CONGRUENCES OF CURVES*

BY

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In the second volume of his *Leçons* DARBOUX has considered congruences of curves defined by the equations,

$$f(x, y, z, a, b) = 0, \qquad \phi(x, y, z, a, b) = 0,$$

where x, y, z are the rectangular coördinates of a point on one of these curves and a, b are the parameters of the curves, and he has established many interesting theorems about congruences of this kind.

In the present discussion we shall consider such congruences when defined by equations of the form,

$$(1) \qquad x = f_1(t, u, v), \qquad y = f_2(t, u, v), \qquad z = f_3(t, u, v),$$

where u and v denote the parameters determining the curves and t is the parameter which determines the points on the curve. By this method of definition the equations of a congruence of curves are given a form similar to that usually adopted in the discussions of rectilinear congruences, so that one proceeds naturally to a generalization of some of the properties of the latter. At the same time the curves, as above defined, are looked upon as the intersections of surfaces, namely, those defined by (1), when u and v respectively are constants.

In §§1, 2 we show how the theorems of DARBOUX can be established when the congruence is defined in this manner. Several examples are given to illustrate the different theorems.

In § 3 we consider the problem of determining a function $\phi(u, v)$ such that the tangents to the curves of the congruence at the points of intersection with the surface, defined by (1) after t has been replaced by ϕ, shall form a normal congruence, or in the second place the ruled surfaces $u =$ const., $v =$ const. of this congruence of tangents shall be developable. It is shown that for every congruence a function exists which furnishes a solution to the first of these problems, and the condition is found which the functions f_1, f_2, f_3 must satisfy in order that ϕ may be constant. However, there does not always exist a function such that the second condition is satisfied. But when the curves $t =$ const.,

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$u =$ const. on the surfaces defined by (1) when v is constant form a conjugate system, and when the curves $t =$ const., $v =$ const. on the surfaces defined by (1) when u is constant are likewise conjugate, ϕ has a constant value and only in this case.

It is next proposed to find a function $\phi(u)$ such that all the tangents to the curves whose parameters satisfy the relation $v = \phi(u)$ form a normal congruence. It is shown that such a function cannot be found for every congruence, but exists only in the case of normal rectilinear congruences when equations (1) define space referred to a triply orthogonal system of surfaces. In closing § 4 we remark that the tangents to all the curves of a congruence form a complex and we find the condition that this complex be linear.

The formulæ (1) define space referred to a triple system of surfaces $t =$ const., $u =$ const., $v =$ const. and in the preceding sections we have considered (1) as defining the congruence C_t of curves of intersection of the surfaces $u =$ const., $v =$ const. It is evident, however, that these equations define equally well the congruence C_u and C_v composed of the intersections of the surfaces $t =$ const., $v =$ const., and $t =$ const., $u =$ const., respectively. Therefore we say that equations (1) define a *triple congruence* of curves. In § 5 the different theorems and equations of the preceding sections are viewed in this three-fold light and several interesting theorems are established, notably that the focal surface of all three congruences is the same surface. The most general triply rectilinear congruence is considered and a few of its properties noted.

In § 6 we consider the triple congruences such that each congruence is cut orthogonally by a family of surfaces, and point out the relation of this problem to that of solving the equations in the study of applicable surfaces.

Finally, in § 7, the conditions are determined which the functions f_1, f_2, f_3 must satisfy in order that the three complexes of tangents be linear.

§ 1. Surfaces of the Congruence. Focal Points and Focal Surfaces.

Consider a congruence of curves defined by equations of the form

$$(1) \qquad x = f_1(t, u, v), \qquad y = f_2(t, u, v), \qquad z = f_3(t, u, v),$$

where the parameters u and v determine the curve and t a point upon the curve. We shall assume that the functions f_1, f_2, f_3 and their derivatives of the first and second orders with respect to these variables are finite and continuous within a suitable region.

It is clear that, when f_1, f_2, f_3 are algebraic, there pass through a given point in space a number, in general limited, of curves of the congruence. For, when particular values are given to x, y, z, the values of t, u, v given by equa-

tions (1) will in general be finite in number. However, when this restriction is not put upon these functions, it will not be unusual to have an infinity of the curves through a point. As an example of this, as DARBOUX remarks,* the totality of tangents to a surface from points on a given curve is a rectilinear congruence having an infinity of lines through each point of the curve.

If we establish a relation between u and v, say

$$v = \phi(u) \tag{2}$$

the curves of the congruence whose parameters satisfy this relation depend upon the single parameter u and consequently form a surface; the coördinates of this surface are given by (1), when v has been replaced by ϕ. This surface will evidently be changed as ϕ is given different forms. With DARBOUX we shall call these *surfaces of the congruence.* It is evident that in the case of rectilinear congruences all these surfaces are ruled.

The equation of the tangent plane to the surface defined by (1) and (2) is

$$\begin{vmatrix} \xi - x & \eta - y & \zeta - z \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \\ \dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial v}\phi'(u) & \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial v}\phi'(u) & \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\phi'(u) \end{vmatrix} = 0, \tag{3}$$

where ξ, η, ζ are current coördinates and the accent denotes differentiation. This equation can be written in the form

$$\begin{aligned} 0 = {} & (\xi - x)\left[\frac{\partial(y, z)}{\partial(t, u)} + \phi'(u)\frac{\partial(y, z)}{\partial(t, v)}\right] \\ & + (\eta - y)\left[\frac{\partial(z, x)}{\partial(t, u)} + \phi'(u)\frac{\partial(z, x)}{\partial(t, v)}\right] + (\zeta - z)\left[\frac{\partial(x, y)}{\partial(t, u)} + \phi'(u)\frac{\partial(x, y)}{\partial(t, v)}\right]. \end{aligned} \tag{4}$$

It is readily found that the anharmonic ratio of the tangent planes to four surfaces of the congruence through a given curve and corresponding to functions ϕ_1, ϕ_2, ϕ_3, ϕ_4 depends only upon these functions, so that we have the theorem of DARBOUX: †

For any four surfaces of a congruence containing the same curve of the congruence the anharmonic ratio of the tangent planes to these surfaces at a point is constant when the point moves along the curve.

