DETERMINATION OF ALL THE SUBGROUPS OF THE KNOWN SIMPLE GROUP OF ORDER 25920*

BY

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Introduction.

The trisection of the periods of hyperelliptic functions of four periods, the determination of the 27 lines on a general cubic surface, and the reduction of a binary sextic to the canonical form $T^2 - U^3$, although apparently unrelated, are not essentially distinct problems from the standpoint of group-theory,† since each is readily reduced to the solution of an algebraic equation whose Galois group is the same simple group of order 25920. This equation has been shown to possess resolvents of degrees 27, 36, 40 (two essentially distinct ones), and 45, but none of degree < 27. The last result was established by JORDAN‡ by an elaborate discussion based on GALOIS's theory of algebraic equations. This result is reëstablished in the present paper, which employs only pure group-theory. All the results mentioned follow from the fundamental theorem (not stated heretofore) *that all the maximal subgroups of the simple* G_{25920} *are conjugate with* G_{960}, G_{720}, G_{648}, H_{648}, *or* G_{576} (§ 70). These five groups, appearing in different notations, play a fundamental rôle in the memoirs of WITTING and BURKHARDT on the geometric and function-theoretic phases of the subject.

Not only in the applications, but also in the theory of groups, the known simple group of order 25920 is of frequent occurrence. In the papers by

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† JORDAN, *Traité des substitutions* (1870), pp. 316–329, 365-369, 666-667; Comptes Rendus (1870), pp. 326–328, 1028; KLEIN, Journal de Mathématiques, ser. 4, vol. 4 (1888), pp. 169–176; WITTING, Mathematische Annalen, vol. 29 (1887), pp. 157–170; MASCHKE, *ibid.*, vol. 33 (1889), pp. 317–344; BURKHARDT, *ibid.*, vol. 35 (1890), pp. 198–296; vol. 38 (1891), pp. 161–224; vol. 41 (1893), pp. 313–343; DICKSON, Comptes Rendus, vol. 128 (1899), pp. 873–5; *Linear Groups* (1901), pp. 303–7.

‡ *Traité*, pp. 319–329. As only typical cases are there treated, much is left for the reader to supply; the case $d = 9$ is not mentioned.

WITTING, MASCHKE, KLEIN, and BURKHARDT, it appears as a quaternary, as a quinary, and as a senary group of linear substitutions with numerical coefficients. Furthermore, it appears in the papers by JORDAN and the writer* as a quaternary abelian group G_{25920} modulo 3, as a quinary orthogonal group O_{25920} modulo 3, as a senary hypoabelian group modulo 2, and as a quaternary hyperabelian group in the $GF[2^2]$. Two sets of generational relations for it have been given, with respectively G_{720} and G_{960} in the foreground.*

Use is made of the results in the following papers by the writer:

I. *Canonical forms of quaternary abelian substitutions in an arbitrary Galois field*, Transactions, vol. 2 (1901), pp. 103–138.

II. *On the subgroups of order a power of p in the quaternary abelian group in the Galois field of order p^n*, Transactions, vol. 4 (1903), pp. 371–386.

III. *The subgroups of order a power of 2 of the simple quinary orthogonal group in the Galois field of order $p^n = 8l \pm 3$*, Transactions, vol. 5 (1904), pp. 1–38.

IV. *Determination of all groups of binary linear substitutions with integral coefficients taken modulo 3 and of determinant unity*, Annals of Mathematics, second series, vol. 5 (1903–4).

V. *Two systems of subgroups of the quaternary abelian group in a general Galois field*, Bulletin of the American Mathematical Society, second series, vol. 10 (1904), pp. 178–184.

To these reference will be made by the corresponding Roman numeral with a subscript to indicate the page. Thus III_8 denotes page 8 of the third paper.

In the treatment of possible subgroups of certain orders in the interval 144–1728, use is made of the papers † by HÖLDER, COLE, LING and MILLER, in which is determined the simplicity or compositeness of all groups of orders < 2000.

Chiefly in the duplicate proofs, use is made of the lists of all transitive substitution-groups of a given degree, with the following reference numbers:

COLE, 1, Bulletin of the American Mathematical Society, ser. 1, vol. 2, (1903), pp. 250–8.

COLE, 2, Quarterly Journal of Mathematics, vol. 27 (1895), pp. 39–50.

MILLER, 1, Quarterly Journal, vol. 28 (1896), pp. 193–231.

MILLER, 2, Proceedings of the London Mathematical Society, vol. 28 (1897), pp. 533–544.

* DICKSON, Proceedings of the London Mathematical Society, vol. 31 (1899), pp. 30–68; vol. 32 (1900), pp. 3–10.

† For references, see American Journal, vol. 22 (1900), p. 13. It may be noted that the orders $792 = 2^3 \cdot 3^2 \cdot 11$ and $1008 = 2^4 \cdot 3^2 \cdot 7$ were overlooked by BURNSIDE.

MILLER, 3, American Journal of Mathematics, vol. 20 (1898), pp. 229–241.

KUHN, Manuscript list of the imprimitive groups of degree 15.

The following table (referred to as "the table") gives the 114 types of non-conjugate subgroups, other than itself and identity, of G_{25920}, a page reference to their definition, the largest subgroup of G_{25920} in which a given subgroup is self-conjugate, and the number of conjugates within G to a given type.

Group.	Def.	Self-conj. only under.	Numb. conjs.	Group.	Def.	Self-conj. only under.	Numb. conjs.
G_2	III_5	G_{576}	45	K_9^*	II_{382}	H_{108}	240
G_2'	III_6	H_{96}	270	K_9^{**}	II_{383}	H_{216}	120
C_3	130	G_{648}	40	G_{10}	140	G_{20}	1296
C_3'	130	H_{216}	120	C_{12}	134	C_{24}	1080
C_3''	130	H_{108}	240	K_{12}	135	G_{24}^*	1080
C_4^3	III_{18}	G_{96}	270	D_{12}'	135	K_{36}^*	720
C_4^5	III_{18}	G_{16}'	1620	D_{12}^*	135	G_{24}^*	1080
G_4^2	III_4	G_{64}	405	G_{12}	136	G_{24}^*	1080
K_4'	III_6	H_{96}	270	G_{12}	136	G_{48}	540
K_4'''	III_6	H_{96}	270	G_{12}'	136	H_{96}	270
K_4^*	III_{15}	G_{48}	540	G_{16}	III_3	G_{960}	27
G_5	139	G_{20}	1296	G_{16}'	III_7	J_{32}^3	810
C_6	133	G_{72}	360	H_{16}'	III_7	J_{32}^3	810
C_6'	133	K_{36}^*	720	H_{16}^3	III_5	G_{64}	405
C_6''	133	K_{36}^*	720	J_{16}^3	III_5	G_{64}	405
C_6^*	133	G_{24}^*	1080	F_{16}	III_{10}	G_{96}	270
D_6	133	G_{36}	720	K_{18}	137	K_{54}	480
D_6'	133	H_{108}	240	K_{18}^*	137	K_{36}^*	720
D_6''	133	K_{36}^*	720	K_{18}^{**}	137	H_{108}	240
G_8	III_5	G_{192}	135	G_{18}^*	137	H_{108}	240
G_8''	III_7	H_{96}	270	H_{18}^*	137	K_{36}^*	720
G_8^3	III_4	H_{192}	135	H_{18}^{**}	137	H_{216}	120
H_8^3	III_4	G_{64}	405	G_{18}^{**}	137	G_{36}^{**}	720
K_8	III_6	J_{32}^3	810	G_{20}	140	G_{20}	1296
J_8	III_{12}	G_{32}	810	C_{24}	135	G_{72}	360
F_8'''	III_{13}	G_{288}	90	G_{24}^*	133	G_{24}^*	1080
L_8	III_{15}	G_{16}'	1620	G_{24}^3	141	G_{48}	540
T_8	III_{15}	G_{16}'	1620	L_{24}	141	G_{48}	540
C_9	II_{385}	H_{27}	960	T_{24}	141	G_{48}	540
K_9	II_{382}	G_{162}	160	G_{24}''	142	H_{96}	270

Group.	Def.	Self-conj. only under.	Numb. conjs.	Group.	Def.	Self-conj. only under.	Numb. conjs.
G_{24}	142	H_{48}	540	G^{**}_{72}	148	G^{**}_{72}	360
L^{*}_{24}	142	G''_{48}	540	G_{80}	157	G_{160}	162
F_{24}	144	G'_{72}	360	G_{81}	II_{372}	G_{162}	160
F'_{24}	144	G'_{72}	360	G_{96}	150	G_{288}	90
F^{*}_{24}	144	G_{288}	90	J_{96}	157	G_{576}	45
G_{27}	II_{377}	G_{648}	40	L_{96}	157	G_{576}	45
H_{27}	II_{377}	G_{81}	320	H_{96}	136	H_{96}	270
K_{27}	II_{377}	H_{648}	40	G_{108}	158	G_{216}	120
G_{32}	III_{5}	G_{576}	45	H_{108}	130	H_{648}	40
J^{3}_{32}	III_{5}	G_{64}	405	K'_{108}	158	H_{216}	120
H^{3}_{32}	III_{5}	G_{64}	405	K''_{108}	158	H_{216}	120
K^{*}_{36}	133	K^{*}_{36}	720	G_{120}	153	G_{120}	216
K^{**}_{36}	147	H_{216}	120	G'_{120}	153	G'_{120}	216
G^{**}_{36}	138	G^{**}_{72}	360	G_{160}	157	G_{160}	162
H^{**}_{36}	147	G^{**}_{72}	360	G_{162}	II_{373}	G_{162}	160
G_{48}	136	G_{48}	540	G_{192}	III_{21}	G_{192}	135
G''_{48}	143	H_{96}	270	H_{192}	161	G_{576}	45
H_{48}	143	H_{96}	270	G_{216}	159	G_{648}	40
H''_{48}	149	H_{96}	270	H_{216}	131	H_{216}	120
F_{48}	150	G_{96}	270	G_{288}	147	G_{576}	45
G_{54}	151	G_{648}	40	H_{324}	163	H_{648}	40
K_{54}	138	H_{108}	240	G_{360}	163	G_{720}	36
K'_{54}	151	H_{216}	120	G_{576}	III_{21}	G_{576}	45
G_{60}	III_{3}	G_{120}	216	G_{648}	II_{372}	G_{648}	40
G'_{60}	153	G'_{120}	216	H_{648}	II_{380}	H_{648}	40
G_{64}	III_{3}	G_{192}	135	G_{720}	163	G_{720}	36
G_{72}	133	G_{72}	360	G_{960}	III_{3}	G_{960}	27

Possible orders of subgroups of G.

1. The number of divisors of $25920 = 2^6 \cdot 3^4 \cdot 5$ is $(6+1)(4+1)(1+1) = 70$. These divisors (of which 30 are < 100) are

1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 27, 30, 32, 36, 40, 45, 48, 54, 60, 64, 72, 80, 81, 90, 96, 108, 120, 135, 144, 160, 162, 180, 192, 216, 240, 270, 288, 320, 324, 360, 405, 432, 480, 540, 576, 648, 720, 810, 864, 960, 1080, 1296, 1440, 1620, 1728, 2160, 2592, 2880, 3240, 4320, 5184, 6480, 8640, 12960, 25920.

Orders immediately excluded.

2. THEOREM. *The group G contains no subgroup of one of the following 16 orders:* 15, 30, 40, 45, 90, 135, 240, 270, 405, 540, 1440, 3240, 4320, 5184, 6480, 8640, 12960.

By SYLOW's theorem, a group of order 15, 40, 45 or 135 contains a single (and hence self-conjugate) subgroup of order 5. But, within G, a Γ_5 is self-conjugate only under a subgroup of Γ_{20} by I_{138}. Likewise, a Γ_{90} contains 1 or 6 conjugate Γ_5; Γ_{270} contains 1 or 6 conjugate Γ_5; Γ_{240} contains 1, 6, or 16; Γ_{540} contains 1, 6, or 36; Γ_{1440} contains 1, 6, 16, 36, or 96; but in no case is the quotient of the order of the group by the number of conjugates a divisor of 20.

By SYLOW's theorem, a Γ_{30} contains 1 or 6 conjugate Γ_5 and 1 or 10 conjugate Γ_3. But any Γ_5 is self-conjugate in at most a Γ_{20}; while a Γ_3 is self-conjugate only under a subgroup of order 648, 216, or 108, (by I_{138} or § 5 below). Hence a subgroup Γ_{30} must contain 6 conjugate Γ_5 and 10 conjugate Γ_3 and hence at least $1 + 24 + 20$ operators.*

A Γ_{405} contains a single Γ_{81} by SYLOW's theorem, whereas within G any Γ_{81} is self-conjugate only under a group of order 162 by II_{373}.

The final six orders for subgroups are excluded since their indices under G are $\leqq 8$, while 25920 does not divide $8!$.

The subgroups of order 3.

3. THEOREM. *Within G, the cyclic subgroups of order 3 fall into 3 distinct sets of conjugate subgroups, representatives of which are*

$$(1) \qquad C_3 = (L_{1,1}), \qquad C_3' = (L_{1,1} L_{2,1}), \qquad C_3'' = (L_{1,-1} L_{2,1}).$$

They are self-conjugate only under G_{648}, H_{216} and H_{108}, respectively.

The theorem follows from I_{138} except as to the characterization of the groups of order 648, 216, 108. The operators of G commutative with $L_{1,1}$ form G_{648} of II_{372}, while $L_{1,1}$ is not conjugate with its inverse by I_{138}. In view of I_{115}, the only homogeneous abelian substitutions commutative with $L_{1,-1} L_{2,1}$ are $U = [k, 0, c, \gamma] T_{2,\pm 1}$. Those transforming $L_{1,-1} L_{2,1}$ into its inverse are $V = UP_{12}$. Hence

$$(2) \qquad H_{108} = \{K_{27}, T_{2,-1}, P_{12}\} \equiv \{[k, 0, c, \gamma], T_{2,-1}, P_{12}\}.$$

Also, by I_{115}, those commutative with $L_{1,1} L_{2,1}$ are the U and V; while those transforming $L_{1,1} L_{2,1}$ into its inverse are seen to be

* Otherwise excluded since Γ_{30} contains a cyclic C_{15}, HÖLDER, Mathematische Annalen, Bd. 43 (1893), p. 412.

$$(3)\qquad \pm\begin{bmatrix} 1 & \gamma_{11} & -\alpha_{21}\alpha_{22} & \gamma_{12} \\ 0 & -1 & 0 & \alpha_{21}\alpha_{22} \\ \alpha_{21} & \gamma_{21} & \alpha_{22} & \gamma_{22} \\ 0 & -\alpha_{12} & 0 & -\alpha_{22} \end{bmatrix}\qquad \begin{array}{l} \alpha_{21}^2 \equiv \alpha_2^2 \equiv 1, \\ \gamma_{21} \equiv \gamma_{11}\alpha_{21} + \gamma_{12}\alpha_{22} + \gamma_{22}\alpha_{21}\alpha_{22}. \end{array}$$

Hence the 108 operators (3) together with those of H_{108} form * H_{216}.

Conjugacies among the operators of H_{108}, and among those of H_{216}.

4. By II_{372}, $U^2 = [-k, 0, c \pm c, -\gamma]$, whence $U^6 = I$. Now $U^2 = U^{-1}$ if and only if the upper signs hold, namely, $U = [k, 0, c, \gamma]$. Hence of the operators U, $[k, 0, c, \gamma]$ is of period 3 if not the identity $[0, 0, 0, 0]$; $[0, 0, c, 0]\,T_{2,-1}$ is of period 2; $[k, 0, c, \gamma]\,T_{2,-1}$ is of period 6 if k and γ are not both $\equiv 0$. The operators $[k, 0, c, \gamma]$ are all commutative by II_{377}. Now I, P_{12}, $T_{2,-1}$ and $P_{12}T_{2,-1}$ transform $[k, 0, c, \gamma]$ into respectively

$$[k, 0, c, \gamma], \qquad [\gamma, 0, c, k], \qquad [k, 0, -c, \gamma], \qquad [\gamma, 0, -c, k].$$

Of these four, those which are distinct form a complete set of conjugates under H_{108}. Next, the operators of period 2 are all conjugate with $T_{2,-1}$, since $T_{2,-1}$ and $[0, 0, -1, 0]$ transform $[0, 0, 1, 0]\,T_{2,-1}$ into $[0, 0, -1, 0]\,T_{2,-1}$ and $T_{2,-1}$ respectively. Consider finally the operators U of period 6. Applying (12) of II_{377}, we find that $[0, 0, -c, 0]$ transforms $[k, 0, c, \gamma]\,T_{2,-1}$ into $[k, 0, 0, \gamma]\,T_{2,-1}$. The latter is transformed into $[\gamma, 0, 0, k]$ by P_{12}.

For $V = [k, 0, c, \gamma]\,T_{2,\pm 1}P_{12}$, we have

$$V^2 = [k + \gamma, 0, c \pm c, k + \lambda].$$

Hence V is of period 2 or 6. Those of period 2 are $[-\gamma, 0, 0, \gamma]\,P_{12}$ and $[-\gamma, 0, c, \gamma]\,T_{2,-1}P_{12}$. The former is transformed into P_{12} by $[0, 0, 0, \gamma]$, the latter into $[0, 0, c, 0]\,T_{2,-1}P_{12} \equiv S_c$ by $[0, 0, 0, \gamma]$. But $[0, 0, -c, 0]$ transforms S_c into $S_0 = P_{12}T_{2,-1}$. The operators V of period 6 are $[k, 0, c, \gamma]\,T_{2,-1}P_{12}$, $k + \gamma \not\equiv 0$, and $[k, 0, c, \gamma]\,P_{12}$, $k + \gamma$ and c not both 0. The first is transformed into $[k + \gamma, 0, 0, 0]\,T_{2,-1}P_{12}$ by $[0, 0, -c, \gamma]$; the second into $[k + \gamma, 0, c, 0]\,P_{12}$ by $[0, 0, 0, \gamma]$.

THEOREM.† *The operators of H_{108} are of period 1, 2, 3, or 6. Those of period 1 or 3 are $[k, 0, c, \gamma]$ and are commutative. Those of period 2*

* By II_{383}, H_{216} is the largest subgroup transforming K_9^{**} into itself.

