{i=1, \ldots, \alpha} \alpha_{ii}^{tk} Y_{ii}^* \quad (k=1, \ldots, n; i=0, \ldots, r-1),
\]

where

\[
\alpha_{ii}^{tk} = \frac{(-1)^t \sum_{s=0}^{r-1} (-1)^s D_s C_{ki}^{(s)} \rho_i^s}{\Delta}.
\]

The coefficients of transformation (9) belong to the domain \( F \). It suffices to show that each \( \alpha_{ii}^{tk} \) is unaltered by the interchange of \( \rho_0 \) with \( \rho_j \) (\( j \) being any one of the series \( 1, 2, \ldots, r - 1 \)), since it is then a symmetric function of \( \rho_0, \rho_1, \ldots, \rho_{r-1} \) with coefficients in \( F \). To show that, for example, it is unaltered by the interchange of \( \rho_0 \) with \( \rho_1 \), we note that under this interchange, \( D_0 \) and \( D_1 \) are interchanged, \( D_s (s > 1) \) is changed into \(- D_s\) while \( C_{ki}^{(s)}(\rho_0) \) and \( C_{ki}^{(s)}(\rho_1) \) are interchanged, and \( C_{ki}^{(s)}(s > 1) \) is unaltered. Hence the factor of \( \alpha \) given by the sum is changed in sign; likewise the factor \( 1/\Delta \).

Moreover, from (7_a) follows at once

\[
\alpha_{ii}^{tk} = 0 \quad (i=m+1, \ldots, n; k=1, \ldots, m; t, l=0, 1, \ldots, r-1).
\]

The group of transformations (9) is therefore of Loewy's form (10), \( \bar{H}_{11} \) being always a matrix of \( rm \) rows and \( rm \) columns. The proof is then readily completed as in Loewy's case (bottom of p. 175 and 176).

The University of Chicago,
April 27, 1903.