From (4) it is seen that when x, y, z satisfy the condition

* *Leçons*, vol. 2, p. 2.

† *Leçons*, vol. 2, p. 3.

$$(5)\qquad \frac{\dfrac{\partial(y,z)}{\partial(t,u)}}{\dfrac{\partial(y,z)}{\partial(t,v)}}=\frac{\dfrac{\partial(z,x)}{\partial(t,u)}}{\dfrac{\partial(z,x)}{\partial(t,v)}}=\frac{\dfrac{\partial(x,y)}{\partial(t,u)}}{\dfrac{\partial(x,y)}{\partial(t,v)}}=\lambda,$$

the equation of the tangent plane is independent of the function ϕ and consequently is the same for all surfaces of the congruence through the corresponding points. This gives the theorem: *

Upon a curve of a congruence there are points at which all the surfaces of the congruence through the curve admit the same tangent plane, whatever be the law according to which the curves have been assembled to generate the surfaces.

With DARBOUX we shall refer to these points as the *focal points* and to their locus as the *focal surface.* In general this focal surface will consist of sheets, whose number will depend upon the number of focal points on the curves.

The equation (5) may be replaced by

$$(6)\qquad \begin{vmatrix} \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix}=0,$$

from which we have the relation

$$(7)\qquad \frac{\partial x}{\partial t}=l\frac{\partial x}{\partial u}+m\frac{\partial x}{\partial v},$$

and similar ones in y and z.

The equation (6) can be looked upon as an equation in t having its coefficients functions of u and v. When u and v are given particular values, the values of t satisfying this equation define the focal points upon the corresponding line. When equation (6) is solved for t and the various solutions are substituted in (1) the latter will define the sheets of the focal surface. For example, suppose that the congruence is rectilinear, in which case (1) can be given the form

$$(8)\qquad x=a_1+a_2t,\qquad y=b_1+b_2t,\qquad z=c_1+c_2t,$$

where $a_1, \cdots, c_2$ are functions of u and v. The corresponding equation (6) is a quadratic in t. When the two roots of this equation are substituted in (8), we have the equations of the two focal sheets.

* DARBOUX, l. c., p. 4.

§ 2. Singular Lines and Singular Surfaces. Principal Surfaces.

It is evident that the character of the functions f_1, f_2, f_3 will determine whether the number of the focal points on a curve is finite or infinite. Again, it may happen that for particular values of u and v, that is for particular lines, the equation (6) will be satisfied identically so that t is indeterminate. In this case every point of the curve is a focal point. When such lines exist, they will in general be isolated; we shall call them *singular lines* of the congruence. In some cases, however, there is a continuum of these lines and consequently a *singular surface.*

Consider for example the congruence defined by

$$
\begin{aligned}
x &= A(a-t)^m(a-u)^n(a-v)^p,\\
y &= B(b-t)^m(b-u)^n(b-v)^p,\\
z &= C(c-t)^m(c-u)^n(c-v)^p,
\end{aligned}
\tag{9}
$$

where A, B, C, a, b, c are constants. For u or v constant these formulæ define a tetrahedral surface, so that the curves of the congruence are the intersections of two families of tetrahedral surfaces. On this account we shall call the congruences defined by (9) *tetrahedral congruences.* Substituting the above values for x, y, z in equation (6), we have

$$
\begin{vmatrix}
(a-u)(a-v) & (b-u)(b-v) & (c-u)(c-v)\\
(a-t)(a-v) & (b-t)(b-v) & (c-t)(c-v)\\
(a-t)(a-u) & (b-t)(b-u) & (c-t)(c-u)
\end{vmatrix} = 0,
\tag{10}
$$

which reduces to

$$
(t-u)(t-v)(u-v)(a-b)(a-c)(b-c) = 0.
\tag{11}
$$

From this we see that the focal points on the line $u=\alpha$, $v=\beta$, where α and β are two unequal constants, are given by $t=\alpha$ and $t=\beta$; but for all the curves whose parameters satisfy the condition $u=v$, the parameter t is indeterminate, and consequently these lines are singular lines. In this case we have a singular surface; its coördinates are

$$
\begin{aligned}
x &= A(a-t)^m(a-u)^{n+p}, \qquad y = B(b-t)^m(b-u)^{n+p},\\
z &= C(c-t)^m(c-u)^{n+p}.
\end{aligned}
\tag{12}
$$

Moreover, the coördinates of the focal surfaces proper have the expressions

$$
\begin{aligned}
x &= A(a-u)^{m+n}(a-v)^p, \qquad y = B(b-u)^{m+n}(b-v)^p,\\
z &= C(c-u)^{m+n}(c-v)^p;
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
x &= A(a-u)^n(a-v)^{m+p}, \qquad y = B(b-u)^n(b-v)^{m+p},\\
z &= C(c-u)^n(c-v)^{m+p}.
\end{aligned}
\tag{14}
$$

From these three sets of expressions we see that the focal surfaces and the singular surface are tetrahedral surfaces. When the focal surface is defined as the locus of all points satisfying equation (6), it has all three of the above surfaces for nappes.

It should be remarked that every congruence defined by (9) does not have a singular surface, for there are certain values of the exponents which do not allow the equality of u and v. Such a case arises when $m = n = p = \frac{1}{2}$; then the congruence is composed of the curves of intersection of two families of confocal quadrics. If m, n, p be replaced by $\frac{1}{2}$ in (13) and (14), the latter take such a form that, if the parameters be eliminated, we have the same equation for the surface in each case and consequently there is only one equation to define the two nappes. This property is common to all congruences for which $m = n = p$.

Instead of solving equation (6) for t and substituting in (1) to get the equations of the focal surface, we can look upon equations (1), (7) and the two similar to the latter as defining the surface. A displacement upon this focal surface will have for projections on the coördinate axes the expression

$$dx = (ldt + du)\frac{\partial x}{\partial u} + (mdt + dv)\frac{\partial x}{\partial v},$$

and similar ones in y and z. Eliminating $ldt + du$ and $mdt + dv$ from these three equations, we get the equation of the tangent plane to the focal surface in a form which in consequence of equation (7) is reducible to the equation of the common tangent plane to the surfaces of the congruence at a focal point. Hence the theorem of DARBOUX: *

The surfaces of the congruence which contain a given curve admit the same tangent plane at a focal point as the focal surface and consequently the curves of the congruence are tangent at all their focal points to the focal surface.