† For an ultimate classification, not needed here, we note that the operators of period 3 fall into the following distinct sets of conjugates:

$$\{[\gamma, 0, 0, \gamma]\}, \quad \{[\gamma, 0, 1, \gamma], [\gamma, 0, -1, \gamma]\}, \quad \{[k, 0, 0, \gamma], [\gamma, 0, 0, k]\}, \quad \{[k, 0, 1, \gamma], [k, 0, -1, \gamma], [\gamma, 0, 1, k], [\gamma, 0, -1, k]\} \quad (k \not\equiv \gamma).$$

The operators of period 6 are conjugate with $[\pm 1, 0, 0, 0]T_{2,-1}$, $[\pm 1, 0, 0, 1]\,T_{2,-1}$, $[-1, 0, 0, -1]\,T_{2,-1}$, $[\pm 1, 0, 0, 0]\,P_{12}$, or $[k', 0', 1, 0]\,P_{12}$, no two of which are conjugate.

are conjugate within H_{108} *with* $T_{2,-1}$, P_{12}, *or* $P_{12}T_{2,-1}$. *Those of period* 6 *are conjugate within* H_{108} *with*

$$[k,0,0,\gamma]\,T_{2,-1},\ [\pm 1,0,0,0]\,T_{2,-1}P_{12},\ [k,0,\gamma,0]P_{12}\quad (k,\gamma \text{ not both} \equiv 0).$$

5. Operator (3) is transformed by $T_{2,-1}$ into a similar operator with α_{21} replaced by $-\alpha_{21}$. We therefore take $\alpha_{21}\equiv +1$. Let first $\alpha_{22}\equiv -1$. The resulting operator (3) is transformed by $L_{1,-\gamma_{11}}L_{2,\gamma_{22}}$ into

$$W_\gamma = \pm\begin{bmatrix} 1 & 0 & 1 & \gamma \\ 0 & -1 & 0 & -1 \\ 1 & -\gamma & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\qquad (\gamma \equiv \gamma_{21}-\gamma_{12}).$$

It is of period 2 if and only if $\gamma\equiv 0$. Now $P_{12}T_{2,-1}$ transforms W_1 into W_{-1}; while $W_1^2=[-1,0,1,1]$ is of period 3 and differs from W_1^{-1}. We obtain therefore two reduced forms: W_0 of period 2 and W_1 of period 6. Let next $\alpha_{22}\equiv +1$. The resulting operator (3) is transformed by $L_{1,-\gamma_{11}}L_{2,-\gamma_{22}}$ into

$$Z_\delta \equiv \pm\begin{bmatrix} 1 & 0 & -1 & \delta \\ 0 & -1 & 0 & 1 \\ 1 & \delta & 1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix}\qquad (\delta \equiv -\gamma_{12}-\gamma_{21}).$$

The latter is transformed into $Z_0\equiv T_{2,-1}W_0$ by the operator $[0,0,-\delta,0]$ of H_{216}. Now $Z_0^2=P_{12}T_{2,-1}$, since W_0 transforms $T_{2,-1}$ into P_{12}. Hence Z_0 is of period 4 in the quotient-group G.

Consider finally the conjugacy of the operators of H_{108} under the group H_{216}. Now W_0 transforms P_{12} into $T_{2,-1}$. Again, W_0 transforms $[k,0,c,\gamma]$ into $[k+\gamma-c,0,k-\gamma,k+\gamma+c]$. Hence W_0 transforms $[k,0,0,\gamma]\,T_{2,-1}$ into $[k+\gamma,\ 0,\ k-\gamma,\ k+\gamma]P_{12}$. The latter is transformed into $[-k-\gamma,0,k-\gamma,0]P_{12}$ by $[0,0,0,k+\gamma]$. Now $-k-\gamma$ and $k-\gamma$ are not both 0 if k and γ are not.

THEOREM. *The operators of* H_{216} *are of period* 1, 2, 3, 4 *or* 6. *Those of period* 1 *or* 3 *are* $[k,0,c,\gamma]$. *Within* H_{216}, *those of period* 2 *are conjugate with* $P_{12}T_{2,-1}$, $T_{2,-1}$ *or* W_0; *those of period* 4 *are conjugate with* $Z_0\equiv T_{2,-1}W_0$; *those of period* 6 *are conjugate with* W_1, $[k,0,0,\gamma]\,T_{2,-1}$, *or* $[\pm 1,0,0,0]\,T_{2,-1}P_{12}$, *where* k *and* γ *are not both* 0.

The subgroups of order 6.

6. THEOREM. *Within* G, *the cyclic subgroups of order* 6 *fall into* 4 *distinct sets of conjugate subgroups, representations of which are*

$$(4)\qquad \begin{aligned} C_6 &= (L_{1,1}T_{1,-1}), & C_6' &= (L_{1,1}L_{2,-1}T_{1,-1}), \\ C_6'' &= (L_{1,1}L_{2,1}T_{1,-1}), & C_6^* &= (P_{12}L_{1,-1}T_{1,-1}). \end{aligned}$$

They are self-conjugate only under G_{72}, K_{36}^*, K_{36}^* *and* G_{24}^*, *respectively:*

$$(5)\qquad G_{72} = \left\{ \pm \begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & \beta & \delta \end{bmatrix}, \quad (\alpha\delta - \beta\gamma \equiv 1) \right\},$$

$$(6)\ K_{36}^* = (K_9^*, T_{2,-1}, P_{12}) = \{[k, 0, 0, \gamma]R, (R = I, T_{2,-1}, P_{1,2}, T_{2,-1}P_{12})\},$$

$$(7)\qquad G_{24}^* = \{C, CT_{2,-1}, CD, CT_{2,-1}D, (C \text{ ranging over } C_6^*)\},$$

$$(7_a)\qquad D = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The theorem follows from I_{138} except as to the characterization of the groups of order 72, 36, 24. By I_{116}, $L_{1,1}T_{1,-1}$ is commutative only with the substitutions of G_{72}. By I_{115}, $L_{1,1}T_{1,-1}$ is not conjugate with its inverse. By I_{116}, $L_{1,1}L_{2,-1}T_{1,-1}$ is commutative only with $[\gamma, 0, 0, \gamma_{22}]T_{2,\pm 1}$; it is transformed into its inverse by P_{12}. Also $L_{1,1}L_{2,1}T_{1,-1}$ is commutative only with $[\gamma, 0, 0, \gamma_{22}]T_{2,\pm 1}$, while it is transformed into $L_{1,1}L_{2,1}T_{2,-1}$, the same as the former in the quotient-group G, by P_{12}. Finally, $P = P_{12}L_{1,-1}T_{1,-1}$ is commutative only with its powers and their products by $T_{2,-1}$, which transforms P into $PT_{1,-1}T_{2,-1}$. The only homogeneous substitutions transforming P into P^{-1} are found to be the 6 operators

$$(8)\qquad \pm \begin{bmatrix} 1 & \gamma & \pm 1 & \pm 1 \pm \gamma \\ 0 & -1 & 0 & \mp 1 \\ \pm 1 & \pm 1 \pm \gamma & -1 & 1 - \gamma \\ 0 & \mp 1 & 0 & 1 \end{bmatrix},$$

which may be written as the products CD, C ranging over C_6^*.

7. Theorem. *Within* G, *the non-cyclic groups of order* 6 *fall into* 3 *distinct sets of conjugate subgroups, representatives of which are*

$$(9)\ D_6 = (L_{1,1}L_{2,1}, W_0),\ D_6' = (L_{1,-1}L_{2,1}, P_{12}),\ D_6'' = (L_{1,-1}L_{2,1}, P_{12}T_{2,-1}).$$

They are self-conjugate only under G_{36}, H_{108}, K_{36}^*, *respectively, where*

(10) $G_{36} = \{ [k, 0, c, k-c]\,\Gamma,\ (\Gamma = I,\ T_{2,-1}P_{12},\ W_0,\ T_{2,-1}P_{12}W_0) \}$.

The subgroups sought are dihedron $G_{2\cdot 3}$ generated by A and B with

$$(11) \qquad A^3 = I, \qquad B^2 = I, \qquad BAB = A^{-1}.$$

We may assume that A is $L_{1,1}$, $L_{1,1}L_{2,1}$ or $L_{1,-1}L_{2,1}$ (§ 3). But $L_{1,1}$ is excluded since it is not conjugate with its inverse within G.

The cyclic group generated by $A = L_{1,-1}L_{2,1}$ is self-conjugate only under H_{108} (§ 3), whose operators of period 2 are conjugate with $T_{2,-1}$, P_{12}, or $P_{12}T_{2,-1}$ (§ 4). The last two transform A into A^{-1}, but $T_{2,-1}$ transforms A into itself and is excluded.

The cyclic group generated by $A = L_{1,1}L_{2,1}$ is self-conjugate only under H_{216} (§ 4), whose operators of period 2 are conjugate with $P_{12}T_{2,-1}$, $T_{2,-1}$, or W_0 (§ 5). The first two transform A into itself and are excluded, while W_0 transforms A into A^{-1}.

An operator which transforms D_6' into itself must transform the subgroup $(L_{1,-1}L_{2,1})$ into itself and hence belong to H_{108}. Moreover, it must transform P_{12} into one of the 3 operators $L_t = L_{1,-t}L_{2,t}P_{12}$ of period 2 in D_6'. Now $U \equiv [k, 0, c, \gamma]\,T_{2,\pm 1}$ transforms P_{12} into $L_{k-\gamma}$; while $V \equiv [k, 0, c, \gamma]\,T_{2,\pm 1}P_{12}$ transforms P_{12} into $L_{\gamma-k}$. Hence H_{108} transforms D_6' into itself.

For D_6'', we seek the operators of H_{108} which transform $P_{12}T_{2,-1}$ into one of the 3 operators $M_t = L_{1,-t}L_{2,t}P_{12}T_{2,-1}$ of period 2 in D_6''. For U the conditions are $c \equiv 0$, $t \equiv k-\gamma$; for V the conditions are $c \equiv 0$, $t \equiv \gamma - k$. Hence D_6'' is self-conjugate only under the group (6).

For D_6, we seek the operators of H_{216} which transform W_0 into one of the 3 operators $N_t \equiv L_{1,t}L_{2,t}W_0$ of period 2 in D_6. For U the conditions are $\pm 1 \equiv +1$, $k \equiv c + \gamma$, $t \equiv -k-\gamma$; for V the conditions are $\pm 1 \equiv -1$, $c \equiv k - \gamma$, $t \equiv -k-\gamma$. For (3) the conditions are

for the upper signs: $\alpha_{22} \equiv -1,\ \alpha_{21} \equiv 1,\ \gamma_{22} \equiv -\gamma_{11},\ t \equiv \gamma_{12} + \gamma_{21} - \gamma_{11}$.

for the lower signs: $\alpha_{22} \equiv -1,\ \alpha_{21} \equiv -1,\ \gamma_{12} \equiv \gamma_{21},\ t \equiv \gamma_{22} - \gamma_{21} - \gamma_{11}$;

The resulting operators (3) are respectively

$$[\gamma_{12}, 0, \gamma_{12} - \gamma_{21}, \gamma_{21}]\,W_0, \qquad [-\gamma_{22}, 0, -\gamma_{22} - \gamma_{11}, \gamma_{11}]\,T_{2,-1}P_{12}W_0.$$

Hence D_6 is self-conjugate only under the group (10).

The subgroups of order 12.

8. THEOREM. *Within G, every cyclic subgroup of order* 12 *is conjugate with*

$$(12) \qquad C_{12} = (M_2 L_{1,1}).$$

The latter is self-conjugate only under the group

$$(13) \qquad C_{24} = (C_{12}, A), \qquad A:\ \xi_2' = \xi_2 + \eta_2,\ \eta_2' = \xi_2 - \eta_2.$$

For proof, see I_{138}. Note that $A^2 = (M_2 L_{1,1})^6 = T_{2,-1}$.

9. THEOREM. *Within G, every non-cyclic commutative subgroup of order 12 is conjugate with K_{12}, every dihedral subgroup of order 12 is conjugate with D'_{12} or else with D^*_{12}, where*

$$(14) \qquad K_{12} = (C^*_6, T_{2,-1}), \qquad D'_{12} = (C'_6, P_{12}), \qquad D^*_{12} = (C^*_6, D).$$

They are self-conjugate only under G^*_{24}, K^*_{36} and G^*_{24}, respectively.

A non-cyclic commutative Γ_{12} or a dihedral Γ_{12} contains a self-conjugate cyclic Γ_6 and an operator of period 2 not in the latter. By § 6 we may take C_6, C'_6, C''_6 or C^*_6 as the Γ_6.

Since C_6 is self-conjugate only under G_{72}, which contains a single operator T_{2-1} of period 2, it is to be excluded.

The groups C'_6 and C''_6 are each self-conjugate only under K^*_{36}, which contains exactly 7 operators of period 2, viz.,

$$T_{2,-1}, \qquad L_{1,-\gamma} L_{2,\gamma} P_{12}, \qquad L_{1,-\gamma} L_{2,\gamma} T_{2,-1} P_{12}.$$

Since $T_{2,-1}$ and $L_{1,-\gamma} L_{2,\gamma}$ lie in C'_6, we may limit the extender to P_{12} and thus obtain D'_{12}. Since $T_{2,-1}$ lies in C''_6, we may limit the extender to $L_{1,-\gamma} L_{2,\gamma} P_{12}$. But $L_{2,\gamma}$ transforms the latter into P_{12} and transforms C''_6 into itself. There results $K''_{12} = (C''_6, P_{12})$. But $P_{12} L_{2,-1}$ transforms K_{12} into K''_{12} since it transforms $P_{12} L_{1,-1} T_{1,-1}$ and $T_{2,-1}$ into the generators $P_{12} L_{1,1} L_{2,1} T_{2,-1}$ and $T_{1,-1}$ of (C''_6, P_{12}).

Finally, C^*_6 is self-conjugate only under G^*_{24}. The only operators of the latter commutative with $P \equiv P_{12} L_{1,-1} T_{1,-1}$ are those of K_{12}; the remaining operators $P^i D$ and $P^i T_{2,-1} D\,(i = 1, \cdots, 6)$ transform P into P^{-1} (§ 6), the first 6 being of period 2 and the last 6 of period 4.

A dihedral Γ_{12} contains a single cyclic Γ_6. Hence D^*_{12} is self-conjugute only under G^*_{24}. Likewise, a substitution commutative with D'_{12} must belong to K^*_{36}. But all the operators of period 2 in the latter belong to D'_{12}. Hence D'_{12} is self-conjugate under K^*_{36}.

Since K_{12} contains the single self-conjugate cyclic group $(L_{1,1} L_{2,1})$ of order 3, an operator transforming K_{12} into itself must belong to H_{216} (§ 3). Since K_{12} contains the single self-conjugate group

$$(15) \qquad \{I, \quad T_{2,-1}, \qquad P_{12} L_{1,1} L_{2,-1} T_{1,-1}, \qquad P_{12} L_{1,1} L_{2,-1}\}$$

of order 4, an operator transforming K_{12} into itself must belong to a group of order 96 or 64 by III, or by the table. Hence K_{12} is transformed into itself by at most 24 operators. But K_{12} is a subgroup of G^*_{24} and hence self-conjugate under it.

10. THEOREM. *Every subgroup simply isomorphic with the group generated by the two operators R and S subject to the generational relations*

$$(16)\qquad R^6 = I, \qquad S^2 = R^3, \qquad S^{-1}RS = R^{-1}$$

is conjugate within G with the group

$$(17)\qquad G_{12}^* = \{P^i,\ P^i T_{2,-1} D,\quad (P \equiv P_{12} L_{1,-1} T_{1,-1},\ i = 1, \cdots, 6)\}.$$

Within G, G_{12}^* is self-conjugate only under G_{24}^*.

The self-conjugate Γ_6 may be taken to be C_6, C_6', C_6'', or C_6^*. Now C_6' and C_6'' are self-conjugate only under K_{36}^*, which contains no operator of period 4, and hence are excluded.

The group C_6 is self-conjugate only under G_{72}. But an operator (5) is of period 4 if and only if $k \equiv 0$, $\alpha + \delta \equiv 0$ (IV), when it becomes

$$(18)\qquad \xi_1' = \xi_1,\ \eta_1' = \eta_1,\ \xi_2' = \alpha\xi_2 + \gamma\eta_2,\ \eta_2' = \beta\xi_2 - \alpha\eta_2 \quad (-\alpha^2 - \beta\gamma \equiv 1).$$

The Γ_{12} must contain 6 operators of period 4 and hence contain every (18). But $L_{1,\lambda} T_{2,\pm 1}$ (18) is neither in C_6 nor of the form (18) if $\gamma \neq 0$, $\beta \neq 0$. Hence also C_6 is to be excluded.

Finally, C_6^* is self-conjugate only under G_{24}^*. Now $P^i T_{2,-1} D$ and no further operators of G_{24}^* are of period 4, where $P \equiv P_{12} L_{1,-1} T_{1,-1}$, since

$$(P^i T_{2,-1} D)^2 = P_{12} L_{1,1} T_{2,-1} L_{2,-1} = P^3.$$

By § 6, $P^i T_{2,-1} D$ transforms P into P^{-1}. Hence G_{12}^* satisfies the conditions.

11. THEOREM. *Within O, every subgroup simply isomorphic with the alternating group on 4 letters is conjugate with one of the two groups*

$$(19)\qquad \begin{aligned} G_{12} = \{I,\ C_1C_3,\ B_3,\ B_3C_1C_3,\ C_2C_4B_iW,\ C_1C_2C_3C_4B_iW,& \\ B_iC_2C_3W^2,\ B_iC_3C_4W^2,\ (i = 2, 4)\},& \end{aligned}$$

$$(20)\qquad G_{12}' = \{\Gamma,\ \Gamma(\xi_2\xi_4\xi_5),\ \Gamma(\xi_2\xi_5\xi_4),\ (\Gamma = I,\ C_2C_4,\ C_2C_5,\ C_4C_5)\}.$$

They are self-conjugate only under the respective groups

$$(21)\qquad G_{48} = \{\Gamma,\ \Gamma B_2 W,\ \Gamma W^2 B_2 \equiv \Gamma B_2 C_1 C_2 W^2,\ (\Gamma \text{ ranging over } G_{16}')\},$$

$$(22)\qquad \begin{aligned} H_{96} = \{\Gamma,\ B_3\Gamma,\ (\xi_1\xi_3)(\xi_4\xi_5)\Gamma,\ (\xi_1\xi_3)(\xi_2\xi_5)\Gamma,\ (\xi_2\xi_4\xi_5)\Gamma,\ (\xi_2\xi_5\xi_4)\Gamma,& \\ (\Gamma \text{ ranging over } G_{16})\}.& \end{aligned}$$

For the self-conjugate *four-group*, we may take G_4^2, K_4', K_4''' or K_4^*. Within O, these are self-conjugate only under G_{64}, H_{96}, H_{96}, G_{48}, respectively. Hence G_4^2 is excluded. The only operators of period 3 in H_{96} are the last 8 operators (20). They must therefore all occur in the group of order 12. They extend K_4' to G_{12}' and K_4''' to H_{48}, defined by (43), so that K_4''' is excluded. The only operators of period 3 in G_{48} are seen (see § 22) to be the last 8 operators (19). Combined with K_4^*, they give G_{12}.

12. *Summary of the subgroups of order* 12. All have now been determined since the five types * of groups of order 12 were examined in §§ 8–11.

THEOREM. *Every existing type of group of order* 12 *is represented among the subgroups of G. Within G, they fall into seven distinct sets of conjugates, two of the dihedron type, two of the type of the alternating group on four letters, and one of each of the three remaining types.*

The subgroups of order 18.