In order that there exist surfaces of the congruence which are such that consecutive curves of the congruence upon these surfaces intersect and thus have an envelope, it is necessary that there be a relation of the form

$$dv = \phi(u, v)\,du, \tag{15}$$

such that †

$$\frac{\dfrac{\partial x}{\partial t}}{\dfrac{\partial x}{\partial u} + \phi\dfrac{\partial x}{\partial v}} = \frac{\dfrac{\partial y}{\partial t}}{\dfrac{\partial y}{\partial u} + \phi\dfrac{\partial y}{\partial v}} = \frac{\dfrac{\partial z}{\partial t}}{\dfrac{\partial z}{\partial u} + \phi\dfrac{\partial z}{\partial v}}. \tag{16}$$

* l. c. p. 5.

† PICARD, *Traité d'Analyse*, vol. 1, p. 294.

Solving for ϕ, we get

$$\phi = \frac{\dfrac{\partial(x, y)}{\partial(u, t)}}{\dfrac{\partial(x, y)}{\partial(t, v)}} = \frac{\dfrac{\partial(y, z)}{\partial(u, t)}}{\dfrac{\partial(y, z)}{\partial(t, v)}} = \frac{\dfrac{\partial(z, x)}{\partial(u, t)}}{\dfrac{\partial(z, x)}{\partial(t, v)}}. \tag{17}$$

We have seen that the latter equalities are satisfied only at the focal points, and that for each focal point there is a solution $t = \psi(u, v)$ of equation (6), which is equivalent to the preceding equations. When such value for t is substituted in (17), we get a corresponding function $\phi(u, v)$ which furnishes a solution of our problem. Hence we have the following theorem similar to one given by DARBOUX: *

Corresponding to the focal points of a curve of a congruence there are surfaces of the congruence, through the given curve, whose consecutive curves have an envelope. And these envelopes lie on the sheets of the focal surface corresponding to the respective focal points.

We call these the *principal surfaces* of the congruence.

From this theorem and a preceding result concerning rectilinear congruences, it follows that every congruence of right lines has two principal surfaces and these are developable.

Let us suppose that the surface defined by (1) with $v =$ const. and passing through a given curve is a principal surface of the congruence. Then from (15) it follows that the corresponding function ϕ is zero. In this case the numerators in (17) are zero. But these expressions are proportional to the direction cosines of the normal to the surface $v =$ const. Hence along the curve on this surface which is the envelope of the curves of the congruence upon this surface the normal is indeterminate, so that this envelope is a singular line for the surface.

An example of this is furnished by the tetrahedral congruence. Thus from (9) we have

$$\frac{\partial(x, y)}{\partial(t, u)} = mnAB(a-t)^{m-1}(a-u)^{n-1}(a-v)^{p}(b-t)^{m-1}(b-u)^{n-1}(b-v)^{p}(u-t)(a-b),$$

and similarly for the other Jacobians. But from (11) we have that the foci are given by $t = u$ or $t = v$. The former makes the above zero, so that we have that the surfaces $v =$ const. are principal surfaces for the congruence and the curve $t = u$ on each of these surfaces is a singular line for the surface. It is to be remarked that in a similar manner the Jacobians $\partial(x, y)/\partial(t, v)$, etc., vanish for $t = v$, so that ϕ is infinite and hence the surfaces defined by (9) when u is constant are principal surfaces for the tetrahedral congruence, and the line $t = v$ is a singular line on each surface. It need hardly be mentioned that

* *Leçons*, vol. 2, p. 7.

these singular lines, and consequently the focal points, are not always real. This fact is evidenced in some of the tetrahedral congruences. Thus the central quadrics may be defined by

$$(18)\quad x^2=\frac{a(a-u)(a-t)}{(a-b)(a-c)},\quad y^2=\frac{b(b-u)(b-t)}{(b-a)(b-c)},\quad z^2=\frac{c(c-u)(c-t)}{(c-a)(c-b)},$$

where $a>b>c$. From these expressions it is seen that, if the tetrahedral surfaces $v=$ const. are central quadrics, the coördinate y of points on the line $t=u$ are imaginary.

§ 3. Certain Rectilinear Congruences Associated with Congruences of Curves.

When the formulæ (1) define space referred to a triply-orthogonal system of surfaces, the curves of the congruence under discussion are the intersections of the surfaces $u=$ const., $v=$ const. of the system. Since the surfaces are triply-orthogonal, these curves cut the surfaces $t=$ const. orthogonally and consequently the tangents to these curves at the points of intersection with one of the surfaces $t=$ const. from a normal congruence. Furthermore, the curves $u=$ const. and $v=$ const. on this surface are the lines of curvature and therefore the ruled surfaces $u=$ const., $v=$ const. of this congruence of tangents are developable. This suggests a two-fold problem of which the preceding case is a solution: *To determine, when possible, a function $\phi(u, v)$ such that if tangents be drawn to the curves of the congruence* (1) *at the points where they are met by the surface,*

$$(19)\qquad \xi=f_1(\phi, u, v),\qquad \eta=f_2(\phi, u, v),\qquad \zeta=f_3(\phi, u, v),$$

1° *these tangents form a normal congruence; or*

2° *the ruled surfaces $u=$ const., $v=$ const. are developable.*

The direction-cosines of the tangent to the curve whose coördinates have the expressions $f_1(u_0, v_0, t)$, $f_2(u_0, v_0, t)$, $f_3(u_0, v_0, t)$ are equal to

$$(20)\qquad \frac{\dfrac{\partial f_1}{\partial t},\ \dfrac{\partial f_2}{\partial t},\ \dfrac{\partial f_3}{\partial t}}{\sqrt{\left[\Sigma\left(\dfrac{\partial f_1}{\partial t}\right)^2\right]}}.$$