13. THEOREM. *The group G contains exactly seven distinct sets of conjugate subgroups of order* 18, *representatives of which are*

$$\begin{aligned}
&K_{18} = (K_9,\ T_{2,-1}), \qquad K_{18}^{*} = (K_9^{*},\ T_{2,-1}), \qquad K_{18}^{**} = (K_9^{**},\ T_{1,-1}),\\
(23)\quad &G_{18}^{*} = (K_9^{*},\ P_{12}), \qquad H_{18}^{*} = (K_9^{*},\ P_{12}T_{2,-1}),\\
&H_{18}^{**} = (K_9^{**},\ P_{12}T_{2,-1}), \qquad G_{18}^{**} = (K_9^{**},\ W_0).
\end{aligned}$$

A Γ_{18} contains a single (self-conjugate) subgroup Γ_9. But within G a cyclic Γ_9 is self-conjugate only under a Γ_{27} by II_{385}. As the Γ_9 we may therefore take one of the non-cyclic groups K_9, K_9^{*}, K_9^{**} (II_{386}).

The group K_9 of the operators $[k, 0, c, 0]$ is self-conjugate only under G_{162} by II_{383}, which contains exactly the 9 operators $[0, \alpha, \gamma, 0]\,T_{2,-1}$ of period 2. If $\alpha = 0$, the latter is transformed into $T_{2,-1}$ by $[0, 0, -\gamma, 0]$; if $\alpha \neq 0$ it is transformed by the substitution $L_{2,\gamma/\alpha}$ of G_{162} into $[0, \alpha, 0, 0]\,T_{2,-1}$. The latter is transformed into $T_{2,-1}$ by the following substitution of G_{162}:

$$(24)\qquad \xi_1' = \xi_1 - \alpha\xi_2, \qquad \eta_2' = \eta_2 + \alpha\eta_1.$$

The group K_9^{*} of the operators $[k, 0, 0, \gamma]$ is self-conjugate only under H_{108} by II_{384}. By § 4 the operators of period 2 of H_{108} are conjugate within H_{108} with P_{12}, $P_{12}T_{2,-1}$, or $T_{2,-1}$. Of the resulting groups $G_{18}^{*}, H_{18}^{*}, K_{18}^{*}$, the first two each contain exactly 3 substitutions of period 2 and the third only one. The 3 of the first and the 3 of the second have the characteristic determinants $(\rho^2 - 1)^2$ and $(\rho^2 + 1)^2$, respectively. Hence no two of these three groups are conjugate under linear transformation.

The group K_9^{**} of the operators $[-\gamma, 0, c, \gamma]$ is self-conjugate only under H_{216} (see foot-note to § 3). The operators of period 2 of H_{216} are conjugate with $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5). No two of the resulting groups (K_9^{**}, Q), $Q = P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0, are conjugate within G. Indeed, $[-\gamma, 0, c, \gamma]\,W_0$ is of period 2 if and only if $c + \gamma \equiv 0$, so that (K_9^{**}, W_0) contains exactly 3

*CAYLEY, American Journal of Mathematics, vol. 11 (1889), pp. 151–3. In his substitution V of $B4$, (dj) should read (gj). In MILLER's list, Quarterly Journal, vol. 28 (1896), p. 255, the group 12_1 should have $(ag)(bh)(ci)(dj)(ek)(fl)$ as the second generator.

operators of period 2, each with the characteristic determinant $(\rho^2+1)^2$. Since $[-\gamma, 0, c, \gamma]T_{2,-1}$ is of period 2 if and only if $\gamma \equiv 0$, the group $(K_9^{**}, T_{2,-1})$ contains exactly 3 operators of period 2, each with the characteristic determinant $(\rho^2-1)^2$. Finally $[-\gamma, 0, c, \gamma]P_{12}T_{2,-1}$ is of period 2 for every c and γ, so that $(K_9^{**}, P_{12}T_{2,-1})$ contains exactly 9 operators of period 2.

14. THEOREM. *Within G, the group K_{18} is self-conjugate only under*

$$(25)\quad K_{54} = (K_{27}, T_{2,-1}) = \{[k, 0, c, \gamma]T_{2,\pm 1} \quad (k, c, \gamma = 0, 1, 2)\}.$$

The operators of K_{18} not in K^9 are $[k, 0, c, 0]T_{2,-1}$. The latter is of period 2 if and only if $k \equiv 0$. The general operator (6) of II_{373} in G_{162} transforms $T_{2,-1}$ into an operator $[0, 0, c, 0]T_{2,-1}$ if and only if $\alpha_{12} \equiv 0$, $\gamma_{12} \equiv c\alpha_{22}$, provided we set $\alpha_{11} = +1$, as we may. The operators (6) with $\alpha_{12} \equiv 0$ form K_{54}.

15. THEOREM. *Within G, the group K_{18}^* is self-conjugate only under K_{36}^*.*

The only operator of period 2 in K_{18}^* is $T_{2,-1}$. It thus remains to determine the operators of H_{108} which are commutative with $T_{2,-1}$. For U of § 3, the condition is $c \equiv 0$. For V the condition is $c \equiv 0$.

16. THEOREM. *Within G, the group G_{18}^* is self-conjugate only under H_{108}.*

The only operators of period 2 in G_{18}^* are $N_t = [-t, 0, 0, t]P_{12}$. But U and V of § 3 transform P_{12} into $N_{k-\gamma}$ and $N_{\gamma-k}$, respectively.

17. THEOREM. *Within G, the group H_{18}^* is self-conjugate only under K_{36}^*.*

The only operators of period 2 in H_{18}^* are $Q_t = [-t, 0, 0, t]P_{12}T_{2,-1}$. Now U transforms $P_{12}T_{2,-1}$ into Q_t if and only if $c \equiv 0$, $t \equiv k-\gamma$; while V transforms $P_{12}T_{2,-1}$ into Q_t if and only if $c \equiv 0$, $t \equiv \gamma - k$.

18. THEOREM. *Within G, the group K_{18}^{**} is self-conjugate only under H_{108}.*

The only operators of period 2 in K_{18}^{**} are $R_t = [0, 0, t, 0]T_{2,-1}$. Now U and V transform $T_{2,-1}$ into $R_{\pm c}$, so that K_{18}^{**} is certainly self-conjugate under H_{108}. But $T_{2,-1}Y = YR_t$, where Y is of the form (3), requires $\alpha_{21} \equiv 0$ and is impossible.

19. THEOREM. *Within G, the group H_{18}^{**} is self-conjugate only under $H_{2.6}$.*

The 9 operators $S_{c,\gamma} = [-\gamma, 0, c, \gamma]P_{12}T_{2,-1}$ belonging to H_{18}^{**} but not to K_9^{**} are all of period 2 (end of § 13). Now U and V transform $P_{12}T_{2,-1}$ into $S_{\pm c, k-\gamma}$ and $S_{\pm c, \gamma-k}$, respectively; while (3) transforms it into $S_{c,\gamma}$ where

$$c \equiv \alpha_{21}(\pm\gamma_{22}-\gamma_{11}) \mp \gamma_{12} - \gamma_{21}, \qquad \gamma \equiv \alpha_{21}(\mp\gamma_{12}-\gamma_{21}) - (\pm\gamma_{22}-\gamma_{11}).$$

20. THEOREM. *Within G, the group G_{18}^{**} is self-conjugate only under*

$$(26)\quad \begin{aligned} G_{36}^{**} &= (G_{18}^{**}, T_{2,-1}P_{12}) \\ &= \{[-\gamma, 0, c, \gamma]R, \ (R = I, W_0, T_{2,-1}P_{12}, W_0T_{2,-1}P_{12})\}, \end{aligned}$$

whose 36 *operators may be exhibited in the explicit form*

$$(27)\qquad \pm\begin{bmatrix} 1 & \gamma_{11} & 0 & \gamma_{12} \\ 0 & 1 & 0 & 0 \\ 0 & \gamma_{12} & 1 & -\gamma_{11} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \pm\begin{bmatrix} 0 & -\gamma_{22} & -1 & \gamma_{12} \\ 0 & 0 & 0 & -1 \\ 1 & \gamma_{12} & 0 & \gamma_{22} \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\pm\begin{bmatrix} 1 & \gamma_{11} & \mp 1 & \gamma_{12} \\ 0 & -1 & 0 & \pm 1 \\ \mp 1 & -\gamma_{12} & -1 & \gamma_{11} \\ 0 & \pm 1 & 0 & 1 \end{bmatrix}.$$

The only operators of period 2 in G_{18}^{**} are $Y_t = [t, 0, t, -t]\, W_0$. We seek the conditions under which $S^{-1} W_0 S = Y_t$, where S belongs to H_{216}. For $S = U$, the upper signs must be taken and $\gamma \equiv -k$, $t \equiv -k - c$. For $S = V$, the lower signs must be taken and $\gamma \equiv -k$, $t \equiv c - \gamma$. For (3), S must be of the third type and $t \equiv -\gamma_{12}$ or γ_{11} according as the upper or the lower signs hold.

The subgroups of orders 10 *and* 20.

21. A group of order 10 or 20 contains a single (self-conjugate) subgroup of order 5. Within G the groups of order 5 are all conjugate, and each is self-conjugate only under a Γ_{20} by I_{138} or as shown below.

Consider the group G_5 generated by K of I_{137}. The homogeneous substitution K is of period 10 since $K^5 = T_{1,-1} T_{2,-1}$. The corresponding operator in the quotient-group G is of period 5. Within G, K, K^2, K^3 and K^4 are all conjugate, there being a single set of conjugate operators of period 5 by I_{138}. The conditions for $KS = SK^2 T_{1,-1} T_{2,-1}$ in the homogeneous group are seen to require that

$$S = \pm\begin{bmatrix} \alpha & \gamma & \beta+\gamma+\delta & -\alpha+\gamma-\delta \\ \beta & \delta & -\alpha+\beta-\delta & \alpha+\beta+\gamma \\ \alpha+\beta+\gamma & -\alpha+\beta-\delta & -\delta & -\beta \\ -\alpha+\gamma-\delta & \beta+\gamma+\delta & -\gamma & -\alpha \end{bmatrix}.$$

The abelian conditions on S then reduce to

$$\beta^2 + \gamma^2 - \alpha^2 - \delta^2 - \alpha\beta - \alpha\gamma - \beta\delta - \gamma\delta \equiv 1,$$
$$-\beta^2 - \gamma^2 - \alpha^2 - \delta^2 + \alpha\beta - \alpha\gamma - \beta\delta + \gamma\delta + \alpha\delta + \beta\gamma \equiv 0.$$

If $\alpha \equiv 0$, we find by addition that $\delta^2 + \beta\delta + \beta\gamma \equiv 1$. Since $\delta \equiv 0$ is excluded, we may set $\delta \equiv +1$, placing the ambiguity in sign before the matrix. From

$$\beta(1+\gamma) \equiv 0, \qquad \beta^2 + \gamma^2 - \beta - \gamma \equiv -1,$$

it follows that $\gamma \neq 1$ or 0. With $\gamma \equiv -1$, $\beta \equiv 0$ or 1, giving respectively

$$S_1 = \pm \begin{vmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{vmatrix}, \qquad S_2 = \pm \begin{vmatrix} 0 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

If $\alpha \equiv 1$, then $(\beta, \gamma, \delta) = (-1, 0, 0), (-1, 1, 0)$, or $(0, 0, -1)$, giving respectively

$$S_3 = \pm \begin{vmatrix} 1 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & -1 & 0 & -1 \end{vmatrix}, \qquad S_4 = \pm \begin{vmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{vmatrix},$$

$$S_5 = \pm \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \end{vmatrix}.$$

The five S_i are of period 4 and no one equals the inverse of another, their squares being distinct. Hence G_5 is self-conjugate only under

(28) $$G_{20} = \{K^t, S_i^j, (i, t = 1, \cdots; j = 1, 2, 3)\}.$$

It contains a single subgroup of order 10:

(29) $$G_{10} = \{K^t, S_i^2, (i, t = 1, \cdots, 5)\}.$$

THEOREM.* *A subgroup of order* 10 *or* 20 *is conjugate within G with G_{10} or G_{20}, respectively. The latter are self-conjugate only under G_{20}.*

The subgroups of order 24.

22. The single† type of group of order 24 which does not contain a subgroup of order 12 is considered in § 30. Consider here the Γ_{24} which have a (self-conjugate) subgroup Γ_{12}. The 7 distinct sets of conjugate Γ_{12} in G are represented by C_{12}, K_{12}, D'_{12}, D^*_{12}, G^*_{12}, G_{12}, G'_{12}; they are self-conjugate only under C_{24}, G^*_{24}, K^*_{36}, G^*_{24}, G^*_{24}, G_{48}, H_{96}, respectively (§§ 8–11). Hence D'_{12} is excluded, while each of the groups C_{12}, K_{12}, D^*_{12}, and G^*_{12} leads to a single Γ_{24}.

* Compare the corresponding investigation on the orthogonal form O, § 46.

† MILLER, Quarterly Journal, vol. 28 (1896), p. 274.

We next determine the Γ_{24} which contain G_{12}. The 8 operators of period 3 in G_{48} are the last 8 operators (19). Their products by $C_1 C_2 C_3 C_4$ evidently give the 8 operators of period 6, viz.,

$$(30) \qquad B_i W, \qquad B_i C_2 C_4 W, \qquad B_i C_1 C_4 W^2, \qquad B_i C_1 C_2 W^2 \quad (i=2, 4).$$

The 19 operators of period 2 are

$$(31) \begin{cases} C_1C_3,\ B_3,\ B_3C_1C_3,\ B_3C_2C_4,\ B_3C_1C_2C_3C_4,\ C_2C_4,\ C_1C_2C_3C_4,\ C_1C_0, \\ \qquad C_3C_0,\ C_1C_5,\ C_3C_5 \\ C_1C_5B_iW,\ C_1C_0B_iW,\ C_iC_5B_iW^2,\ C_iC_0B_iW^2 \qquad (i=2, 4). \end{cases}$$

The 12 operators of period 4 are

$$(32) \quad \begin{matrix} B_3C_iC_5, & B_3C_iC_0, & B_jC_3C_5W^2, & B_jC_3C_0W^2, \\ B_2C_4C_5W, & B_2C_4C_0W, & B_4C_2C_5W, & B_4C_2C_0W \end{matrix} \quad (i=1, 3; j=2, 4).$$

Now C_2C_4, C_1C_5 and C_1C_0 extend G_{12} to the respective groups

$$(33) \quad G_{24}^3 = \begin{Bmatrix} \Gamma,\ C_1C_3\Gamma,\ C_2C_4\Gamma,\ C_1C_2C_3C_4\Gamma \quad (\Gamma=I,\ B_3,\ B_2W, B_4W) \\ B_iC_1C_2W^2, B_iC_2C_3W^2, B_iC_1C_4W^2, B_iC_3C_4W^2 \quad (i=2,4) \end{Bmatrix},$$

$$(34) \quad L_{24} = \begin{Bmatrix} \Gamma,\ C_1C_3\Gamma,\ C_1C_5\Gamma,\ C_3C_5\Gamma,\ B_iC_2C_3W^2,\ B_iC_3C_4W^2, \\ C_1C_0B_iW,\ C_3C_0B_iW,\ C_2C_4B_iW,\ C_1C_2C_3C_4B_iW, \\ C_iC_0B_iW^2,\ C_2C_5B_4W^2,\ C_4C_5B_2W^2 \end{Bmatrix},$$

$$(35) \quad T_{24} = \begin{Bmatrix} \Gamma,\ C_1C_3\Gamma,\ C_1C_0\Gamma,\ C_3C_0\Gamma,\ B_iC_2C_3W^2,\ B_iC_3C_4W^2, \\ C_1C_5B_iW,\ C_3C_5B_iW,\ C_2C_4B_iW,\ C_1C_2C_3C_4B_iW, \\ C_iC_5B_iW^2,\ C_2C_0B_4W^2,\ C_4C_0B_2W^2 \end{Bmatrix},$$

where in (34) and (35), $\Gamma = I,\ B_3$; $i = 2, 4$. Now all the operators (31) occur in these three groups; all the operators (32) occur in the last two groups. No two of the three groups are conjugate within O since * G_{24}^3 contains G_8^3 self-conjugately, while L_{24} contains three groups conjugate with L_8, and T_{24} three groups conjugate with T_8.

To determine the Γ_{24} which contain G'_{12}, we note that the operators of period 2 in H_{96} are those of G_{16} together with

$$(36) \qquad (\xi_1\xi_3)(\xi_r\xi_s)C \quad (C=I,\ C_1C_3,\ C_rC_s,\ C_1C_3C_rC_s;\ r,\ s=2, 4, 5).$$

Those of period 4 are given when C ranges over the remaining 12 operators of G_{16}. Now C_1C_3, C_1C_2, C_2C_3, B_3 and $C_1C_3B_3$ extend G'_{12} to the respective groups.

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* There are exactly 7 operators of period 2 in G_{24}^3 and 9 in each L_{24} and T_{24}.

$$G''_{24} = \{\Gamma,\ \Gamma(\xi_2\xi_4\xi_5)^{\pm 1},\ (\Gamma = I,\ C_1C_3,\ C_2C_4,\ C_5C_0,\ C_2C_5,\ C_4C_5,\ C_2C_0,\ C_4C_0)\}, \tag{37}$$

$$G_{24} = \{\Gamma,\ \Gamma(\xi_2\xi_4\xi_5)^{\pm 1},\ (\Gamma = I,\ C_1C_2,\ C_1C_4,\ C_2C_4,\ C_1C_5,\ C_2C_5,\ C_4C_5,\ C_3C_0)\}, \tag{38}$$

$$\{\Gamma, \Gamma(\xi_2\xi_4\xi_5)^{\pm 1}, (\Gamma = I,\ C_2C_3,\ C_2C_4,\ C_3C_4,\ C_2C_5,\ C_3C_5,\ C_4C_5,\ C_1C_0)\}, \tag{39}$$

$$L^*_{24} = \{\Gamma,\ \Gamma(\xi_2\xi_4\xi_5)^{\pm 1},\ \Gamma B_3,\ \Gamma(\xi_1\xi_3)(\xi_2\xi_5),\ \Gamma(\xi_1\xi_3)(\xi_4\xi_5),\ (\Gamma = I,\ C_2C_4,\ C_2C_5,\ C_4C_5)\}. \tag{40}$$

$$\left\{\Gamma_1,\ \Gamma_1(\xi_2\xi_4\xi_5)^{\pm 1},\ \Gamma_2 B_3,\ \Gamma_2(\xi_1\xi_3)(\xi_2\xi_5),\ \Gamma_2(\xi_1\xi_3)(\xi_4\xi_5),\ \left(\begin{array}{l}\Gamma_1 = I,\ C_2C_4,\ C_2C_5,\ C_4C_5\\ \Gamma_2 = C_1C_3,\ C_5C_0,\ C_2C_0,\ C_4C_0\end{array}\right)\right\}. \tag{41}$$

All the operators (36) lie in these five groups. Of the operators of period 4, the square of CB_3 is C_2C_4 if $C = C_2C_5$, C_4C_5, C_2C_0 or C_4C_0; the square of $C(\xi_1\xi_3)(\xi_2\xi_5)$ is C_2C_5 if $C = C_2C_4$, C_4C_5, C_2C_0 or C_5C_0; the square of $C(\xi_1\xi_3)(\xi_4\xi_5)$ is C_4C_5 if $C = C_2C_4$, C_2C_5, C_4C_0 or C_5C_0. But these 12 operators of period 4, whose squares belong to G'_{12} and hence extend G'_{12} to a Γ_{24}, belong to the groups (40) and (41). The squares of the remaining operators of period 4 in H_{96} do not belong to G'_{12} and hence such an operator of period 4 extends G'_{12} to a Γ_{48}. Finally, all the substitutions $\Gamma(\xi_2\xi_4\xi_5)^{\pm 1}$, Γ ranging over G_{16}, of period 3 or 6 in H_{96} belong to the groups (37)–(39).

Now B_3 transforms (39) into (38); C_1C_5 transforms (41) into (40).

23. THEOREM. *Within G, the group C_{24} is self-conjugate only under G_{72}.*

The group C_{24} defined by (14) is the direct product of the cyclic group $C_3 = (L_{1,1})$ by a group Γ_8 affecting only ξ_2 and η_2. By IV the latter is composed of the identity, one operator $T_{2,-1}$ of period 2, and 6 operators of period 4. Hence C_{24} contains a single cyclic subgroup C_6. But the latter is self-conjugate only under G_{72}, the direct product of C_3 and a binary group Γ_{24} having Γ_8 as a self-conjugate subgroup. Note that Γ_8 is of the type F'''_8.

24. THEOREM. *Within G, the group G^*_{24} is self-conjugate only under itself.*

Of the operators of G^*_{24}, every $P^iT_{2,-1}D$ is of period 4; P, P^5, $PT_{2,-1}$, $P^5T_{2,-1}$, $P^2T_{2,-1}$ and $P^4T_{2,-1}$ are of period 6; P^2 and P^4 are of period 3;

$$P^3, \qquad T_{2,-1}, \qquad P^3T_{2,-1}, \qquad P^iD \qquad (i=0, 1, \cdots, 5),$$

are of period 2. Hence G^*_{24} contains exactly 3 cyclic subgroups of order 6, $C^*_6 = (P)$, $C''_6 = (P^2T_{2,-1})$, and $(PT_{2,-1})$. The latter is transformed into C''_6 by D, which transforms P into P^{-1}, and $T_{2,-1}$ into $P^3T_{2,-1}$. Now

C_6^* and C_6'' are not conjugate under G (§6). Hence an operator of G which transforms G_{24}^* into itself must transform C_6^* into itself and hence belong to G_{24}^* (§6).

25. THEOREM. *Within O, the group G_{24}^3 is self-conjugate only under G_{48}.*

The only subgroup of order 8 of G_{24}^3 is G_8^3 (§22). The latter is self-conjugate only under $H_{192} = (G_{64}, W)$ by III_{29}. Now B_2, $C_1 C_4$ and $B_2 C_1 C_4$ transform $B_2 W$ into $C_3 C_4 W$, $B_3 W$ and $C_1 C_2 B_4 W$, respectively, none of which belong to G_{24}^3. Hence the product of any operator of G_{48} (which transforms G_{24}^3 into itself by §22) by B_2, $C_1 C_4$ or $B_2 C_1 C_4$ does not transform G_{24}^3 into itself. But $\Gamma C_1 C_4$, ΓB_2, $\Gamma B_2 C_1 C_4$, where Γ ranges over G_{16}', give all the operators of G_{64} not in G_{16}'.

26. THEOREM. *Within O, L_{24} and T_{24} are self-conjugate only under G_{48}.*

The group L_{24} contains exactly 3 conjugate subgroups of order 8, one of which is L_8. The latter is self-conjugate only under G_{16}' by III_{35}. Hence at most $3 \cdot 16$ operators of O transform L_{24} into itself. But L_{24} is a subgroup of G_{48}. The proof for T_{24} follows by replacing L_{24}, L_8 by T_{24}, T_8.

27. THEOREM. *Within O, the group G_{24}'' is self-conjugate only under H_{96}.*

Indeed, by III_{29}, the self-conjugate subgroup G_8'' of G_{24}'' is self-conjugate only under H_{96}.

28. THEOREM. *Within O, the group L_{24}^* is self-conjugate only under* *

$$G_{48}''' = \{ G_{16}'', (\xi_2 \xi_4 \xi_5) \}. \tag{42}$$

The group L_{24}^* defined by (40) contains 9 operators of period 2 and hence 3 subgroups of order 8. Now B_2 transforms L_{24}^* into $\{ L_8, (\xi_1 \xi_3 \xi_5) \}$. But L_8 is self-conjugate only under G_{16}' by III_{35}. Now $C_2 C_4$, which extends L_8 to G_{16}', transforms L_8 and $(\xi_1 \xi_3 \xi_5)$ each into itself. Hence $\{ L_8, (\xi_1 \xi_3 \xi_5) \}$ is self-conjugate only under $\{ G_{16}', (\xi_1 \xi_3 \xi_5) \}$, which is transformed into (42) by B_2.

29. THEOREM. *Within O, the group G_{24} is self-conjugate only under*

$$H_{48} = \{ G_{16}, (\xi_2 \xi_4 \xi_5) \}. \tag{43}$$

The group G_{24} defined by (38) is transformed by $(\xi_1 \xi_4)(\xi_3 \xi_5)$ into

$$\{ G_8, (\xi_1 \xi_2 \xi_3) \}. \tag{44}$$

Now G_8 is self-conjugate only under G_{192} by III_{30}. But the only even substitutions on ξ_1, ξ_2, ξ_3, ξ_4 which transform into itself the cyclic group generated by $(\xi_1 \xi_2 \xi_3)$ are the powers of the latter. Hence (44) is self-conjugate only under the group $\{ G_{16}, (\xi_1 \xi_2 \xi_3) \}$ of order 48. Transforming it by $(\xi_1 \xi_4)(\xi_3 \xi_5)$, we obtain H_{48}.

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* We readily see that the order of (42) is 48 and that it is a subgroup of H_{96}.

30. THEOREM. *Within O, a subgroup simply isomorphic with the group of order 24 generated by four operators subject to the generational relations*

$$A^4 = I,\ B^2 = A^2,\ B^{-1}AB = A^{-1},\ C^3 = I,\ C^{-1}AC = B,\ C^{-1}BC = AB$$

is conjugate with one of the three groups

$$(45)\quad F_{24} = \{F_8''', (\xi_2\xi_4\xi_3)\},\quad F'_{24} = \{F_8''', W\},\quad F^*_{24} = \{F_8''', (\xi_2\xi_4\xi_3)W\}.$$

As the subgroup generated by A and B we may take F_8'''. Now $(\xi_2\xi_4\xi_3)$, W, and each B_i transform F_8''' into itself. But $(\xi_2\xi_4\xi_3)$, $(\xi_2\xi_3\xi_4)$, and B_2 transform $B_2C_1C_4$ into $B_3C_1C_2$, $B_4C_1C_3$, and $B_2C_2C_3$, respectively; B_3 transforms $B_3C_1C_2$ into $B_3C_3C_4$; B_4 transforms $B_4C_1C_3$ into $B_4C_2C_4$. Hence we may take $A = B_2C_1C_4$. Next, B_4 is commutative with $B_2C_1C_4$, and transforms $B_4C_1C_3$ and $B_3C_1C_2$ into $B_4C_2C_4$ and $B_3C_3C_4$, respectively. Hence we may take $B = B_3C_1C_2$ or $B_4C_1C_3$. Since F_8''' is self-conjugate, C must leave ξ_5 fixed (III_{32}).

For $B = B_3C_1C_2$, the conditions $AC = CB$ and $BC = CAB$ give

$$C = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ -\alpha_{14} & \alpha_{13} & -\alpha_{12} & \alpha_{11} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{14} & -\alpha_{13} & 0 \\ -\alpha_{13} & -\alpha_{14} & \alpha_{11} & \alpha_{12} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix},$$

subject to the single condition $\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \alpha_{14}^2 \equiv 1$. If one of the α_{1j} is $\equiv 0$, then three are, the resulting substitutions of period 3 being

$$(\xi_2\xi_4\xi_3),\qquad (\xi_1\xi_2\xi_3)C_2C_3,\qquad (\xi_1\xi_3\xi_4)C_3C_4,\qquad (\xi_1\xi_4\xi_2)C_2C_4.$$

But $B_2C_1C_3$, $B_3C_2C_3$, $B_4C_1C_2$, which transform F_8''' into itself, transform $(\xi_2\xi_4\xi_3)$ into $(\xi_1\xi_3\xi_4)C_3C_4$, $(\xi_1\xi_4\xi_2)C_2C_4$, $(\xi_1\xi_2\xi_3)C_2C_3$.

Let next each $\alpha_{1j} \neq 0$ in C. The conditions for $C^2 = C^{-1}$ are

$$\alpha_{11} \equiv 1 - \alpha_{12}\alpha_{14} - \alpha_{12}\alpha_{13} - \alpha_{13}\alpha_{14},\qquad -\alpha_{14} \equiv -1 + \alpha_{11}\alpha_{12} + \alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{13},$$

$$-\alpha_{12} \equiv -1 + \alpha_{11}\alpha_{13} + \alpha_{11}\alpha_{14} + \alpha_{13}\alpha_{14},\quad -\alpha_{13} \equiv -1 + \alpha_{11}\alpha_{12} + \alpha_{11}\alpha_{14} + \alpha_{12}\alpha_{14}.$$

Eliminating α_{11} from the last three, we get

$$(\alpha_{12}\alpha_{13} - 1)(\alpha_{14} + 1) \equiv 0,\qquad (\alpha_{13}\alpha_{14} - 1)(\alpha_{12} + 1) \equiv 0,$$
$$(\alpha_{12}\alpha_{14} - 1)(\alpha_{13} + 1) \equiv 0.$$

If $\alpha_{13} \equiv 1$, the conditions give merely $\alpha_{12} \equiv \alpha_{14}$, $\alpha_{11} = \alpha_{12}$. If $\alpha_{13} \equiv -1$, they reduce to $(\alpha_{12} + 1)(\alpha_{14} + 1) \equiv 0$, $\alpha_{11} = \alpha_{12}\alpha_{14}$. The resultant operators are

$$\begin{bmatrix} \alpha & \alpha & 1 & \alpha & 0 \\ -\alpha & 1 & -\alpha & \alpha & 0 \\ -\alpha & \alpha & \alpha & -1 & 0 \\ -1 & -\alpha & \alpha & \alpha & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \alpha_2\alpha_4 & \alpha_2 & -1 & \alpha_4 & 0 \\ -\alpha_4 & -1 & -\alpha_2 & \alpha_2\alpha_4 & 0 \\ -\alpha_2 & \alpha_2\alpha_4 & \alpha_4 & 1 & 0 \\ 1 & -\alpha_4 & \alpha_2\alpha_4 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[(\alpha_2+1)(\alpha_4+1)\equiv 0].$$

Call them $C_{(\alpha)}$ and C_{α_2, α_4}, respectively. Now $C_{-1,-1} = W$, so that we have the group (F_8''', W). Next $C_{-1,1} = C_1 C_4 W C_3 C_4$, which $C_1 C_3$ transforms into $WB_2 C_1 C_4$, an operator of (F_8''', W). Again, $C_{1,-1} = C_1 C_2 W C_2 C_4$, which $C_2 C_3$ transforms into $WB_3 C_1 C_2$, an operator of (F_8''', W). But W transforms F_8''' into itself and $C_{(-1)}$ into $C_{1,-1}$. Finally, $C_{(+1)} = W^2(\xi_2\xi_3\xi_4)$.

For $B = B_4 C_1 C_3$, the conditions $AC' = C'B$ and $BC' = C'AB$ give

$$C' = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{13} & \alpha_{14} & -\alpha_{11} & -\alpha_{12} & 0 \\ \alpha_{14} & -\alpha_{13} & \alpha_{12} & -\alpha_{11} & 0 \\ -\alpha_{12} & \alpha_{11} & \alpha_{14} & -\alpha_{13} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

subject to the single condition $\alpha_{11}^2 + \alpha_{12}^2 + \alpha_{13}^2 + \alpha_{14}^2 \equiv 1$. If one of the α_{1j} is $\equiv 0$, then three are, the resulting substitutions of period 3 being

$$(\xi_1\xi_4\xi_3), \qquad (\xi_1\xi_2\xi_4)C_2C_4, \qquad (\xi_1\xi_3\xi_2)C_1C_2, \qquad (\xi_2\xi_3\xi_4)C_2C_3.$$

But the last three are the transforms of $(\xi_1\xi_4\xi_3)$ by $B_4C_1C_2$, $B_3C_2C_3$, $B_2C_1C_3$, respectively. The resulting group $[(F_8''', (\xi_1\xi_4\xi_3)]$ is transformed into F_{24} by B_2.

Let next each $\alpha_{1j} \neq 0$ in C'. The conditions for $C'^2 = C'^{-1}$ are

$$\alpha_{11} = 1 + \alpha_{12}\alpha_{13} - \alpha_{12}\alpha_{14} + \alpha_{13}\alpha_{14}, \qquad \alpha_{13} = -1 + \alpha_{11}\alpha_{12} + \alpha_{11}\alpha_{14} + \alpha_{12}\alpha_{14},$$

$$\alpha_{14} = 1 - \alpha_{11}\alpha_{12} + \alpha_{11}\alpha_{13} + \alpha_{12}\alpha_{13}, \quad -\alpha_{12} = -1 - \alpha_{11}\alpha_{13} + \alpha_{11}\alpha_{14} - \alpha_{13}\alpha_{14}.$$

Eliminating α_{11} from the last three, we get

$$(\alpha_{12}\alpha_{14}-1)(1-\alpha_{13}) \equiv 0, \quad (\alpha_{12}\alpha_{13}+1)(1+\alpha_{14}) \equiv 0, \quad (\alpha_{13}\alpha_{14}+1)(1+\alpha_{12}) \equiv 0.$$

If $\alpha_{13} \equiv 1$, the conditions reduce to $(\alpha_{12}+1)(\alpha_{14}+1) \equiv 0$ and $\alpha_{11} = \alpha_{12}\alpha_{14}$. If $\alpha_{13} \equiv -1$, they reduce to $\alpha_{14} = \alpha_{12}$, $\alpha_{11} = \alpha_{12}$. The resulting operators are

$$\begin{bmatrix} \alpha & \alpha & -1 & \alpha & 0 \\ -1 & \alpha & -\alpha & -\alpha & 0 \\ \alpha & 1 & \alpha & -\alpha & 0 \\ -\alpha & \alpha & \alpha & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \alpha_2\alpha_4 & \alpha_2 & 1 & \alpha_4 & 0 \\ 1 & \alpha_4 & -\alpha_2\alpha_4 & -\alpha_2 & 0 \\ \alpha_4 & -1 & \alpha_2 & -\alpha_2\alpha_4 & 0 \\ -\alpha_2 & \alpha_2\alpha_4 & \alpha_4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$[(\alpha_2+1)(\alpha_4+1) \equiv 0]$.

Call them $C'_{(\alpha)}$ and C'_{α_2,α_4} respectively. Now W transforms $C'_{(+1)}$ into $W(\xi_2\xi_4\xi_3)B_2C_2C_3$, which belongs to F^*_{24}. Next, $C'_{(-1)} = W^2C_1C_4$, which C_1C_2 transforms into $B_3C_3C_4W^2$, an operator of F'_{24}. Next, $C'_{-1,-1}$ equals $W^2B_3C_1C_2C_3C_4$, which C_1C_3 transforms into W^2. Again, $C'_{1,-1}$ equals $W^2B_4C_2C_4$, which belongs to F_{24}. Finally,

$$C'_{-1,1} = W^2B_2C_3C_4,$$

which is transformed by C_1C_4 into $B_4C_2C_4W^2$, an operator of F'_{24}.

Each of these three contains F'''_8 self-conjugately and hence a single subgroup of order 8. Hence a group which transforms one of them into itself must transform F'''_8 into itself and therefore leave ξ_5 fixed. An operator of the latter is of the form W^aJ, where J belongs to J_{96}, merely permuting ξ_1^2, ξ_2^2, ξ_3^2, ξ_4^2. Hence it cannot transform W into an operator of F_{24}. Hence F_{24} and F'_{24} are not conjugate.

31. Theorem. *Within O, F_{24} and F'_{24} are self-conjugate only under*

$$(46) \qquad G'_{72} = \{F'''_8, (\xi_2\xi_4\xi_3), W\}.$$

Since $(\xi_2\xi_4\xi_3)$ and W are commutative, while each transforms F'''_8 into itself, the groups F_{24} and F'_{24} are each self-conjugate under G'_{72}.

To show that there are no further operators transforming F_{24} into itself, we note that C_1C_2 and C_3C_4, C_1C_3 and C_2C_4, C_1C_4 and C_2C_3, transform $(\xi_2\xi_4\xi_3)$ into $(\xi_2\xi_4\xi_3)C_2C_3$, $(\xi_2\xi_4\xi_3)C_3C_4$, $(\xi_2\xi_4\xi_3)C_2C_4$, respectively, while B_2 transforms $(\xi_2\xi_4\xi_3)$, C_2C_3, C_3C_4, C_2C_4 into $(\xi_1\xi_3\xi_4)$, C_1C_4, C_3C_4, C_1C_3, respectively. Hence $C_1C_4B_2$ and $C_2C_3B_2$ transform $(\xi_2\xi_4\xi_3)$ into $(\xi_1\xi_3\xi_4)C_1C_3$, while no other $C_iC_jB_2$ transforms $(\xi_2\xi_4\xi_3)$ into an operator of F_{24}. Now $C_1C_4B_2 = B_2C_2C_3$ and $C_2C_3B_2 = B_2C_1C_4$ belong to F'''_8. In this way it may be verified that every operator EC, where E is an even permutation of ξ_1, ξ_2, ξ_3, ξ_4 and C is a product of an even number of the C_i, must belong to F_{24} if it transforms $(\xi_2\xi_4\xi_3)$ into an operator of F_{24}.

To show that there are no operators other than those of G'_{72} which transform F'_{24} into itself, we note that B_2 and B_3 transform W into $B_2C_3C_4W$ and $B_3C_2C_4W$, respectively, neither of which occurs in F'_{24}. But B_2 extends

F_8''' to a group of order 16, which B_3 extends to G_{32}. In view of (47), F'_{24} is self-conjugate only under a group of order $\frac{1}{4}\cdot 288$.

32. THEOREM. *Within O, the group F^*_{24} is self-conjugate only under*

(47) $$G_{288} = \{ G_{32}, (\xi_2\xi_3\xi_4), W\}.$$

Now F^*_{24} is self-conjugate under G_{72}. Moreover, B_2 and B_3 transform $(\xi_2\xi_4\xi_3)W$ into $B_2C_2C_3(\xi_2\xi_4\xi_3)W$ and $B_3C_3C_4(\xi_2\xi_4\xi_3)W$, respectively, each belonging to F^*_{24}. But B_2 and B_3 extend F_8''' to G_{32}.