From the theory of rectilinear congruences we know that the necessary and sufficient condition that the congruence of tangents be normal to a surface is that *

$$\frac{\Sigma\dfrac{\partial f_1}{\partial t}\,d\xi}{\sqrt{\left[\Sigma\left(\dfrac{\partial f_1}{\partial t}\right)^2\right]}}+dl=0,$$

* Bianchi, *Lezioni*, p. 256; German edition, p. 268.

where l is the distance from the point (ξ, η, ζ) to the point where the line cuts an orthogonal surface. If we put

$$(21) \qquad \Sigma\left(\frac{\partial f_1}{\partial t}\right)^2 = T_1, \qquad \Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial u} = T_2, \qquad \Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial v} = T_3,$$

the above expression takes the form

$$(22) \qquad T_1 dt + T_2 du + T_3 dv + \sqrt{T_1}\, dl = 0.$$

When, in particular, the surface defined by (19) is orthogonal to the congruence the above equation reduces to

$$(23) \qquad T_1 dt + T_2 du + T_3 dv = 0.$$

Any function $t = \phi(u, v)$ which satisfies this equation gives a solution of our problem. In order that ϕ may involve an arbitrary constant, that is, in order that there may be an infinity of surfaces cutting the congruence in such a way that the tangents to the curves at the points of intersection are normal to the surfaces, it is necessary and sufficient that the functions T_1, T_2, T_3 satisfy identically the equation *

$$(24) \qquad T_1\left(\frac{\partial T_2}{\partial v} - \frac{\partial T_3}{\partial u}\right) + T_2\left(\frac{\partial T_3}{\partial t} - \frac{\partial T_1}{\partial v}\right) + T_3\left(\frac{\partial T_1}{\partial u} - \frac{\partial T_2}{\partial t}\right) = 0.$$

This is the condition found by BELTRAMI to be the necessary and sufficient condition that the curves be cut orthogonally by a family of surfaces.

We will consider now the general case given by equation (22). Replace t by $\phi(u, v)$; then the quantity

$$\frac{1}{\sqrt{T_1}}\left[\left(T_2 + T_1\frac{\partial\phi}{\partial u}\right)du + \left(T_3 + T_1\frac{\partial\phi}{\partial v}\right)dv\right]$$

must be an exact differential. Expressing this condition we get the equation

$$(25) \qquad \begin{aligned}\left(T_1\frac{\partial T_1}{\partial v} - 2T_1\frac{\partial T_3}{\partial t} + T_2\frac{\partial T_1}{\partial t}\right)\frac{\partial\phi}{\partial u} + \left(2T_1\frac{\partial T_2}{\partial t} - T_2\frac{\partial T_1}{\partial t} - T_1\frac{\partial T_1}{\partial u}\right)\frac{\partial\phi}{\partial v} \\ + \left(2T_1\frac{\partial T_2}{\partial v} - T_2\frac{\partial T_1}{\partial v} - 2T_1\frac{\partial T_3}{\partial u} + T_3\frac{\partial T_1}{\partial u}\right) = 0.\end{aligned}$$

Hence, given any congruence whatever, a function ϕ can be found which furnishes a solution of the problem. And the orthogonal surfaces are given by the equations

$$x = \xi + \frac{l}{\sqrt{T_1}}\frac{\partial f_1}{\partial t}, \qquad y = \eta + \frac{l}{\sqrt{T_1}}\frac{\partial f_2}{\partial t}, \qquad z = \zeta + \frac{l}{\sqrt{T_1}}\frac{\partial f_3}{\partial t},$$

where l is given by quadrature from (22). Hence we have the theorem:

* FORSYTH, *Treatise on Differential Equations*, p. 251.

Given any congruence whatever, there is an infinity of surfaces which cut the curves in points such that the tangents to the curves at these points form a normal congruence. The family of surfaces is given by the integration of a partial differential equation of the first order and the orthogonal surfaces by a quadrature.

From equation (25) it follows that the necessary and sufficient condition that the surfaces $t =$ const. furnish a solution of the problem is that the functions T_1, T_2, T_3 satisfy the equation

$$2T_1\frac{\partial T_2}{\partial v} - T_2\frac{\partial T_1}{\partial v} - 2T_1\frac{\partial T_3}{\partial u} + T_3\frac{\partial T_1}{\partial u} = 0. \tag{26}$$

It is evident that to this class belong the curves of intersection of triply-orthogonal surfaces; they correspond to the solution $T_2 = T_3 = 0$.

We proceed now to the second part of the problem. The rectilinear congruence of tangents is given by

$$x = \xi + \rho\frac{\partial f_1}{\partial t}, \qquad y = \eta + \rho\frac{\partial f_2}{\partial t}, \qquad z = \zeta + \rho\frac{\partial f_3}{\partial t}. \tag{27}$$

Comparing these expressions with (8) and recalling (17), we find that the necessary and sufficient condition that there exist a function $\phi(u, v)$ such that the developables of the congruence (27) be given by $u =$ const. and $v =$ const. is

$$\begin{vmatrix} \frac{\partial^2 f_1}{\partial t^2} & \frac{\partial^2 f_2}{\partial t^2} & \frac{\partial^2 f_3}{\partial t^2} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \end{vmatrix}\frac{\partial \phi}{\partial u} + \begin{vmatrix} \frac{\partial^2 f_1}{\partial t\,\partial u} & \frac{\partial^2 f_2}{\partial t\,\partial u} & \frac{\partial^2 f_3}{\partial t\,\partial u} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \end{vmatrix} = 0, \tag{28}$$

and

$$\begin{vmatrix} \frac{\partial^2 f_1}{\partial t^2} & \frac{\partial^2 f_2}{\partial t^2} & \frac{\partial^2 f_3}{\partial t^2} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix}\frac{\partial \phi}{\partial v} + \begin{vmatrix} \frac{\partial^2 f_1}{\partial t\,\partial v} & \frac{\partial^2 f_2}{\partial t\,\partial v} & \frac{\partial^2 f_3}{\partial t\,\partial v} \\ \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \end{vmatrix} = 0. \tag{29}$$

From these expressions for $\partial\phi/\partial u$ and $\partial\phi/\partial v$ it is evident that the condition of integrability is not satisfied by every congruence. Since this condition of integrability involves the three functions f_1, f_2, f_3, two of the latter can be chosen arbitrarily and the determination of the third requires the integration of a partial differential equation of the third order in three variables. When this

condition is satisfied, ϕ is found by a quadrature, and, as it involves an arbitrary constant, there will be an infinity of surfaces furnishing a solution of the problem.