The subgroups of order 36.

33. THEOREM. *Within G, the subgroups of order* 36 *are conjugate with one of the four: K^*_{36}, G^{**}_{36},*

(48) $$K^{**}_{36} = (K^{**}_{18}, P_{12}), \qquad H^{**}_{36} = (H^{**}_{18}, T_{2,-1}W_0).$$

A Γ_{36} contains either a Γ_{18} or else a Γ_{12} of the non-cyclic commutative type.* Within G, the latter is self-conjugate only under a Γ_{24} (§ 9). *Hence a subgroup Γ_{36} of G contains a Γ_{18}.* For the latter we may take one of the 7 groups (23). But K_{18} is self-conjugate only under K_{54} (§ 14) and hence is excluded. The group G^{**}_{18} leads to G^{**}_{36} only (§ 20), while K^*_{18} and H^*_{18} lead to K^*_{36} only (§ 15, § 17).

Next, G^*_{18} and K^{**}_{18} are self-conjugate only under H_{108} (§ 16, § 18). The operators which extend one of these Γ_{18} to a subgroup Γ_{36} may be limited to the operators of period 2 of H_{108} or to those of period 6 whose squares belong to Γ_{18}. The former are conjugate within H_{108} with P_{12}, $P_{12}T_{2,-1}$, $T_{2,-1}$. Since P_{12} belongs to G^*_{18}, there results a single Γ_{36}:

$$(G^*_{18}, T_{2,-1}) = (K^*_9, P_{12}, T_{2,-1}) = K^*_{36}.$$

Since $T_{2,-1}$ belongs to K^{**}_{18}, there results only $(K^{**}_{18}, P_{12}) = K^{**}_{36}$. Consider next the operators of period 6 of H_{108} given at the end of § 4. Their squares are $[-k, 0, 0, -\gamma]$, $[\pm 1, 0, 0, \pm 1]$, and $[k, 0, -\gamma, k]$, respectively. The first two of these belong to G^*_{18}, while the third does only when $\gamma \equiv 0$. The resulting operators of period 6 are $[k, 0, 0, \gamma]T_{2,-1}$, $[\pm 1, 0, 0, 0]T_{2,-1}P_{12}$ and $[\pm 1, 0, 0, 0]P_{12}$; they either belong to G^*_{18} or extend it to $(G^*_{18}, T_{2,-1}) = K^*_{36}$. The only ones of the above squares which belong to $K^{**}_{18} = \{[-\gamma, 0, c, \gamma]T_{2,\pm 1}\}$ are $[\gamma, 0, 0, -\gamma]$ and $[0, 0, -\gamma, 0]$. The resulting operators of period 6 are $[-\gamma, 0, 0, \gamma]T_{2,-1}$ and $[0, 0, \gamma, 0]P_{12}$, where $\gamma \neq 0$, the first belonging to K^{**}_{18} and the second extending it to $(K^{**}_{18}, P_{12}) = K^{**}_{36}$.

Finally H^{**}_{18} is self-conjugate only under H_{216} (§ 19). The operators of period 2 of H_{216} are conjugate within it with $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5), the

* MILLER, Quarterly Journal, vol. 28 (1896), p. 283.

first belonging to H_{18}^{**}, the second and third extending H_{18}^{**} to K_{36}^{**} and G_{36}^{**}. The operators of period 4 are conjugate with $T_{2,-1}W_0$ (§ 5), which extends H_{18}^{**} to H_{36}^{**}. The operators of period 6 are conjugate with W_1, $[k, 0, 0, \gamma]T_{2,-1}$ or $[\pm 1, 0, 0, 0]T_{2,-1}P_{12}$ (§ 5). Their squares are $[-1, 0, 1, 1]$, $[-k, 0, -\gamma]$, $[\pm 1, 0, 0, \pm 1]$, the last not belonging to H_{18}^{**} and the second belonging to it only when $k = -\gamma$. The resulting operators $W_1 \equiv [1, 0, -1, -1]W_0$ and $[-\gamma, 0, 0, \gamma]T_{2,-1}$ extend H_{18}^{**} to $(H_{18}^{**}, W_0) = G_{36}^{**}$ and $(H_{18}^{**}, T_{2,-1}) = K_{36}^{**}$, respectively.

34. THEOREM. *Within G, the group K_{36}^{*} is self-conjugate only under itself.*

Since K_{36}^{*} is a subgroup of H_{108}, under which K_9^* is self-conjugate, K_9^* is the only group of order 9 contained in K_{36}^*, by SYLOW's theorem. Hence K_{36}^* is self-conjugate only under a subgroup of H_{108}. Since $[0, 0, 1, 0]$ extends K_9^* to K_{27}, it extends K_{36}^* to H_{108}; but it transforms $T_{2,-1}$ into $[0, 0, 1, 0]T_{2,-1}$, which is not in K_{36}^*.

35. THEOREM. *Within G, the group K_{36}^{**} is self-conjugate only under H_{216}.*

Since K_{36}^{**} is a subgroup of H_{216}, the largest group in which K_9^{**} is self-conjugate, K_9^{**} is the only group of order 9 in K_{36}^{**}. Hence K_{36}^{**} is self-conjugate only under H_{216} or a subgroup of it. Now $[1, 0, 0, 0]$ extends K_{36}^{**} to H_{108} and transforms P_{12} into $[-1, 0, 0, 1]P_{12}$, which belongs to K_{36}^{**}. Also, W_0 extends H_{108} to H_{216} and transforms P_{12} into $T_{2,-1}$. Hence K_{36}^{**} is certainly self-conjugate under H_{216}.

36. THEOREM. *Within G, the group G_{36}^{**} is self-conjugate only under*

$$G_{72}^{**} = (G_{36}^{**}, P_{12}) = (K_9^{**}, T_{2,-1}, P_{12}, W_0). \tag{49}$$

As in the preceding section, G_{36}^{**} is self-conjugate only under a subgroup of H_{216}. Now P_{12} transforms W_0 into $W_0T_{2,-1}P_{12}$ and hence extends G_{36}^{**} to a group of order 72. Now $[1, 0, 0, 0]$, which extends this G_{72}^{**} to H_{216}, transforms W_0 into $[0, 0, 1, 1]W_0$, which does not lie in G_{36}^{**}.

37. THEOREM. *Within G, the group H_{36}^{**} is self-conjugate only under G_{72}^{**}.*

As in §35, H_{36}^{**} is self-conjugate only under a subgroup of H_{216}. Now W_0, which extends H_{108} to H_{216}, transforms $T_{2,-1}W_0$ into its inverse $W_0T_{2,-1}$. Hence H_{36}^{**} is certainly self-conjugate under $(H_{36}^{**}, W_0) = G_{72}^{**}$. The latter is extended to H_{216} by $[1, 0, 0, 0]$, which transforms $T_{2,-1}W_0$ into $T_{2,-1}[0, 0, 1, 1]W_0$ (§ 36). The latter equals $[0, 0, -1, 1]T_{2,-1}W_0$ and is not in H_{36}^{**} since $[0, 0, -1, 1]$ is not in K_9^{**}.

38. THEOREM. *No two of the groups K_{36}^*, K_{36}^{**}, G_{36}^{**}, H_{36}^{**} are conjugate within G.*

For the first three groups, the result follows from §§34–36. Neither K_{36}^* nor K_{36}^{**} contains operators of period 4, being subgroups of H_{108} (§4). The same is true for G_{36}^{**} since $(K_9^{**}, P_{12}T_{2,-1}) = H_{18}^{**}$ and $(K_9^{**}, W_0) = G_{18}^{**}$ have no operators of period 4, while

$$[-\gamma, 0, c, \gamma]\, P_{12} T_{2,-1} W_0 = \pm \begin{vmatrix} -1 & c+\gamma & 1 & \gamma - c \\ 0 & 1 & 0 & -1 \\ 1 & c-\gamma & 1 & c+\gamma \\ 0 & -1 & 0 & -1 \end{vmatrix}$$

is not of the form (3) if $\alpha_{22} \equiv +1$ and hence is not of period 4 (§ 5). But H_{36}^{**} contains $T_{2,-1} W_0$, of period 4.

The subgroups of order 48.

39. It is first shown that *every subgroup* Γ_{48} *contains a* Γ_{24}. A Γ_{48} not containing a Γ_{24} has * 16 cyclic Γ_3 and hence a self-conjugate Γ_{16}. The latter may be taken to be G_{16} or F_{16} by III. For $\Gamma_{16} = G_{16}$, Γ_{48} is a subgroup of G_{960} and hence is conjugate with H_{48} of § 29, which contains the substitution $(\xi_2 \xi_4 \xi_5) C_1 C_8$ of period 6. For $\Gamma_{16} = F_{16}$, Γ_{48} is a subgroup of G_{96}. Since G_{96} contains at most 16 conjugate Γ_3, all of them must belong to Γ_{48}. Hence the latter is $\{F_{16}, W(\xi_2\xi_4\xi_3)\}$, which contains $W(\xi_2\xi_4\xi_3) B_3$ of period 12, its cube being $B_2 C_2 C_4$.

We consider in turn for Γ_{24} the types of non-conjugate subgroups of order 24. Now C_{24}, F_{24}, F'_{24}, and G^*_{24} are excluded, being self-conjugate only under G_{72}, G'_{72}, G'_{72}, and G^{**}_{24}, respectively. Again, G^3_{24}, L_{24} and T_{24} lead only to G_{48}; G_{24} leads only to H_{48}; L^*_{24} leads only to G''_{48}.

The group G''_{24} is self-conjugate only under H_{96}. The operators of period 2 belonging to H_{96} but not to G''_{24} are given by (36) and

$$(50) \qquad C_1 C_2,\ C_1 C_4,\ C_1 C_5,\ C_2 C_3,\ C_3 C_4,\ C_3 C_5,\ C_1 C_0,\ C_3 C_0.$$

Any one of the set (36) extends G''_{24} to G''_{48}, given by (42); any one of the set (50) extends G''_{24} to H_{48}, given by (43). Next, $C(\xi_1\xi_3)(\xi_r\xi_s)$, where C ranges over the operators of G_{16} other than I, $C_1 C_3$, $C_r C_s$, $C_1 C_3 C_r C_s$, give the operators of period 4 of H_{96}. Their squares all belong to G''_{24}. Now $C_1 C_2 B_3$ extends G''_{24} to

$$(51) \qquad H''_{48} = \{H''_{16}, (\xi_2\xi_4\xi_5)\},$$

composed of the substitutions of G''_{24} and ΣB_3, $\Sigma(\xi_1\xi_3)(\xi_2\xi_5)$, $\Sigma(\xi_1\xi_3)(\xi_4\xi_5)$, where Σ ranges over the set (50). It contains all the CB_3 of period 4 except $C_2 C_5 B_3$, $C_4 C_5 B_3$, $C_2 C_0 B_3$, $C_4 C_0 B_3$, each of which extends G'''_{24} to $(G'''_{24}, B_3) = G'''_{48}$; it contains all the $C(\xi_1\xi_3)(\xi_2\xi_5)$ of period 4 except for $C = C_2 C_4$, $C_4 C_5$, $C_2 C_0$, $C_5 C_0$; while, for these, $C(\xi_1\xi_3)(\xi_2\xi_5)$ extends G''_{24} to G'''_{48}; likewise for the $C(\xi_1\xi_3)(\xi_4\xi_5)$. Finally, the operators of period 6 of H_{96} are the $\Gamma(\xi_2\xi_4\xi_5)^{\pm 1}$, where Γ ranges over the operators of G_{16} other than I,

* MILLER, Quarterly Journal, vol. 30 (1899), p. 245.

C_2C_4, C_2C_5, C_4C_5, which furnish operators of period 3 belonging to G''_{24}. Hence the squares of those of period 6 belong to G''_{24}. Any $\Sigma(\xi_2\xi_4\xi_5)^{\pm 1}$, where Σ is one of the operators (50), extends G''_{24} to H_{48}; the remaining ones belong to G''_{24}.

Finally, F^*_{24} is self-conjugate only under G_{288}. To the latter corresponds (*Linear Groups*, § 189), the abelian group

$$(52)\qquad G'_{288} = \left\{ \pm \begin{bmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \end{bmatrix} \quad \begin{bmatrix} ad - bc \equiv 1 \\ a_1d_1 - b_1c_1 \equiv 1 \end{bmatrix} \right\},$$

since the second compound of the general operator (52) leaves Y_{12} and Y_{34} unaltered. By IV the binary group Γ_{24} has a single set of conjugate subgroups of each of orders 2, 3, 4, 6, 8, but no group of order 12. Hence every subgroup of order 48 of (52) is conjugate with $\pm(\Gamma_4, \Gamma_{24})$. Hence if a subgroup of G_{288} is of order 48 and contains F^*_{24}, it is conjugate with

$$(53)\qquad F_{48} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_3)W, \Gamma W^2(\xi_2\xi_3\xi_4), (\Gamma \text{ ranging over } F_{16})\}.$$

The subgroups of order 48 *are conjugate with* G_{48}, H_{48}, G''_{48}, H''_{48} *or* F_{48}.

40. THEOREM. *Within* O, *the group* G_{48} *is self-conjugate only under itself.*

An operator which transforms G_{48} into itself must transform its 3 subgroups of order 16 amongst themselves. By III_{28}, G'_{16} is self-conjugate within O only under $J^3_{32} = \{G'_{16}, C_1C_2\}$. Now C_1C_2 transforms B_2W into $WC_1C_2C_3C_4$, which does not belong to G_{48}.

41. THEOREM. *Within* O, *the group* H_{48} *is self-conjugate only under* H_{96}.

By III, bottom of p. 20, H_{48} is self-conjugate only under a subgroup of G_{960}. But the only even substitutions on $\xi_1, \cdots, \xi_5$ which transform $(\xi_2\xi_4\xi_5)$ into itself or its inverse are the powers of $(\xi_2\xi_4\xi_5)$ and $(\xi_1\xi_3)(\xi_2\xi_4)$, $(\xi_1\xi_3)(\xi_2\xi_5)$, $(\xi_1\xi_3)(\xi_4\xi_5)$.

42. THEOREM. *Within* O, *the group* G''_{48} *is self-conjugate only under* H_{96}.

By the foot-note to § 28, G''_{48} is a subgroup of H_{96}. Now G''_{16} is one of 3 conjugates under G''_{48} and is self-conjugate only under a subgroup Γ_{32} of O.

43. THEOREM. *Within* O, *the group* H''_{48} *is self-conjugate only under* H_{96}.

By the method of § 42, the proof follows from § 39.

44. THEOREM. *Within* O, *the group* F_{48} *is self-conjugate only under*

$$(54)\qquad G_{96} = \{\Gamma, \Gamma(\xi_2\xi_4\xi_3)W, \Gamma W^2(\xi_2\xi_3\xi_4), (\Gamma \textit{ ranging over } G_{32})\}.$$

Indeed, F_{16} is self-conjugate under F_{48}, while within O it is self-conjugate only under G_{96} by III_{37}. For another proof, see end of § 39.

The subgroups of order 54.

45. THEOREM. *Within* G, *the groups of order* 54 *fall into* 3 *distinct sets of conjugate subgroups. As representatives, we may take*

(55) $G_{54} = (G_{27}, T_{2,-1}), \quad K_{54} = (K_{27}, T_{2,-1}), \quad K'_{54} = (K_{27}, P_{12}T_{2,-1}).$

They are self-conjugate only under the respective groups G_{648}, H_{108}, H_{216}.

A group of order 54 contains a self-conjugate group of order 27. The latter may be assumed to be H_{27}, K_{27} or G_{27}, which are self-conjugate within G only under G_{81}, H_{648}, G_{648}, respectively (II, pp. 378–380). Hence H_{27} is excluded.

The only operators of period 2 in G_{648} are $[0, a, c, 0]T_{2,-1}$, as shown directly or by § 50. Each is therefore (§ 13) conjugate with $T_{2,-1}$ within G_{648}, which has G_{162} as a subgroup.

In order that the general operator (19) of II_{380} in H_{648} shall equal its inverse S^{-1} or $S^{-1}T_{1,-1}T_{2,-1}$, the conditions are, respectively,

$$\alpha_{11} \equiv \delta_{11}, \quad \alpha_{22} \equiv \delta_{22}, \quad \alpha_{21} \equiv \delta_{12}, \quad \delta_{21} \equiv \alpha_{12}, \quad \gamma_{22} \equiv \gamma_{11} \equiv 0, \gamma_{12} \equiv -\gamma_{21};$$

$$\alpha_{11} \equiv -\delta_{11}, \alpha_{22} \equiv -\delta_{22}, \alpha_{21} \equiv -\delta_{12}, \delta_{21} \equiv -\alpha_{12}, \gamma_{21} \equiv \gamma_{12}.$$

In the first case, the abelian conditions (see (19) of II_{380}) give

$$\alpha_{11}^2+\alpha_{12}\delta_{12} \equiv \alpha_{22}^2+\alpha_{12}\delta_{12} \equiv 1, \qquad \alpha_{12}(\alpha_{11}+\alpha_{22}) \equiv \delta_{12}(\alpha_{11}+\alpha_{22}) \equiv \gamma_{12}(\alpha_{11}+\alpha_{22}) \equiv 0,$$

so that the substitutions of period 2 are

$$\pm\begin{bmatrix} 1 & 0 & \alpha_{12} & \gamma_{12} \\ 0 & 1 & 0 & 0 \\ 0 & -\gamma_{12} & -1 & 0 \\ 0 & \alpha_{12} & 0 & -1 \end{bmatrix}, \qquad \pm\begin{bmatrix} 1 & 0 & 0 & \gamma_{12} \\ 0 & 1 & 0 & \delta_{12} \\ \delta_{12} & -\gamma_{12} & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$\pm\begin{bmatrix} 0 & 0 & 1 & \gamma_{12} \\ 0 & 0 & 0 & 1 \\ 1 & -\gamma_{12} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The first is transformed into a like substitution Σ with $\alpha_{12} \equiv 0$ by

$$A_{\alpha_{12}}: \qquad \xi_1' = \xi_1 - \alpha_{12}\xi_2, \qquad \eta_2' = \eta_2 + \alpha_{12}\eta_1,$$

which belongs to H_{648}; while Σ is transformed into $T_{2,-1}$ by

$$B_{\gamma_{12}}: \qquad \xi_1' = \xi_1 - \gamma_{12}\eta_2, \qquad \xi_2' = \xi_2 - \gamma_{12}\eta_1,$$

likewise in H_{648}. Transforming the second by $A_{\delta_{12}}$, we obtain one of the third type. The third type is transformed into P_{12} by $L_{2,\gamma_{12}}$, which belongs to H_{648}. But H_{648} contains Z_0 (§ 5) which transforms P_{12} into $T_{1,-1}$, identical with $T_{2,-1}$ in the quotient-group.