We have remarked that, when the formulæ (1) define a triply orthogonal system of surfaces, a solution of the above problem is given by

$$t = \phi = \text{const.}$$

From (28) and (29) we have that the necessary and sufficient condition that the congruence admits this solution is that f_1, f_2, f_3 are particular solutions of equations of the form

$$(30) \qquad \begin{aligned} \frac{\partial^2 \theta}{\partial t\, \partial u} + a_1 \frac{\partial \theta}{\partial t} + b_1 \frac{\partial \theta}{\partial u} &= 0, \\ \frac{\partial^2 \theta}{\partial t\, \partial v} + a_2 \frac{\partial \theta}{\partial t} + b_2 \frac{\partial \theta}{\partial v} &= 0. \end{aligned}$$

Hence *the necessary and sufficient condition that the tangents to the curves defined by* (1) *at their intersections with a surface* $t = \text{const.}$ *form a congruence for which the ruled surfaces* $u = \text{const.}$ *and* $v = \text{const.}$ *are developable is that the parametric curves on the two families of surfaces defined by* (1), *when* u *and* v *respectively are constant, form a conjugate system.**

From (28) and (29) it is seen that for any value whatever of ϕ to satisfy the conditions of the problem f_1, f_2, f_3 must satisfy, in addition to the equations (30), two equations of the form

$$\frac{\partial^2 \theta}{\partial t^2} + \alpha_1 \frac{\partial \theta}{\partial t} + \beta_1 \frac{\partial \theta}{\partial u} = 0,$$

$$\frac{\partial^2 \theta}{\partial t^2} + \alpha_2 \frac{\partial \theta}{\partial t} + \beta_2 \frac{\partial \theta}{\partial v} = 0.$$

This shows that the curves $u = \text{const.}$ on the surfaces $v = \text{const.}$ and the curves $v = \text{const.}$ on the surfaces $u = \text{const.}$ must be asymptotic, that is the curves of intersection of these two families must be asymptotic for both surfaces. This can be satisfied only by these lines being rectilinear, so that the congruence is rectilinear. Hence the only case where ϕ can be chosen at will is the evident case where the triple system is formed of the two families of developables of a normal congruence and the surfaces normal to the latter.

* Two curves upon a surface are said to have conjugate directions at a point, when their tangents are parallel to conjugate diameters of the Dupin indicatrix for the point. A double family of curves forms a conjugate system, when the two curves through any point of the surface have conjugate directions at the point. The necessary and sufficient condition that a parametric system be conjugate is that the point equation be of the form (30).

§ 4. Normal Rectilinear Congruences and Linear Complexes Associated with Congruences of Curves.

We propose now the problem of finding a surface S of the congruence, given by a relation of the form $v = \phi(u)$, such that the tangents to the curves satisfying this relation form a normal congruence. This is equivalent to saying that the curves $u =$ const. on this surface are geodesics. The linear element of S is

$$ds^2 = \Sigma\left(\frac{\partial f_1}{\partial t}\right)^2 dt^2 + 2\left(\Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial u} + \phi'\Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial v}\right)dt\,dv + \Sigma\left(\frac{\partial f_1}{\partial u} + \phi'\frac{\partial f}{\partial v}\right)^2 du^2,$$

where the prime denotes differentiation. The necessary and sufficient condition that the lines $u =$ const. be geodesics is *

$$2E\frac{\partial F}{\partial t} - F\frac{\partial E}{\partial t} - E\frac{\partial E}{\partial u} = 0, \tag{31}$$

where, as usual, E, F, G are the coefficients in the expression for the linear element of the surface. When the above expressions are substituted in this equation, we get the condition

$$\phi'(u) = \frac{\Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial u}\cdot\Sigma\frac{\partial f_1}{\partial t}\frac{\partial^2 f_1}{\partial t^2} - \Sigma\left(\frac{\partial f_1}{\partial t}\right)^2\cdot\Sigma\frac{\partial f_1}{\partial u}\frac{\partial^2 f_1}{\partial t^2}}{\Sigma\left(\frac{\partial f_1}{\partial t}\right)^2\left[\Sigma\frac{\partial f_1}{\partial v}\frac{\partial^2 f_1}{\partial t^2} + \Sigma\frac{\partial f_1}{\partial t}\frac{\partial^2 f_1}{\partial t\,\partial v}\right] - \Sigma\frac{\partial f_1}{\partial t}\frac{\partial f_1}{\partial v}\cdot\Sigma\frac{\partial f_1}{\partial t}\frac{\partial^2 f_1}{\partial t^2}}. \tag{32}$$

From this expression it follows that for the existence of a function ϕ satisfying the condition of the problem it is necessary and sufficient that the right-hand member of this equation be a function of u alone or a constant; moreover, when this condition is satisfied, ϕ is given by a quadrature. It is readily seen that this condition is not satisfied in general, so that every congruence does not furnish a solution of this problem.

We note that this condition is satisfied when

$$\frac{\partial^2 f_1}{\partial t^2} = \frac{\partial^2 f_2}{\partial t^2} = \frac{\partial^2 f_3}{\partial t^2} = 0,$$

that is, when

$$x = a_1 + a_2 t, \qquad y = b_1 + b_2 t, \qquad z = c_1 + c_2 t, \tag{33}$$

where $a_1, a_2, \cdots, c_2$, are functions of u and v; this congruence is evidently rectilinear. Let a_2, b_2, c_2 satisfy the condition

$$a_2^2 + b_2^2 + c_2^2 = 1,$$

and consequently be the direction-cosines of the lines. When this fact is noted in (32) we find that $\phi'(u)$ is indeterminate. This is what should be expected; for,

* Bianchi, *Lezioni*, p. 146; German translation, p. 150.

if v be replaced by any function of u in (33) the latter define a ruled surface and consequently the lines $u =$ const. are geodesics.