In the second case, the abelian conditions give

$$-\alpha_{11}^2+\alpha_{12}\delta_{12}\equiv-\alpha_{22}^2+\alpha_{12}\delta_{12}\equiv 1,\quad \alpha_{12}(\alpha_{11}+\alpha_{22})\equiv 0,\quad \delta_{12}\equiv(\alpha_{11}+\alpha_{22})\equiv 0,$$

so that the substitutions of period 2 are

$$\pm\begin{bmatrix} \alpha_{11} & \gamma_{11} & \alpha_{12} & \gamma_{12} \\ 0 & -\alpha_{11} & 0 & \delta_{12} \\ -\delta_{12} & \gamma_{12} & -\alpha_{11} & \gamma_{22} \\ 0 & -\alpha_{12} & 0 & \alpha_{11} \end{bmatrix},\qquad (\gamma_{11}\delta_{12}-\alpha_{11}\gamma_{12}+\alpha_{12}\gamma_{22}\equiv 0).$$

Transforming by $A_{-\alpha_{11}\delta_{12}}$, we obtain a similar substitution with $\alpha_{11}\equiv 0$. Transforming the latter by $B_{\gamma_{11}\alpha_{12}}$, we obtain a similar substitution with $\alpha_{11}\equiv\gamma_{11}\equiv 0$. Transforming by the substitution $L_{2,\alpha_{12}\gamma_{12}}$ of H_{648}, we have also $\gamma_{12}\equiv 0$. Hence $\gamma_{22}\equiv 0$, so that the final substitution is $P_{12}T_{2,-1}$. Since the latter has the characteristic determinant $(\rho^2+1)^2$, while the only operators of period 2 in K_{54}, viz., $[0, 0, c, 0]T_{2,-1}$, have the characteristic determinant $(\rho^2-1)^2$, the resulting groups K_{54} and K'_{54} are not conjugate.

To find the largest group in which K'_{54} is self-conjugate, we seek the operators (19) of II_{380} which transform $P_{12}T_{2,-1}$ into one of the 9 operators $[-\gamma, 0, c, \gamma]P_{12}T_{2,-1}$ of period 2. The conditions are

$$\alpha_{21}\equiv\pm\alpha_{12}\quad \alpha_{22}\equiv\mp\alpha_{11},\quad \delta_{21}\equiv\pm\delta_{12},\quad \delta_{22}\equiv\mp\delta_{11},$$
$$c\delta_{11}+\gamma\delta_{21}\equiv\pm\gamma_{12}-\gamma_{21},\quad \gamma\delta_{11}-c\delta_{21}\equiv\gamma_{11}\pm\gamma_{22}.$$

The latter serve merely to determine c and γ uniquely, since the determinant $\delta_{11}^2+\delta_{21}^2$ will be seen to be $\neq 0$. The abelian conditions reduce to

$$\alpha_{11}\delta_{11}+\alpha_{12}\delta_{12}\equiv 1,\quad \alpha_{12}\delta_{11}-\alpha_{11}\delta_{12}\equiv 0,\quad \alpha_{11}\gamma_{21}+\alpha_{12}\gamma_{22}\mp\alpha_{12}\gamma_{11}\pm\alpha_{11}\gamma_{12}\equiv 0.$$

For $\alpha_{11}\equiv 0$, then $\alpha_{12}\equiv\delta_{12}$, $\delta_{11}\equiv 0$, $\gamma_{22}\equiv-\gamma_{11}$. For $\alpha_{11}\equiv 1$, either $\alpha_{12}\equiv\delta_{12}\equiv 0$, $\delta_{11}\equiv 1$, or $\alpha_{12}\equiv\pm 1$, $\delta_{12}\equiv\mp 1$, $\delta_{11}\equiv -1$. The resulting group is H_{216}.

For K_{54} we seek the operators (19) of II_{380} which transform $T_{2,-1}$ into one of the operators $[0, 0, c, 0]T_{2,-1}$ of period 2. The conditions are

$$\alpha_{12}\equiv\alpha_{21}\equiv\delta_{12}\equiv\delta_{21}\equiv 0,\qquad \gamma_{12}\equiv c\delta_{22},\qquad \gamma_{21}\equiv c\delta_{11};$$
$$\alpha_{11}\equiv\alpha_{22}\equiv\delta_{11}\equiv\delta_{22}\equiv 0,\qquad \gamma_{11}\equiv c\delta_{21},\qquad \gamma_{22}\equiv c\delta_{12},$$

according as the upper or lower signs are taken. In the first case, the abelian conditions reduce to $\alpha_{11}\delta_{11}\equiv 1$, $\alpha_{22}\delta_{22}\equiv 1$, the resulting substitutions being the

U of § 3. In the second case, the abelian conditions reduce to $\alpha_{12}\delta_{12} \equiv 1$, $\alpha_{21}\delta_{21} \equiv 1$, the resulting substitutions being the V of § 3. The U and the V form H_{108}.

The subgroups of order 60.

46. Theorem. *The subgroups of order* 60 *of* O *fall into two distinct sets of conjugates, represented by* G_{60} *of* III_3 *and* G'_{60}, *which are self-conjugate* only under* G_{120} *and* G'_{120}, *respectively, where†*

$$(56)\qquad G'_{60} = \{(\xi_1\xi_2\xi_3\xi_4\xi_5),\ Q\},\qquad G_{120} = \{G_{60},\ \Sigma\},\qquad G'_{120} = \{G'_{60},\ \Sigma\},$$

Q and Σ being respectively

$$(57)\qquad \begin{bmatrix} -1 & -1 & -1 & 0 & 1 \\ -1 & 0 & -1 & -1 & 1 \\ -1 & -1 & 0 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix},\qquad \begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & -1 \\ 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 & -1 \end{bmatrix}.$$

Since no subgroup of order a divisor of < 60 of 60 is self-conjugate in a Γ_{60} by the earlier results (see the table), a subgroup Γ_{60} must be simple and hence simply isomorphic with the alternating group $G_{60}^{(5)}$ on 5 letters.‡

Within O, all the operators of period 5 are conjugate by I. Hence we may assume that Γ_{60} contains the linear substitution $A = (\xi_1\xi_2\xi_3\xi_4\xi_5)$. The cyclic group G_5 generated by it is seen (§ 21) to be self-conjugate only under

$$(58)\qquad G'_{20} = \{A,\ \Sigma\} = \{A^i,\ A^i\Sigma,\ A^i\Sigma^2,\ A^i\Sigma^3\ (i = 0, 1, 2, 3, 4)\}.$$

Since $\Sigma^2 = (\xi_1\xi_2)(\xi_3\xi_5)$, the only operators of period 2 in G'_{20} are

$$(\xi_1\xi_2)(\xi_3\xi_5),\quad (\xi_1\xi_3)(\xi_4\xi_5),\quad (\xi_1\xi_4)(\xi_2\xi_3),\quad (\xi_1\xi_5)(\xi_2\xi_4),\quad (\xi_2\xi_5)(\xi_3\xi_4),$$

which are transformed into each other transitively by the powers of A. They belong to Γ_{60} since it contains 5 operators of period 2 which transform G_5 into itself. Now Γ_{60} is simply isomorphic with the abstract group generated by A_1 and B where

$$(59)\qquad A_1^5 = B^2 = (A_1B)^3 = I.$$

*Cf. Proceedings of the London Mathematical Society, vol. 31 (1899), p. 53.

† Note that $\Sigma = W^2(\xi_1\xi_2\xi_3\xi_4\xi_5)W^2(\xi_1\xi_5\xi_4\xi_2\xi_3)C_1C_2C_3C_4$, $Q = C_3C_4WC_2C_5(\xi_1\xi_5)(\xi_2\xi_4)\Sigma^{-1}$, so that Σ and Q belong to O.

‡ To give another proof, a Γ_{60} contains 1 or 6 conjugate Γ_5. But a Γ_5 is self-conjugate only under a Γ_{20} within O. Hence a subgroup Γ_{60} contains 6 conjugate Γ_5 and is simply isomorphic with $G_{60}^{(5)}$ (Burnside, *The Theory of Groups*, pp. 107–108).

These relations are satisfied when $A_1 = A$, $B = (\xi_1 \xi_2)(\xi_3 \xi_4)$, whence $(\xi_1 \xi_4)(\xi_2 \xi_3) B$ becomes $(\xi_1 \xi_3)(\xi_2 \xi_4)$ of period 2. Hence the normalized Γ_{60}, which contains A and $(\xi_1 \xi_4)(\xi_2 \xi_3)$, may be generated by A and a substitution B such that B, $(\xi_1 \xi_4)(\xi_2 \xi_3) B$ and AB are of periods 2, 2 and 3, respectively. Now the condition that an orthogonal substitution shall be of period 2 is that its matrix be symmetrical with respect to the main diagonal. Hence B and $(\xi_1 \xi_4)(\xi_2 \xi_3) B$ are both of period 2 if and only if

$$B = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} & \alpha_{13} & \alpha_{25} \\ \alpha_{13} & \alpha_{23} & \alpha_{22} & \alpha_{12} & \alpha_{25} \\ \alpha_{14} & \alpha_{13} & \alpha_{12} & \alpha_{11} & \alpha_{15} \\ \alpha_{15} & \alpha_{25} & \alpha_{25} & \alpha_{15} & \alpha_{55} \end{bmatrix}.$$

The orthogonal conditions on the 1st and 4th rows, 2d and 3d rows, give

$$(60) \qquad -\alpha_{11}\alpha_{14} - \alpha_{12}\alpha_{13} + \alpha_{15}^2 \equiv 0, \qquad -\alpha_{12}\alpha_{13} - \alpha_{22}\alpha_{23} + \alpha_{25}^2 \equiv 0.$$

The orthogonal condition on the last row gives $\alpha_{55}^2 - \alpha_{15}^2 - \alpha_{25}^2 \equiv 1$, whence

$$\alpha_{55} \equiv \pm 1, \ \alpha_{15} \equiv \alpha_{25} \equiv 0; \qquad \text{or} \qquad \alpha_{55} \equiv 0, \ \alpha_{15} \not\equiv 0, \ \alpha_{25} \not\equiv 0.$$

In the first case, we find that

$$(AB)^{-1} = \begin{bmatrix} 0 & \cdots & \pm 1 \\ \alpha_{11} & \cdots & 0 \\ \alpha_{12} & \cdots & 0 \\ \alpha_{13} & \cdots & 0 \\ \alpha_{14} & \cdots & 0 \end{bmatrix}, \qquad (AB)^2 = \begin{bmatrix} \pm\alpha_{14} & \cdots & \alpha_{12}^2 - \alpha_{11}\alpha_{13} \\ \pm\alpha_{13} & \cdots & \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{22} + \alpha_{11}\alpha_{23} \\ \pm\alpha_{12} & \cdots & \alpha_{13}^2 + \alpha_{12}\alpha_{23} + \alpha_{11}\alpha_{22} \\ \pm\alpha_{11} & \cdots & \alpha_{13}\alpha_{14} + \alpha_{12}\alpha_{13} + \alpha_{11}\alpha_{12} \\ 0 & \cdots & \pm\alpha_{14} \end{bmatrix}.$$

Equating these when the lower signs are taken, we get $\alpha_{14} \equiv 0$, $\alpha_{13} \equiv -\alpha_{11}$, $\alpha_{12} \equiv 0$, $\alpha_{11}\alpha_{13} \equiv 1$, which are contradictory since -1 is a quadratic non-residue of 3. For the upper signs,

$$\alpha_{14} \equiv 0, \ \alpha_{13} \equiv \alpha_{11}, \ \alpha_{12}^2 - \alpha_{11}\alpha_{13} \equiv 1, \ \alpha_{12}\alpha_{13} + \alpha_{12}\alpha_{22} + \alpha_{11}\alpha_{23} \equiv 0,$$
$$\alpha_{13}^2 + \alpha_{12}\alpha_{23} + \alpha_{11}\alpha_{22} \equiv 0.$$

Hence $\alpha_{12}^2 - \alpha_{11}^2 \equiv 0$, so that $\alpha_{11} \equiv \alpha_{13} \equiv 0$, $\alpha_{12} \not\equiv 0$, $\alpha_{22} \equiv 0$, $\alpha_{23} \equiv 0$. Hence

$$B = (\xi_1 \xi_2)(\xi_3 \xi_4) \qquad \text{or} \qquad (\xi_1 \xi_2)(\xi_3 \xi_4) C_1 C_2 C_3 C_4.$$

The latter is excluded since its product by $(\xi_1 \xi_2 \xi_3 \xi_4 \xi_5)$ on the left is not of period 3.

In the second case $(\alpha_{55} \equiv 0)$, we equate the coefficients in the first and fifth rows of the matrices for $(AB)^{-1}$ and $(AB)^2$, and get

$\alpha_{15} \equiv 1 + \alpha_{11}\alpha_{25} + \alpha_{12}\alpha_{25} + \alpha_{13}\alpha_{15}$, $\quad \alpha_{25} \equiv \alpha_{11}\alpha_{15} + \alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{13} + \alpha_{13}\alpha_{14} + \alpha_{14}\alpha_{15}$,

$\alpha_{14} \equiv 1 - \alpha_{15}\alpha_{25}$, $\quad \alpha_{13} \equiv 1 + \alpha_{12}\alpha_{15} + \alpha_{13}\alpha_{25} + \alpha_{14}\alpha_{25}$, $\quad 0 \equiv \alpha_{12}^2 - \alpha_{11}\alpha_{13} - \alpha_{14}\alpha_{15}$,

$\alpha_{25} \equiv \alpha_{13}^2 + \alpha_{12}\alpha_{15} + \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{23} + \alpha_{14}\alpha_{25}$, $\quad \alpha_{12} \equiv \alpha_{22}\alpha_{15} + \alpha_{23}\alpha_{25} + \alpha_{13}\alpha_{25} + \alpha_{15}\alpha_{25}$,

$\alpha_{15} \equiv \alpha_{13}\alpha_{15} + \alpha_{11}\alpha_{23} + \alpha_{12}\alpha_{22} + \alpha_{12}\alpha_{13} + \alpha_{14}\alpha_{25}$, $\quad \alpha_{11} \equiv \alpha_{23}\alpha_{15} + \alpha_{22}\alpha_{25} + \alpha_{12}\alpha_{25} + \alpha_{15}\alpha_{25}$.

From the first, third and fourth it follows that

$$\alpha_{12} \equiv 1 - \alpha_{15} - \alpha_{15}\alpha_{25} + \alpha_{13}\alpha_{15} - \alpha_{13}\alpha_{15}\alpha_{25}, \quad \alpha_{11} \equiv -1 + \alpha_{15} - \alpha_{25} - \alpha_{15}\alpha_{25} - \alpha_{13}\alpha_{15}.$$

Eliminating α_{11}, α_{12}, α_{14} from the second condition of the preceding set and from the first relation (60), we get, respectively,

$$\alpha_{13}^2(\alpha_{15} - 1)(1 - \alpha_{25}) + \alpha_{13}(\alpha_{15}\alpha_{25} + \alpha_{15} - \alpha_{25} + 1) - \alpha_{25} \equiv 0,$$

$$\alpha_{13}^2(\alpha_{15}\alpha_{25} - \alpha_{15}) + \alpha_{13}(\alpha_{15}\alpha_{25} - \alpha_{15} - \alpha_{25} - 1) + 1 + \alpha_{15} - \alpha_{25} \equiv 0.$$

Eliminating α_{13}^2, we get $(\alpha_{13} + \alpha_{25})(\alpha_{15} + 1) \equiv 0$. If $\alpha_{15} \equiv -1$, the second becomes

$$\alpha_{13}^2(1 - \alpha_{25}) + \alpha_{13}\alpha_{25} - \alpha_{25} \equiv 0.$$

By trial, α_{13} is neither 0 nor -1. Hence $\alpha_{13} \equiv 1$, $\alpha_{25} \equiv 1$. Then $\alpha_{14} \equiv -1$, $\alpha_{11} \equiv -1$, $\alpha_{12} \equiv 0$. The last four of the above set of 9 conditions then reduce to $\alpha_{22} \equiv \alpha_{23} \equiv -1$. But the resulting substitution AB is not of period 3. Hence must $\alpha_{15} \equiv +1$, $\alpha_{13} \equiv -\alpha_{25}$. Hence $\alpha_{25} \equiv +1$, $\alpha_{13} \equiv -1$. Then $\alpha_{11} \equiv -1$, $\alpha_{12} \equiv -1$, $\alpha_{14} \equiv 0$. The last four of the above set of 9 conditions then reduce to $\alpha_{22} + \alpha_{23} \equiv -1$. For these values of α_{ij}, the necessary and sufficient condition that AQ shall have period 3 is $\alpha_{22} \equiv 0$. The resulting substitution B is Q given by (57).

Since G_5 is self-conjugate only under G_{20}, while Γ_{60} contains but 6 conjugate groups of order 5, it follows that at most 120 operators of O transform Γ_{60} into itself. That G_{60} is self-conjugate under G_{120} follows from the fact that Σ, an operator of period 4 defined by (57), transforms $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ into its square, and $(\xi_1\xi_2)(\xi_3\xi_4)$ into $(\xi_2\xi_4)(\xi_3\xi_5)$. That G'_{60} is self-conjugate under G'_{120} follows since Σ transforms Q into $(\xi_1\xi_5)(\xi_2\xi_4)Q(\xi_5\xi_4\xi_3\xi_2\xi_1)$, an operator of G'_{60}.

Finally, G_{60} and G'_{60} are not conjugate within O. If an operator T of O transforms G'_{60} into G_{60} and any subgroup G'_5 of G'_{60} into G_5 and if S be one of the existing operators of G'_{120} which transforms G'_{60} into itself and G'_5 into G_5, then $S^{-1}T$ transforms G_5 into itself and G'_{60} into G_{60}. Hence $S^{-1}T$ belongs to G_{20}, a subgroup of G'_{120}. Hence T belongs to G'_{120} and transforms G'_{60} into itself, contrary to hypothesis.

The subgroups of order 72.

47. Every Γ_{72} contains 1 or 4 conjugate Γ_9. If a Γ_9 is self-conjugate, the quotient-group Γ_{72}/Γ_9 is a group of order 8 having a subgroup of order 4, so

that Γ_{72} has a subgroup of order 36. Let next Γ_9 be one of 4 conjugate subgroups of Γ_{72}. The group Γ_g which transforms into itself each of the four is self-conjugate under Γ_{72} and g is a divisor, less than 18, of 18. Evidently $g \geqq 3 = 72/4!$, and $g \neq 9$. If $g = 6$, Γ_g must be cyclic, since a non-cyclic subgroup of order 6 of G is self-conjugate only under a group of order 36 or 108 (§ 7). Moreover, the cyclic Γ_6 are self-conjugate only under subgroups conjugate with G_{72}, K_{36}^*, or G_{24}^* (§ 6). Hence we may take $\Gamma_6 = C_6$, $\Gamma_{72} = G_{72}$. Finally, if $g = 3$, Γ_{72}/Γ_g is simply isomorphic with the symmetric group on 4 letters, so that Γ_{72} contains a subgroup of order 36. *Hence* Γ_{72} *is conjugate with* G_{72} *or else contains a subgroup of order* 36.