Again, we remark that when the congruence is given by a triply-orthogonal system the equation (31) reduces to

$$\frac{\partial E}{\partial u} = 0,$$

so that E is a function of t alone. From this it follows that the parameter t can be so chosen that we have

$$E = \Sigma \left(\frac{\partial f_1}{\partial t} \right)^2 = 1.$$

In this case the square of the linear element on the surfaces $v =$ const. and $u =$ const. would take the respective forms

$$ds^2 = dt^2 + \lambda du^2,$$

$$ds^2 = dt^2 + \mu dv^2,$$

where λ denotes a function of t and u, and μ a function of t and v. From this we see that the intersections of the surfaces $u =$ const., $v =$ const., that is the curves of the congruence, would be geodesics on both surfaces, which is possible only in case these curves be rectilinear.* Hence the only triply-orthogonal system furnishing a solution of this problem is composed of the developable surfaces of a normal congruence and the orthogonal surfaces.

We have incidentally from the preceding discussion:

A family of surfaces of MONGE† *cannot form part of a triply-orthogonal system unless the other surfaces are the developables of the normals to the former.*

The totality of the tangents to the curves of the congruence (1) is a complex. If the equations of a line in this complex are given in the PLÜCKER form

$$bz - cy = p, \qquad cx - az = q, \qquad ay - bx = r,$$

we have

$$a = \frac{\partial f_1}{\partial t}, \qquad b = \frac{\partial f_2}{\partial t}, \qquad c = \frac{\partial f_3}{\partial t},$$

$$p = f_3 \frac{\partial f_2}{\partial t} - f_2 \frac{\partial f_3}{\partial t}, \qquad q = f_1 \frac{\partial f_3}{\partial t} - f_3 \frac{\partial f_1}{\partial t}, \qquad r = f_2 \frac{\partial f_1}{\partial t} - f_1 \frac{\partial f_2}{\partial t}.$$

The necessary and sufficient condition that this complex be linear is that there exist six constants a_1, b_1, c_1, α_1, β_1, γ_1, such that the equation

* BIANCHI, *Lezioni*, p. 206; German translation, p. 214.

† DARBOUX, *Leçons*, vol. 1, p. 103.

$$(34)\qquad \begin{aligned}(a_1 - \beta_1 f_3 + \gamma_1 f_2)\frac{\partial f_1}{\partial t} + (b_1 - \gamma_1 f_1 + \alpha_1 f_3)\frac{\partial f_2}{\partial t} \\ + (c_1 - \alpha_1 f_2 + \beta_1 f_1)\frac{\partial f_3}{\partial t} = 0\end{aligned}$$

is satisfied identically. From this we see that given any six constants and two functions f_2, f_3, the determination of f_1, so that (1) defines a congruence whose tangents form a linear complex, requires the integration of a linear equation.

§5. Triple Congruences.

We have remarked that the equations (1) define three congruences C_t, C_u, C_v, according as t, u and v are considered as the parameters of points on the curve. Since equation (6) is symmetrical with respect to t, u, v, it follows that this equation holds for the congruences C_u and C_v as well as for C_t. Recalling the interpretation of this equation, we have the theorem:

In any triple congruence of curves the focal surface is the same for all three congruences.

Consider a point M of the focal surface and the curves Γ_t, Γ_u, Γ_v through this point. All the surfaces of the congruence C_t through Γ_t are tangent to one another and the focal surface at M; likewise for the surfaces of the congruences C_u and C_v through Γ_u and Γ_v respectively. Hence:

All the surfaces of the three congruences through the respective lines of these congruences meeting the focal surface in the same point are tangent to one another and the focal surface at the point. And these respective lines are all tangent to the focal surface.

The most general triply rectilinear congruence is defined by

$$\begin{aligned} x &= (\alpha_1 uv + \beta_1 u + \gamma_1 v) + (a_1 uv + b_1 u + c_1 v + d_1)t, \\ y &= (\alpha_2 uv + \beta_2 u + \gamma_2 v) + (a_2 uv + b_2 u + c_2 v + d_2)t, \\ z &= (\alpha_3 uv + \beta_3 u + \gamma_3 v) + (a_3 uv + b_3 u + c_3 v + d_3)t. \end{aligned}$$

We have shown elsewhere * that these lines are the intersections of a triple system of hyperbolic paraboloids and furthermore that they furnish the only example of a triple system of surfaces cutting one another along asymptotic lines. In consequence of the above theorem we have the following:

In a triply rectilinear congruence the three lines through a point on the focal surface lie in a plane—the tangent plane to the focal surface at this point.

Consider now a singular line of C_t. Then for $u = \alpha$, $v = \beta$, where (α, β) determine the singular line, the equation (6) is independent of t. For the con-

* Bulletin of the American Mathematical Society, ser. 2, vol. 7, p. 303.

gruence C_u the point of parameter $u = \alpha$ on every curve of the congruence lying on the surface $v = \beta$ is a focal point and therefore the singular line of C_t lies on one of the focal sheets of C_u. We have then this theorem:

The singular lines of any congruence lie on the focal sheets of two other congruences which form with the first a triple congruence. And when one of these congruences has a singular surface, it is one of the sheets of the focal surfaces of the other congruences.

For example, we consider the triple tetrahedral congruence defined by (9). The singular surfaces of C_u and C_v are given by (14) and (13) respectively, and the focal sheet by (12), (13) and (12), (14), respectively.

From the preceding results it follows that the focal surface must be looked upon as comprising the singular lines and singular surfaces, if the theorem to the effect that the focal surface is the same for all three congruences is to be perfectly general. Hence, if C_t has isolated singular lines which do not lie on the focal sheets, C_u and C_v will have curves for some of the focal nappes.