The subgroup K_{36}^* is excluded, since it is self-conjugate only under itself (§ 34). Each of the groups G_{36}^{**} and H_{36}^{**} leads to G_{72}^{**} only (§§ 36, 37). Finally, K_{36}^{**} is self-conjugate only under H_{216} (§ 35). A substitution which extends K_{36}^{**} to a Γ_{72} must be an operator of H_{216} of period 2, 4, or 6, whose square belongs to K_{36}^{**} (§ 5). For those of period 2, we may take $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5), of which the first two belong to K_{36}^{**} and are excluded. But W_0 extends K_{36}^{**} to G_{72}^{**}. Those of period 4 are conjugate with $T_{2,-1}W_0$, whose square $P_{12}T_{2,-1}$ belongs to K_{36}^{**}; it extends the latter to G_{72}^{**}. As at the end of § 33, we may restrict the operators of period 6 to $W_1 = [1, 0, -1, -1]\, W_0$ and $[-\gamma, 0, 0, \gamma]\, T_{2,-1}$, the latter belonging to K_{36}^{**} and the former extending it to G_{72}^{**}. *Hence* Γ_{72} *is conjugate with* G_{72}^{**} *if it contains a subgroup of order* 36.

Since $T_{2,-1}$ is the only operator of period 2 in G_{72}, an operator which transforms G_{72} into itself must be commutative with $T_{2,-1}$ and hence by I_{109} be one of the operators A or AP_{12}, where A is an operator (52). Considering the binary substitutions on ξ_1 and η_1, we must have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \qquad (ad - bc \equiv 1),$$

k to be suitably chosen. Hence $c \equiv 0$, $a \equiv kd$. Since $ad \equiv 1$, we have $a \equiv d \equiv \pm 1$, $k \equiv 1$. Placing the ambiguity of sign in front of the matrix for A, we may take $a \equiv d \equiv 1$. Hence A belongs to G_{72}. But P_{12} does not transform G_{72} into itself. *Hence* G_{72} *is self-conjugate only under itself.*

Since G_{72}^{**} is a subgroup of H_{216}, the largest subgroup of G in which K_9^{**} is self-conjugate, K_9^{**} is the only subgroup of order 9 of G_{72}^{**}. Hence the latter is self-conjugate only under a subgroup of H_{216}. Now $[1, 0, 0, 0]$, which extends G_{72}^{**} to H_{216}, transforms W_0 into $[0, 0, 1, 1]\, W_0$, which does not belong to G_{72}^{**}. *Hence* G_{72}^{**} *is self-conjugate only under itself.*

That G_{72} and G_{72}^{**} are not isomorphic follows from a consideration of their operators of period 2 or of their subgroups of order 9.

THEOREM. *Within* G *there are exactly two distinct sets of conjugate subgroups of order* 72. *As representatives we may take* G_{72} *and* G_{72}^{**}; *each is self-conjugate only under itself.*

The subgroups of order 80.

48. A Γ_{80} contains 1 or 16 conjugate Γ_5, the first alternative being here excluded (§ 2). Having 64 operators of period 5, Γ_{80} contains a single Γ_{16}, which is therefore self-conjugate. Hence Γ_{16} is conjugate with G_{16} (III). If G_{16} be taken as Γ_{16}, Γ_{80} must be a subgroup of G_{960} by III, bottom of p. 20. Every operator of period 5 in the latter is of the form PC, where C belongs to G_{16} and P is a cyclic permutation of $\xi_1, \cdots, \xi_5$. Within G_{960}, P is conjugate with $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ or its inverse. Hence Γ_{80} is conjugate with

$$G_{80} = \{ G_{16}, (\xi_1\xi_2\xi_3\xi_4\xi_5) \}. \tag{61}$$

A substitution commutative with G_{80} must be commutative with G_{16} and hence belong to G_{960}. Of the even substitutions on $\xi_1, \cdots, \xi_5$ only the powers of $S = (\xi_1\xi_2\xi_3\xi_4\xi_5)$ transform the latter into itself, none transform S into either S^2 or S^3, while $(\xi_2\xi_5)(\xi_3\xi_4)$ transforms S into S^4.

THEOREM. *Within G, every subgroup of order 80 is conjugate with G_{80}. The latter is self-conjugate only under*

$$G_{160} = \{ G_{16}, (\xi_1\xi_2\xi_3\xi_4\xi_5), (\xi_2\xi_5)(\xi_3\xi_4) \}. \tag{62}$$

Subgroups of order 96.

49. Any Γ_{96} contains 1 or 3 conjugate Γ_{32}. In the first case, we may take G_{32} as Γ_{32}, so that Γ_{96} is a subgroup of G_{576}, the group of all substitutions replacing ξ_5 by $\pm\xi_5$. An operator of period 3 of G_{576} must replace ξ_5 by $+\xi_5$ and hence belong to $G_{288} = [G_{32}, W, (\xi_2\xi_3\xi_4)]$. We may therefore limit Γ_{96} to the groups obtained by extending G_{32} by respectively W, $(\xi_2\xi_3\xi_4)$, $W(\xi_2\xi_3\xi_4)$, $W(\xi_2\xi_4\xi_3)$. But C_1C_5 transforms G_{32} into itself and $W(\xi_2\xi_3\xi_4)$ into $W^2(\xi_2\xi_3\xi_4)$, the inverse of the last extender. The three remaining groups are G_{96} of § 44 and

$$J_{96} = [G_{32}, (\xi_2\xi_3\xi_4)], \qquad L_{96} = (G_{32}, W). \tag{63}$$

Since each has a single subgroup of order 32, an operator transforming one of them into itself or another must transform G_{32} into itself and hence belong to G_{576}. Now G_{96}, J_{96} and L_{96} are each self-conjugate in G_{288}. Moreover, C_1C_5 transforms W into W^2, $(\xi_2\xi_3\xi_4)$ into itself, and G_{32} into itself. *Hence J_{96} and L_{96} are each self-conjugate only under the subgroup G_{576} of O, and are not conjugate within O; but G_{96} is self-conjugate only under G_{288}.*

Let next Γ_{96} contain 3 conjugate Γ_{32}. Let Γ_g be the group of the operators of Γ_{96} which transform each Γ_{32} into itself. Then g is a divisor of 32, $g < 32$, $g \geqq 96/3\,!$. Hence $g = 16$. Since Γ_{16} is self-conjugate in Γ_{96}, Γ_{16} is conjugate with F_{16} or G_{16}. Taking $\Gamma_{16} = F_{16}$, we have $\Gamma_{96} \equiv G_{96}$. Taking $\Gamma_{16} = G_{16}$, we have Γ_{96} as a subgroup of G_{960}. Within the latter, Γ_{96} is conjugate with

H_{96}, defined by (22). We first transform one of the operators of period 3 of G_{60} into $(\xi_2\xi_4\xi_5)$ and proceed as in § 41.

By III, bottom of p. 20, H_{96} is self-conjugate only under a subgroup of G_{960}. But the only even substitutions on $\xi_1, \cdots, \xi_5$ which transform $(\xi_1\xi_3)(\xi_2\xi_4)$, $(\xi_1\xi_3)(\xi_2\xi_5)$, $(\xi_1\xi_3)(\xi_4\xi_5)$ amongst themselves are these three and the powers of $(\xi_2\xi_4\xi_5)$. *Hence H_{96} is self-conjugate only under itself within O.*

The Subgroups of order 108.

50. Any Γ_{108} contains 1 or 4 conjugate Γ_{27}. Let first Γ_{27} be self-conjugate. Then Γ_{108}/Γ_{27} has a subgroup of order 2, so that Γ_{108} has a self-conjugate subgroup Γ_{54}. For the latter we may take G_{54}, K_{54} or K'_{54} (§ 45). But K_{54} leads to H_{108} only (§ 45).

Next, G_{54} is self-conjugate (§ 45) only under the group G_{648} of the operators (2) of II_{372}. The square of (2) is*

$$\pm\begin{vmatrix} 1 & -k-\gamma a^2+\beta c^2+ac(\alpha-\delta) & a+a\alpha+c\beta & c+c\delta+a\gamma \\ 0 & 1 & 0 & 0 \\ 0 & (1+\alpha)(\alpha c-\gamma a)+\gamma(\beta c-\delta a) & \alpha^2+\beta\gamma & \gamma(\alpha+\delta) \\ 0 & (1+\delta)(\beta c-\delta a)+\beta(\alpha c-\gamma a) & \beta(\alpha+\delta) & \delta^2+\beta\gamma \end{vmatrix} \quad (\alpha\delta-\beta\gamma\equiv 1).$$

This operator belongs to G_{54} only when $\alpha\equiv\delta$, $\beta\equiv\gamma\equiv 0$, whence (2) belongs to G_{54}, or $\alpha\equiv-\delta$, $\alpha^2+\beta\gamma\equiv\pm 1$, so that $\pm$ must be $-$. But there exists (IV) a binary substitution of determinant unity which transforms $\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & -\alpha\end{smallmatrix}\right)$ into $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$. Hence, in the second case, (2) is conjugate within G_{648} with $[k, a, c, 0]M_2$, where M_2 replaces ξ_2 by η_2, and η_2 by $-\xi_2$. Hence every Γ_{108} defined by G_{54} is conjugate with

$$(64)\qquad G_{108}=(G_{54}, M_2).$$

Finally, K'_{54} is self-conjugate only under H_{216} (§ 45). Within H_{216} the operators of period 2 are conjugate with $P_{12}T_{2,-1}$, $T_{2,-1}$ or W_0 (§ 5), the first belonging to K'_{54}, the second and third extending it to respectively H_{108} and

$$(65)\qquad K'_{108}=(K'_{54}, W_0)=(K_{27}, P_{12}T_{2,-1}, W_0).$$

The operators of period 4 of H_{216} are conjugate with $T_{2,-1}W_0$ (§ 5), whose square $P_{12}T_{2,-1}$ belongs to K'_{54}, so that it extends K'_{54} to

$$(66)\qquad K''_{108}=(K'_{54}, T_{2,-1}W_0)=(K_{27}, P_{12}T_{2,-1}, T_{2,-1}W_0).$$

Consider lastly the operators of period 6 given in the theorem of § 5. Their squares (§ 33, end) all belong to K_{27}, so that none are excluded. But

* Hence (2) is of period 2 only if $\beta\equiv\gamma\equiv 0$, $\alpha\equiv\delta\equiv-1$, $k\equiv 0$, giving $[0, a, c, 0]T_{2,-1}$.

$W_1 = [1, 0, -1, -1]\, W_0$ extends K'_{54} to K'_{108}, $[k, 0, 0, \gamma]\, T_{2,-1}$ extends K'_{54} to H_{108}, while $[\pm 1, 0, 0, 0]\, T_{2,-1} P_{12}$ belongs to K'_{54}.

We pass now to the case in which Γ_{27} is one of 4 conjugate subgroups. The group Γ_g of the operators of Γ_{108} which transform each of the four Γ_{27} into itself is self-conjugate under Γ_{108} and g is a divisor < 27 of 27, while $g \geqq 108/4!$. Hence $g = 9$, so that Γ_9 is conjugate with K_9^* or K_9^{**} (II_{386}). In the first case, Γ_{108} is conjugate with H_{108}. In the second case, Γ_{108} is conjugate with a subgroup of H_{216}. Since Γ_{108}/Γ_9 is simply isomorphic with the alternating group on 4 letters, which contains a self-conjugate group of order 4, Γ_{108} contains a self-conjugate group of order 36. By §§ 34–37, this must be of the type K_{36}^{**}, which is self-conjugate under H_{216}. Applying a suitable transformation within H_{216}, we may assume that Γ_{108} contains also K_{27}, with which all the subgroups of order 27 of H_{216} are conjugate, so that $\Gamma_{108} = H_{108}$.

Since H_{108} contains a single group K_{27} of order 27, and since K_{27} is self-conjugate only under H_{648}, H_{108} is self-conjugate only under a subgroup of H_{648}. But H_{108} is self-conjugate under H_{648}.

Since G_{108} contains a single group G_{27} of order 27 and since G_{27} is self-conjugate only under G_{648}, G_{108} is self-conjugate only under a subgroup of G_{648}. The latter affects ξ_2 and η_2 by a binary group of 24 substitutions $\left(\begin{smallmatrix}\alpha & \gamma\\ \beta & \delta\end{smallmatrix}\right)$ of determinant unity. Within the latter, the group $\{I, T_{2,-1}, M_2, T_{2,-1}M_2\}$ is self-conjugate only under a group of order 8 (IV). Hence G_{108} is self-conjugate only under

$$(67)\quad G_{216} = \{\text{operators (2) of } \mathrm{II}_{372} \text{ with } \alpha \equiv \delta \equiv \pm 1,\ \beta \equiv \gamma \equiv 0, \text{ or else } \alpha + \delta \equiv 0\}.$$

Finally, K'_{108} and K''_{108} are self-conjugate only under H_{216}.

Within G, *the subgroups of order* 108 *are conjugate with* H_{108}, G_{108}, K'_{108} *or* K''_{108}. *These are self-conjugate only under* H_{648}, G_{216}, H_{216}, H_{216}, *respectively.*

The subgroups of order 120.

51. THEOREM.* *Within* G, *every subgroup of order* 120 *is conjugate with* G_{120} *or* G'_{120}. *The latter are self-conjugate only under themselves.*

Every Γ_{120} is composite. By the earlier results (see table), a Γ_{120} has no self-conjugate subgroup of order < 60. We may therefore assume that it contains G_{60} or G'_{60}, whence Γ_{120} is G_{120} or G'_{120}, respectively (§ 46).

* Second proof: A Γ_{120} contains 1 or 6 conjugate Γ_5, the first case being here excluded. Since Γ_{120} thus contains a Γ_{20} but no self-conjugate subgroup lying in Γ_{20}, it is (DYCK) expressible as a transitive substitution-group on 6 letters and hence is simply isomorphic with the symmetric group on 5 letters. A third proof depends upon the 5 or 15 conjugate Γ_8, there being no primitive group of order 120 on 15 letters (MILLER, 2); while of the two imprimitive groups, one has substitutions of period 15 and is excluded, and the other is isomorphic with $G_{120}^{(5)}$ (KUHN).

Exclusion of the order 144.

52. Every Γ_{144} is composite.* By the earlier results, a group of order a divisor < 144 of 144 is self-conjugate in a group of order a multiple of 144 only when it is conjugate with G_2, F_8''', F_{24}^*, or G_{32}. Of these, F_8''' and F_{24}^* are self-conjugate only under G_{288}, which has no subgroup of order 144 (see end of § 39). Again, G_2 and G_{32} are self-conjugate only under G_{576}, which corresponds to the abelian group (A, P_{12}), A being defined by (52). Since the group G'_{288} of the A has no subgroup of index 2, Γ_{144} must contain $A'P_{12}$ and exactly one fourth of the operators A, the latter consequently (IV) forming a group given by the direct product (Γ_6, Γ_{24}) of a Γ_6 on one pair of the variables ξ_i, η_i and a Γ_{24} on the other pair. But $A'P_{12}$ transforms (Γ_6, Γ_{24}) into a group (Γ_{24}, Γ_6). Hence Γ_{144} does not exist.

The subgroups of order 160.

53. A Γ_{160} contains 1 or 5 conjugate Γ_{32}. By III, a subgroup Γ_{32} of G is self-conjugate only under a Γ_{64} or a Γ_{576}. The group Γ_g transforming each of the 5 Γ_{32} into itself is self-conjugate under Γ_{160} and g is a divisor < 32 of 32. But by III no subgroup of order 2, 4, or 8 is self-conjugate under a Γ_{160}. Hence $g = 16$ and Γ_{16} is conjugate with G_{16}, Γ_{160} with a subgroup of G_{960}. Proceeding as in § 48, we find that Γ_{160} is conjugate with G_{160}.

The 5 subgroups of order 32 of G_{160} are conjugate within G with J_{32}^k of III_5, which is self-conjugate only under G_{64}. Hence G_{160} is transformed into itself by at most 320 operators of G. (This also follows from the fact that G_{160} contains 16 conjugate Γ_5.) But G contains no Γ_{320} (§ 59).

THEOREM. *Within G, every subgroup of order* 160 *is conjugate with* G_{160}, *which is self-conjugate only under itself.*

The subgroups of order 162.

54. A Γ_{162} contains a single Γ_{81}. The latter may be taken to be G_{81} of II_{372}, whence the former is G_{162} of II_{373}, since G_{81} is self-conjugate only under G_{162}.

THEOREM. *Within G, every subgroup of order* 162 *is conjugate with* G_{162}, *which is self-conjugate only under itself.*

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*The proof may be modified so as not to assume that Γ_{144} is composite. It has 1, 3, or 9 conjugate Γ_{16}, the first case being excluded. If Γ_{16} is one of 3 conjugates the largest group Γ_g transforming each into itself is of order 24 and hence conjugate with F_{24}^* (§§ 22–32). If Γ_{16} is one of 9 conjugates, g is a divisor < 16 of 16. If $g = 8$, Γ_g is conjugate with F_8'''. Also $g \neq 4$ by III. If $g = 2$, Γ_2 is conjugate with $\{I, T_{2,-1}\}$, so that Γ_{144} is a subgroup of (A, P_{12}). If $g = 1$, Γ_{144} is simply isomorphic with a transitive substitution-group on 9 letters. Such a group contains substitutions of period 8 (COLE, 1) and is here excluded.

Exclusion of the order 180.

55. A Γ_{180} is composite.* By the table, no subgroup of order $\leqq 90$ of G is self-conjugate in a group of order divisible by 180.

The subgroups of order 192.

56. THEOREM. *Within O, there are exactly two distinct sets of conjugate subgroups of order* 192, *represented by G_{192} and H_{192}; G_{192} is self-conjugate only under itself, and H_{192} only under G_{576}.*

We may take G_{64} as one of the conjugate subgroups of order 2^6 in Γ_{192}. If G_{64} is self-conjugate, $\Gamma_{192} = G_{192}$ by III_{21}. Next, let G_{64} be one of 3 conjugate subgroups. The group Γ_g of the operators of Γ_{192} which transform into itself each of the three groups of order 64 is self-conjugate under Γ_{192}. Moreover, $g < 64$ and $g \geqq 192/3\,!$, so that $g = 32$. Hence Γ_g is conjugate with G_{32}. Taking $\Gamma = G_{32}$, we have Γ_{192} as a subgroup of G_{576}, Under the latter, G_{64} has exactly 3 conjugate subgroups: G_{64}, $W^{-1}G_{64}W$, $W^{-2}G_{64}W^2$. Hence $\Gamma_{192} = (G_{64}, W) = H_{192}$.

A substitution which transforms G_{192} into itself must belong to G_{960} by III, bottom of p. 20. But the even substitutions on $\xi_1, \cdots, \xi_5$ which transform G_{192} into itself do not alter ξ_5. Hence G_{192} is self-conjugate only under itself.