We have seen that the necessary and sufficient condition that the tangents to the curves of C_t at the points of intersection with a surface $t =$ const. form a congruence for which the ruled surfaces $u =$ const. and $v =$ const. are the developables is that the parametric lines on the two families of surfaces, defined by (1) when u and v respectively are given constant values, form conjugate systems. Similar results are true for C_u and C_v. Moreover, the conditions that C_u and C_t possess this property carry with them the conditions for C_v. In consequence of this we have the theorem:

The necessary and sufficient condition that the tangents to the curves of intersection of any two families of surfaces of a triple system at the points of intersection with a surface of the third family form a rectilinear congruence whose developables cut the surfaces of the third family in the same curves as the first two families of surfaces is that the system be triply conjugate.

It is well known that on the tetrahedral surface defined by

$$x = \alpha(a-u)^m(a-v)^n, \quad y = \beta(b-u)^m(b-v)^n, \quad z = \gamma(c-u)^m(c-v)^n,$$

the curves $u =$ const., $v =$ const. form a conjugate system. Hence the triple tetrahedral congruences (9) possess the property referred to in the above theorem.

§ 6. Triple Normal Congruences of Curves.

We have found the conditions (24) which the functions f_1, f_2, f_3 in (1) must satisfy in order that there may exist a family of surfaces cutting the curves of the congruence C_t orthogonally. Similar conditions must be satisfied for the same to be true of C_u and C_v.

Thus, we put

$$(35)\qquad U_1 = \Sigma \frac{\partial x}{\partial u}\frac{\partial x}{\partial t}, \qquad U_2 = \Sigma \left(\frac{\partial x}{\partial u}\right)^2, \qquad U_3 = \Sigma \frac{\partial x}{\partial u}\frac{\partial x}{\partial v},$$

and

$$(36)\qquad V_1 = \Sigma \frac{\partial x}{\partial v}\frac{\partial x}{\partial t}, \qquad V_2 = \Sigma \frac{\partial x}{\partial v}\frac{\partial x}{\partial u}, \qquad V_3 = \Sigma \left(\frac{\partial x}{\partial v}\right)^2.$$

The equations for the determination of $u = \chi(t, v)$ and $v = \psi(t, u)$ giving orthogonal surfaces are

$$U_1 dt + U_2 du + U_3 dv = 0,$$

$$V_1 dt + V_2 du + V_3 dv = 0,$$

and the conditions that the integrals involve an arbitrary constant are

$$(37)\qquad U_1\left(\frac{\partial U_2}{\partial v} - \frac{\partial U_3}{\partial u}\right) + U_2\left(\frac{\partial U_3}{\partial t} - \frac{\partial U_1}{\partial v}\right) + U_3\left(\frac{\partial U_1}{\partial u} - \frac{\partial U_2}{\partial t}\right) = 0,$$

$$(38)\qquad V_1\left(\frac{\partial V_2}{\partial v} - \frac{\partial V_3}{\partial u}\right) + V_2\left(\frac{\partial V_3}{\partial t} - \frac{\partial V_1}{\partial v}\right) + V_3\left(\frac{\partial V_1}{\partial u} - \frac{\partial V_2}{\partial t}\right) = 0.$$

Comparing (21), (35) and (36), we note that

$$(39)\qquad U_1 = T_2, \qquad V_1 = T_3, \qquad V_2 = U_3.$$

We see then that the three equations (24), (37) and (38) must be satisfied simultaneously in order that each of the congruences have a family of surfaces cutting them orthogonally.

The first of these equations is satisfied by $T_2 = T_3 = 0$, and from (23) we have that for this case the surfaces $t =$ const. cut the curves of C_t orthogonally. In consequence of (39) we have that equation (37) also is satisfied when U_3 is zero and then the surfaces $u =$ const. cut the curves C_u orthogonally. When these conditions are satisfied, equation (38) vanishes and the surfaces orthogonal to C_v are $v =$ const. From this we have that the congruences in this case are given by the intersections of the surfaces of a triply orthogonal system.

Again we note that these equations are satisfied by

$$T_1 = T_2 = T_3 = U_2 = U_3 = V_3.$$

But in this case the linear element of space is a perfect square and hence the surfaces $t =$ const., $u =$ const., $v =$ const. are developables circumscribing an imaginary circle at infinity.*

The equations of conditions are also satisfied, when these six functions satisfy the eight equations:

* DARBOUX, *Leçons*, vol. 1, p. 148, footnote.

$$\frac{\partial T_2}{\partial v}=\frac{\partial T_3}{\partial u}=\frac{\partial U_3}{\partial t}, \qquad \frac{\partial T_3}{\partial t}=\frac{\partial T_1}{\partial v}, \qquad \frac{\partial T_1}{\partial u}=\frac{\partial T_2}{\partial t}, \qquad \frac{\partial U_2}{\partial v}=\frac{\partial U_3}{\partial u},$$

$$\frac{\partial T_2}{\partial u}=\frac{\partial U_2}{\partial t}, \qquad \frac{\partial U_3}{\partial v}=\frac{\partial V_3}{\partial u}, \qquad \frac{\partial V_3}{\partial t}=\frac{\partial T_2}{\partial v}.$$

A solution of these equations is given by

$$T_1=f(t), \quad U_2=\phi(u), \quad V_3=\psi(v), \quad T_2=av+b, \quad T_3=au+c,$$
$$U_3=at+d,$$

where f, ϕ, ψ are arbitrary functions and a, b, c, d are arbitrary constants. The surfaces orthogonal to the congruence C_t are given by (1), when t is replaced by a function of $auv+bu+cv+e$, the form of the function depending upon f; similarly for the surfaces orthogonal to the congruences C_u and C_v.

It is readily found that the above equations are satisfied also by

$$T_2=F_1(u,t), \qquad T_3=F_2(v,t), \qquad U_3=F_3(u,v),$$

$$U_2=\int\frac{\partial F_3}{\partial u}dv+\int\frac{\partial F_1}{\partial u}dt+\psi(u),$$

$$T_1=\int\frac{\partial F_2}{\partial t}dv+\int\frac{\partial F_1}{\partial t}du+\phi(t),$$

$$V_3=\int\frac{\partial F_2}{\partial v}dt+\int\frac{\partial F_3}{\partial v}du+\omega(v),$$

where F_1, F_2, F_3, ϕ, ψ, ω, are arbitrary functions. As in the preceding case the orthogonal surfaces are given by quadratures, but the operations are far less simple.

But after a system of functions satisfying the above equations of condition have been found the determination of the corresponding congruence requires the solutions of the equations

$$\Sigma\left(\frac{\partial x}{\partial t}\right)^2=T_1, \qquad \Sigma\left(\frac{\partial x}{\partial u}\right)^2=U_2, \qquad \Sigma\left(\frac{\partial x}{\partial v}\right)^2=V_3,$$

$$\Sigma\frac{\partial x}{\partial t}\frac{\partial x}{\partial u}=T_2, \qquad \Sigma\frac{\partial x}{\partial t}\frac{\partial x}{\partial v}=T_3, \qquad \Sigma\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}=U_3.$$

This determination is seen to involve as a particular case the problem of finding all surfaces with a given linear element, so that one realizes the difficulties which arise in a general solution of this problem.

§ 7. Triple Linear Complexes.

We have found that when the functions f_1, f_2, f_3 satisfy an equation of the form

$$(40)\quad (a_1-\beta_1 f_3+\gamma_1 f_2)\frac{\partial f_1}{\partial t}+(b_1-\gamma_1 f_1+\alpha_1 f_3)\frac{\partial f_2}{\partial t}+(c_1-\alpha_1 f_2+\beta_1 f_1)\frac{\partial f_3}{\partial t}=0,$$

where $a_1, \cdots, \gamma_1$, are constants, the complex of the tangents to the congruence C_t is linear. In like manner for the complexes of tangents to the curves of C_u and C_v the following equations must be satisfied:

$$(41)\quad \begin{aligned}(a_2-\beta_2 f_3+\gamma_2 f_2)\frac{\partial f_1}{\partial u}+(b_2-\gamma_2 f_1+\alpha_2 f_3)\frac{\partial f_2}{\partial u}\\ +(c_2-\alpha_2 f_2+\beta_2 f_1)\frac{\partial f_3}{\partial u}=0,\end{aligned}$$

and

$$(42)\quad \begin{aligned}(a_3-\beta_3 f_3+\gamma_3 f_2)\frac{\partial f_1}{\partial v}+(b_3-\gamma_3 f_1+\alpha_3 f_3)\frac{\partial f_2}{\partial v}\\ +(c_3-\alpha_3 f_2+\beta_3 f_1)\frac{\partial f_3}{\partial v}=0,\end{aligned}$$

where $a_2, \cdots, \gamma_3$, are constants.

The equations (40) and (41) may be written in the form*

$$(43)\qquad \frac{\partial f_1}{\partial t}=P_1 f_1+Q_1, \qquad \frac{\partial f_1}{\partial u}=P_2 f_1+Q_2,$$

where

$$P_1=\frac{\gamma_1\dfrac{\partial f_2}{\partial t}-\beta_1\dfrac{\partial f_3}{\partial t}}{a_1-\beta_1 f_3+\gamma_1 f_2}, \qquad Q_1=-\frac{(b_1+\alpha_1 f_3)\dfrac{\partial f_2}{\partial t}+(c_1-\alpha_1 f_2)\dfrac{\partial f_3}{\partial t}}{a_1-\beta_1 f_3+\gamma_1 f_2},$$

and similarly for P_2 and Q_2. Expressing the condition of integrability of (43) and making use of the same in the reduction, we have

$$\left(\frac{\partial P_1}{\partial u}-\frac{\partial P_2}{\partial t}\right)f_1+\left(P_1 Q_2-P_2 Q_1+\frac{\partial Q_1}{\partial u}-\frac{\partial Q_2}{\partial t}\right)=0,$$

or for the sake of brevity,

$$(44)\qquad L_1 f_1+M_1=0.$$

Three cases arise for consideration according as f_2 and f_3 do or do not satisfy the two partial differential equations

$$L_1=0, \qquad M_1=0.$$

* The cases where any of the coefficients of the differential quotients in the above equation vanish are of no interest and will be excluded.

If both of these equations are satisfied, we have simply to determine f_1 from (43) by quadratures and then the tangents to the congruences C_t and C_u form linear complexes.

If $M_1 = 0$, but $L_1 \neq 0$, the complexes are not linear for any expression of f_1; also for the case $L_1 = 0$, $M \neq 0$.

Finally, if both L_1 and M_1 are not zero, the necessary and sufficient condition that the two complexes be linear is that f_2 and f_3 satisfy the partial differential equations of the third order

$$\frac{\partial}{\partial t}\frac{M_1}{L_1} = \frac{P_1 M_1}{L_1} - Q_1, \qquad \frac{\partial}{\partial u}\frac{M_1}{L_1} = \frac{P_2 M_1}{L_1} - Q_2;$$

and in this case we take $-M_1/L_1$ for f_1.

In order that all three complexes be linear we must have (43) and

$$\frac{\partial f_1}{\partial v} = P_3 f_1 + Q_3,$$

where the forms of P_3 and Q_3 are readily found. Now the conditions of integrability are (44) and two other equations of similar form which we shall write

$$L_2 f_1 + M_2 = 0, \qquad L_3 f_1 + M_3 = 0.$$

Considerations similar to the preceding show that for all three complexes to be linear the functions f_2 and f_3 must satisfy six partial differential equations. Since the above results hold alike for f_1, f_2 and f_3, we have the theorem:

Associated with every triple congruence of curves there are three complexes of straight lines formed by the tangents to these curves. For one of these complexes to be linear two of the functions f_1, f_2, f_3 may be chosen arbitrarily and the third is found by quadratures; for two of them to be linear it is necessary and sufficient that two of the functions satisfy partial differential equations of the second or third orders and the third function is determined by quadratures or directly; and, finally, for all three to be linear two of the functions must satisfy six partial differential equations of the second or third orders and the third follows as above.

PRINCETON, *January*, 1903.