A substitution S which transforms H_{192} into itself must replace ξ_5 by $\alpha_{5s}\xi_s$, $s \geqq 5$, by III_{20}. Likewise, S^{-1} must replace ξ_5 by $\alpha_{5t}\xi_t$, $t \leqq 5$, so that S replaces ξ_t by $\pm\xi_5$. Hence $S = S'(\xi_5\xi_s\cdots\xi_t)$, where S' leaves ξ_5 and ξ_s unaltered except in sign. Then $s = 5$; for if not, $S' = CS''$, where C is a product of the C_i, and S'' permutes the variables other than ξ_5 and ξ_s, so that S transforms $(\xi_1\xi_3)(\xi_2\xi_4)$ into a substitution permuting certain variables including ξ_5 and hence not in H_{192}. It has now been shown that S belongs to G_{576}. Inversely, $(\xi_2\xi_3\xi_4)$ extends H_{192} to G_{576} and transforms H_{192} into itself.

The subgroups of order 216.

57. A Γ_{216} has 1 or 4 conjugate Γ_{27}. Let first Γ_{27} be self-conjugate. Since Γ_{216}/Γ_{27} has a subgroup of order 4, Γ_{216} contains a Γ_{108}. Since the latter contains Γ_{27} self-conjugately, it is conjugate with H_{108}, G_{108}, K'_{108} or K''_{108} (§ 50). The last three lead only to G_{216} and H_{216} (§ 51, end). If $\Gamma_{108} = H_{108}$, Γ_{216} is a subgroup of H_{648}. But H_{648}/K_{27} is simply isomorphic with the symmetric group

* We may proceed otherwise. A Γ_{180} contains 1, 4, or 10 conjugate Γ_9. By II, the subgroups of order 9 are self-conjugate only under groups of orders 27, 108, 162, 216. Hence a subgroup Γ_{180} contains 10 conjugate Γ_9. The group Γ_g transforming each into itself is self-conjugate under Γ_{180} and $g < 18$. As before $g \neq 9$. Also, $g \neq 6$, $g \neq 3$, $g \neq 2$ by §§ 6, 7, 3 and III. Hence Γ_{180} is simply isomorphic with a transitive substitution-group on 10 letters. But no such group exists (COLE, 2).

on 4 letters, whose subgroups of order 8 are all conjugate (V). Hence Γ_{216} is conjugate with H_{216}, and H_{216} is self-conjugate only under itself within G. Let next Γ_{27} be one of 4 conjugate subgroups. The group Γ_g transforming each into itself is self-conjugate under Γ_{216} and g is a divisor < 54 of 54, while $g \geqq 216/4!$. Hence $g = 9, 18$, or 27. If $g = 9$, Γ_g is conjugate with K_9^{**} and Γ_{216} with H_{216} (II_{383}). If $g = 18$, Γ_g has a single Γ_9, consequently self-conjugate under Γ_{216}. If $g = 27$, we are led to the first case.

THEOREM. *The subgroups of order* 216 *are conjugate with* G_{216} *or* H_{216}, *the former being self-conjugate only under* G_{648} *and the latter only under itself.*

The subgroups of order 288.

58. Since * a Γ_{288} has a self-conjugate subgroup of order $\leqq 144$, it is conjugate with G_{288} or with a subgroup of G_{576} (by the table). In the latter case also, it is conjugate with G_{288} (§ 52).

By III_{20}, a substitution which transforms G_{288} into itself belongs to G_{576}.

THEOREM. *Every subgroup of order* 288 *of* G *is conjugate with* G_{288}, *which is self-conjugate only under* G_{576}.

Exclusion of the order 320.

59. A Γ_{320} contains 1 or 5 conjugate Γ_{64}. But a subgroup Γ_{64} of G is self-conjugate only under a Γ_{192} (III_{21}). Hence Γ_{64} is one of 5 conjugates. The group Γ_g transforming each into itself is self-conjugate under Γ_{320} and g is a divisor < 64 of 64, $g \geqq 320/5!$. Now the values 4, 8, 32 of g are excluded by III. For $g = 16$, we may take $\Gamma_{16} = G_{16}$, whence Γ_{320} is a subgroup of G_{960}. Then Γ_{320}/Γ_{16} would be simply isomorphic with a subgroup of the alternating group on 5 letters (III_3).

The subgroups of order 324.

60. A Γ_{324} has 1 or 4 conjugate Γ_{81}, one of which may be taken to be G_{81}. The latter, being self-conjugate only under G_{162}, is one of 4 conjugate subgroups of Γ_{324}. If Γ_g be the group transforming each of the 4 into itself, g is a divisor < 81 of 81 and $g \geqq 324/4!$. Hence $g = 27$. We may take $\Gamma_{27} = G_{27}$, or K_{27}, whence Γ_{324} is a subgroup of G_{648} or H_{648}, respectively. But G_{648}/G_{27} has

* For a second proof, we note that a Γ_{288} contains 1, 3, or 9 conjugate Γ_{32}. If Γ_{32} is self-conjugate it is conjugate with G_{32}, and Γ_{288} with a subgroup of G_{576} (III). If one of 3 conjugates, the group transforming each into itself is of order 48, so that Γ_{288}/Γ_{48} has a subgroup of order 3, and Γ_{288} a subgroup Γ_{144}, contrary to § 52. If Γ_{32} is one of 9 conjugates, g is a divisor < 32 of 32. But $g = 16$ and $g = 4$ are excluded by III; $g = 8$ leads to a group conjugate with F_8''', whence Γ_{288} is conjugate with G_{288}. For $g = 2$, we may take $\Gamma_2 = \{I, T_{2,-1}\}$, whence $\Gamma_{288} = G_{288}$ (§ 47). If $g = 1$, Γ_{288} is simply isomorphic with a transitive substitution group on 9 letters, which is impossible (COLE, 1).

no subgroup of index 2. However H_{648}/K_{27} contains a (single) subgroup of order 12 (V). Hence Γ_{324} is conjugate with

$$(68) \qquad H_{324} = \{\text{ operators } (19) \text{ of } \mathrm{II}_{380} \text{ with } \delta_{11}\delta_{22} - \delta_{12}\delta_{21} \equiv 1\}.$$

The number of operators of G commutative with H_{324} is evidently 4×162.

THEOREM. *Every subgroup of order* 324 *of* G *is conjugate with* H_{324}, *which is self-conjugate only under* H_{648}.

The subgroups of order 360.

61. A subgroup Γ_{360} has no self-conjugate subgroup of order $\leqq 180$ in view of the earlier results (see the table). Hence it is simply isomorphic with the alternating group G^6_{360} and contains subgroups of order 60. Now $(\xi_1\xi_4)(\xi_5\xi_6)$ extends G^5_{60} to G^6_{360}; its product by $(\xi_1\xi_4)(\xi_2\xi_3)$ on the left is of period 2; its product by $(\xi_1\xi_2\xi_3\xi_4\xi_5)$ on the left is of period 3. Hence we may assume that Γ_{360} has the normalized subgroup Γ_{60} of § 46 and an operator B of period 2 such that $(\xi_1\xi_4)(\xi_2\xi_3)B$ is of period 2 and $(\xi_1\xi_2\xi_3\xi_4\xi_5)B$ of period 3. Hence $B = (\xi_1\xi_2)(\xi_3\xi_4)$ or Q (§ 46).

THEOREM. *Within* O, *every subgroup of order* 360 *is conjugate with*

$$(69) \qquad G_{360} = \{G_{60}, G'_{60}\} = \{(\xi_1\xi_2\xi_3\xi_4\xi_5), (\xi_1\xi_2)(\xi_3\xi_4), Q\}.$$

The simple group G_{360} *is self-conjugate only under*

$$(70) \qquad G_{720} = \{G_{360}, \Sigma\}.$$

Exclusion of the order 432.

62. A Γ_{432} is composite. But no subgroup of order < 432 is self-conjugate in a group of order divisible by 432 (see table).*

* Second proof. A Γ_{432} has 1, 4, or 16 conjugate Γ_{27}. The first case is excluded by III. Let first Γ_{27} be one of 4 conjugates. The group Γ_g transforming each into itself is self-conjugate under Γ_{432} and g is a divisor < 108 of 108, while $g \geqq 432/4! = 18$. But $g \neq 27$ as before, $g \neq 18$ by §§ 14–20, $g \neq 36$ by §§ 33–37, $g \neq 54$ by § 45. Let next Γ_{27} be one of 16 conjugates. Then g is a divisor < 27 of 27. But $g \neq 9$ by II, $g \neq 3$ by § 3. If $g = 1$, Γ_{432} is simply isomorphic with a transitive group on 16 letters. Having no self-conjugate Γ_{16} by III, it is not primitive (MILLER, 3, p. 229). It is not imprimitive in view of the following theorem and proof communicated to me August 22 by Professor G. A. MILLER: *There exists no imprimitive group of degree* 16 *and of order* 432, 864, 1296, *or* 2592. For, an imprimitive group of degree 16 must have 8, 4, or 2 systems of imprimitivity, which are permuted according to a transitive group of degree 8, 4, or 2, respectively. The latter group may be assumed primitive. Since a primitive group of degree 8 has its order divisible by 7, the imprimitive groups under consideration would have either 4 or 2 systems. In the former case, these systems would be permuted according to G^4_{12} or G^4_{24}. The head would be divisible by 9, which is impossible since its order could not be divisible by 144. Hence the required group must contain just 2 systems of imprimitivity and hence have an intransitive subgroup of index 2 with constituents of degree 8. This is impossible since the order of the head would be divisible by 27 so that each constituent of the head would have an order divisible by 9. The order of such a constituent would therefore be divisible by 32, so that the order of the entire group would be divisible by 64.

Exclusion of the order 480.

63. A composite* Γ_{480} may be assumed (from the table) to contain G_{16} and to be a subgroup of G_{960}, whereas G_{960}/G_{16} is simply isomorphic with $G_{60}^{(5)}$ and has no subgroup of order 30.

The subgroups of order 576.

64. A subgroup Γ_{576}, being composite, must be conjugate† with G_{576} by the table of the earlier results. Now G_{576} contains exactly 3 groups conjugate with G_{64}, since the latter is self-conjugate only under G_{192}, a subgroup of G_{576} (III_{21}).

THEOREM. *Every subgroup of order* 576 *of* O *is conjugate with* G_{576}', *which is self-conjugate only under itself.*

The subgroups of order 648.

65. We may assume that Γ_{648} contains G_{81}, which is self-conjugate only under G_{162} (II_{373}). Hence Γ_{648} contains 4 groups conjugate with G_{81}. The group Γ_g transforming each of the 4 into itself is self-conjugate under Γ_{648} and g is a divisor < 162 of 162, $g \geqq 648/4!$. Hence $g = 81$, 54, or 27. As above, $g \neq 81$. If $g = 54$, Γ_{648} is conjugate with G_{648} (§ 45). If $g = 27$, Γ_{648} is conjugate with G_{648} or H_{648} (II_{379}, II_{380}). Evidently the largest subgroup of G which transforms Γ_{648} into itself is of order 4×162.

THEOREM. *The subgroups of order* 648 *of* G *are conjugate with* G_{648} *or* H_{648}, *each of which is self-conjugate only under itself.*

The subgroups of order 720.

66. A subgroup Γ_{720}, necessarily composite, contains no self-conjugate subgroup of order < 360 by the table of the earlier results. We may thus assume that Γ_{720} contains G_{360}, so that it is identical with G_{720} (§ 61).

THEOREM.‡ *Every subgroup of order* 720 *of* O *is conjugate with* G_{720}, *which is self-conjugate only under itself.*

It may be shown that G_{720} is simply isomorphic with the symmetric group on 6 letters.‡

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* Second proof. A Γ_{480} has 1, 6, 16, or 96 conjugate Γ_5; 1, 4, 10, 16, 40, or 160 conjugate Γ_3. But a subgroup Γ_5 of G is self-conjugate only under a Γ_{20}, a Γ_3 only under a group of order a divisor of 648. Hence Γ_{480} contains 384 operators of period 5 and either 80 or 320 operators of period 3, and 32 further operators of a Γ_{32}.

† Second proof. A Γ_{576} has 1, 3, or 9 conjugate Γ_{64}, the first case being excluded. If 3 conjugates, the group Γ_g transforming each into itself is of order 96, so that Γ_{576} is conjugate with G_{576} (§ 49). If 9 conjugates, g is a divisor < 64 of 64. If $g = 32$ or 2, Γ_{576} is conjugate with G_{576} by III. Also, $g \neq 16$, $g \neq 8$, $g \neq 4$. Finally, $g \neq 1$, since there is no transitive $G_{576}^{(9)}$ (COLE, 1).

‡ Cf. Proceedings of the London Mathematical Society, vol. 31 (1899), pp. 30–68; ser. 2, vol. 1 (1903–4), pp. 283–4.

Exclusion of the orders 810 *and* 864.

67. Subgroups of these orders are excluded * by the table, being composite.

The subgroups of order 960.

68. A subgroup Γ_{960}, being composite, must be conjugate † with G_{960} by the table. Now G_{960} is self-conjugate only under itself by III, bottom of p. 20.

Exclusion of the orders 1080, 1296, 1620, *and* 1728.

69. Subgroups of these orders are composite, ‡ but contain no self-conjugate subgroups (table).

* To give another proof for 810, we may assume that Γ_{810} contains G_{81}, which is self-conjugate only under G_{162}. Hence Γ_{810} contains 10 groups conjugate to G_{81}. But no subgroup of O of order 27, 9, or 3 is self-conjugate in a Γ_{810} by II. Hence Γ_{810} is simply isomorphic with a transitive substitution-group on 10 letters, whereas no such group exists (COLE, 2).

To give another proof for 864, we note that Γ_{864} contains 1, 4, or 16 conjugate Γ_{27}, the first case being excluded by II. Let first there be 4 conjugates and denote by Γ_g the largest subgroup of Γ_{864} transforming each into itself. Then g is a divisor of 216, $g \geqq 36 = 864/4!$. The resulting values 216, 108, 72, 54, 36 are excluded by the earlier results. Suppose next there are 16 conjugates. Then g is a divisor < 54 of 54. But 27, 18, 9, 6, 3, 2 are excluded by the earlier results. The case $g = 1$ is excluded by MILLER 3, and the foot-note to § 62.

† Another proof follows from a consideration of the 5 or 15 conjugate subgroups of order 64, there being no primitive group $G_{960}^{(15)}$ (MILLER, 2) and no such imprimitive group (KUHN).

‡ Second proof for 1080. A Γ_{1080} contains 1, 6, 36, or 216 conjugate Γ_5. But a subgroup Γ_5 is self-conjugate only under a Γ_{20}. Hence Γ_5 is one of 216 conjugates, so that Γ_{1080} contains 864 operators of period 5. It contains 1, 4, 10, or 40 conjugate Γ_{27}, the first two cases being excluded by II. Let first there be 10 conjugates. No group of order g, where $1 < g \leqq 108$, is self-conjugate under a Γ_{1080} by the table. Moreover, there is no transitive group $G_{1080}^{(10)}$ (COLE, 2). Hence there are 40 conjugate Γ_{27}. If they are of either of the types G_{27}, K_{27}, they form a complete set of conjugates under G_{25920}. Now the group K_{27} of the $[k, 0, c, \gamma]$ contains $L_{1,1} = [1, 0, 0, 0]$, $L_{1,1} L_{2,1} = [1, 0, 0, 1]$, $L_{1,-1} L_{2,1} = [-1, 0, 0, 1]$, so that the conjugates to K_{27} contain $2(40 + 120 + 240)$ operators of period 3. The group G_{27} of the $[k, a, c, 0]$ contains $L_{1,1}$ and also $[-1, 0, -1, 0]$, into which $L_{1,-1} L_{2,1} = [-1, 0, 0, 1]$ is transformed by the abelian substitution

$$\xi_1' = \xi_1, \quad \eta_1' = \eta_1 - \eta_2, \quad \xi_2' = \xi_2 + \xi_1, \quad \eta_2' = \eta_2.$$

Hence the conjugates to G_{27} contain at least $2(40 + 240)$ operators of period 3. Hence the conjugate Γ_{27} are of neither of the types G_{27}, K_{27}, and hence of the type H_{27}. By II_{379}, the subgroups of order 9 of H_{27} are the commutative K_9 and three conjugate cyclic C_9. Within G_{25920}, each C_9 is self-conjugate only under H_{27} by II_{385}. Hence a C_9 is one of 40 distinct conjugates within Γ_{1080}, so that the latter contains $6 \times 40 = 240$ operators of period 9. But there were only 216 operators of period $\neq 5$.

Second proof for 1296. A Γ_{1296} contains 1, 4, or 16 conjugate Γ_{81}, the first two cases being excluded by II. The largest subgroup of Γ_{1296} which transforms each of the 16 Γ_{81} into itself is of order 1 by II. But there exists no primitive G_{1296}^{16} having no self-conjugate Γ_{16} (MILLER, 3, p. 229). There exists no imprimitive G_{1296}^{16} by the foot-note to § 62.

Second proof for 1620 (the lengthy case of JORDAN, *Traité*, pp. 322–9). A Γ_{1620} contains 1, 4, or 10 conjugate Γ_{81}. But a subgroup Γ_{81} is self-conjugate only under a G_{162}. Hence there are 10 conjugates. Now Γ_{1620} has no self-conjugate subgroup of order 2, 3, 6, 9, 18, or 27, and hence none of order 54. But there exists no transitive group $G_{1620}^{(10)}$ (COLE, 2).

Second proof for 1728. A subgroup of G_{25920} of index 15 must be maximal (§ 2), and hence requires the existence of a primitive $G_{25920}^{(15)}$, contrary to MILLER, 2.

Exclusion of the orders 2160, 2592, 2880.

70. If a subgroup of one of these orders exists, it must be maximal (§ 2). But there exists no primitive group of order 25920 on 12 letters (MILLER, 1), nor on 10 letters * (COLE, 2), nor on 9 letters (COLE, 1).

Maximal subgroups of G_{25920}.

71. THEOREM. *Every maximal subgroup of* G_{25920} *is conjugate with one of the five groups:* G_{960}, G_{720}, G_{648}, H_{648}, G_{576}.

By the table the only subgroups self-conjugate only under themselves are the above five and the following non-maximal ones: G_{20} (in G_{720}), G_{24}^* (in H_{648}, since its generators $P_{12} L_{1,-1} T_{1,-1}$, $T_{2,-1}$, and D are all of the form (19) of II_{380}), K_{36}^* (in H_{108}), G_{48} (in G_{576}, since its substitutions replace ξ_5 by $\pm \xi_5$), G_{72} (in G'_{288} of § 39), H_{96} (in G_{960}), G_{120} and G'_{120} (in G_{720}), G_{160} and G_{192} (in G_{960}), G_{162} and H_{212} (in H_{648}).

THE UNIVERSITY OF CHICAGO,
August 21, 1903.

* Second proof for 2592. A Γ_{2592} contains 1, 4, or 16 conjugate Γ_{81}, the first two cases being here excluded. But a G_{2592}^{16} is neither primitive (MILLER, 3) nor imprimitive (foot-note to § 62).