THE GROUPS OF ORDER $p^3 q^\beta$*

BY

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§ 1. *Introduction.*

The researches of FROBENIUS† and BURNSIDE,‡ familiar to all students of the theory of groups, have established the non-existence of simple groups of orders pq^β and $p^2 q^\beta$, p and q being different primes, and the consequent solvability of all groups of these orders. In the following paper I show that all groups of order $p^3 q^\beta$ are compound, and therefore also solvable. For convenience of reference I place at the beginning of the discussion the two following theorems of which repeated use is made in the subsequent reasoning:

I. *A simple group $\mathfrak{G}$ of order $g = p^\alpha q^\beta$, p and q being different primes and $p > q$, cannot contain any subgroup $\mathfrak{H}$ of order h whose index g/h in $\mathfrak{G}$ is $< p^2$.*

In view of the results of FROBENIUS and BURNSIDE cited above, we assume that $\alpha > 2$.

The case $g/h < p$ is trivial. Here every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$ transforms every conjugate of $\mathfrak{H}$ into itself and $\mathfrak{G}$ is certainly compound. Again, for $g/h = p$, $h = p^{\alpha-1} q^\beta$, every subgroup of order $p^{\alpha-1}$ in $\mathfrak{H}$ transforms $\mathfrak{H}$ and therefore every conjugate of $\mathfrak{H}$ into itself, and $\mathfrak{G}$ is again certainly compound.

Suppose, now, that $\mathfrak{K}$ is the largest subgroup of $\mathfrak{G}$ that contains $\mathfrak{H}$ and is less that $\mathfrak{G}$. Since $\mathfrak{G}$ is simple, $\mathfrak{K}$ is invariant under only those elements of $\mathfrak{G}$ which are contained in $\mathfrak{K}$. The index g/k of $\mathfrak{K}$ in $\mathfrak{G}$ is $\leqq g/h$ and $> p$; $\mathfrak{K}$ has g/k distinct conjugates in $\mathfrak{G}$.

1) Let $g/k = pq^i (0 < q^i < p)$, $k = p^{\alpha-1} q^{\beta-i}$. Having less than p^2 conjugates in $\mathfrak{G}$, $\mathfrak{K}$ contains a subgroup of order $p^{\alpha-1}$ from every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$. Every subgroup of order $p^{\alpha-1}$ of $\mathfrak{K}$ is permutable with every subgroup $\mathfrak{L}$ of order $g^{\beta-i}$ of $\mathfrak{K}$. $\mathfrak{L}$ is invariant under a subgroup of $\mathfrak{G}$ of order $g^{\beta-i+1}$, and this transforms $\mathfrak{K}$ into at least one conjugate $\mathfrak{K}'$ of $\mathfrak{K}$, different from $\mathfrak{K}$.

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† FROBENIUS: Berliner Sitzungsberichte, 1895, p. 185; cf. Acta Mathematica, vol. 26 (1902), p. 198.

‡ BURNSIDE: *Theory of Groups*, p. 348. Cf. JORDAN, Liouville's Journal, ser. 5, vol. 4 (1898), pp. 21–26.

$\mathfrak{K}$ and $\mathfrak{K}'$, having $\mathfrak{L}$ in common, have no group of order $p^{\alpha-1}$ in common. Then $\mathfrak{L}$ is permutable with two different groups of order $p^{\alpha-1}$ contained in the same group $\mathfrak{A}$ of order p^α, and is therefore permutable with $\mathfrak{A}$. $\mathfrak{L}$ and $\mathfrak{A}$ generate a group of order $p^\alpha q^{\beta-i}$ contained in $\mathfrak{G}$ and having $< p$ conjugates in $\mathfrak{G}$.

2) Let $g/k = q^j (p < q^j < p^2)$, $k = p^\alpha q^{\beta-j}$. As in 1), every conjugate of $\mathfrak{K}$ contains a subgroup of order $p^{\alpha-1}$ from every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$, and every subgroup $\mathfrak{L}$ of order $q^{\beta-j}$ of $\mathfrak{K}$ occurs also in a conjugate $\mathfrak{K}'$ of $\mathfrak{K}$ different from $\mathfrak{K}$. $\mathfrak{K}$ and $\mathfrak{K}'$ have in common the group $\mathfrak{L}$ and a group of order $p^{\alpha-1}$ from every group $\mathfrak{A}$ of $\mathfrak{K}$ or $\mathfrak{K}'$; their greatest common subgroup $\mathfrak{D}$ is of order $p^{\alpha-1}q^{\beta-j}$. All the conjugates of $\mathfrak{D}$ in $\mathfrak{K}$ or $\mathfrak{K}'$ are obtained by transforming $\mathfrak{D}$ by any group $\mathfrak{A}$ of $\mathfrak{K}$ or $\mathfrak{K}'$. Hence all the subgroups of order $p^{\alpha-1}$ in $\mathfrak{D}$ are common to all the conjugates of $\mathfrak{D}$ in $\mathfrak{K}$ and $\mathfrak{K}'$. The subgroups of order $p^{\alpha-1}$ of $\mathfrak{D}$ generate a group invariant in $\mathfrak{K}$ and in $\mathfrak{K}'$, and therefore in a group $\mathfrak{M}$ contained in $\mathfrak{G}$ and containing $\mathfrak{K}$ and $> \mathfrak{K}$. Then $\mathfrak{M} = \mathfrak{G}$, and $\mathfrak{G}$ is compound.

II. *If a simple group* $\mathfrak{G}$ *of order* $g = p^\alpha q^\beta$, p *and* q *being, as in* I, *different primes and* $p > q$, *contains a subgroup* $\mathfrak{H}$ *of order* $p^i q^j$ *where* $1 < q^{\beta-j} < p$, *then* $\mathfrak{H}$ *is contained in a subgroup* $\mathfrak{K}$ *of* $\mathfrak{G}$ *of order* $p^{i+x}q^\beta$ $(x \geqq 0, i + x < \alpha)$.

Suppose that the largest group containing $\mathfrak{H}$ and contained in $\mathfrak{G}$ and $< \mathfrak{G}$ is $\mathfrak{K}$ of order $k = p^{i+x}q^{j+y}(j + y < \beta)$; $\mathfrak{K}$ is invariant under only those elements of $\mathfrak{G}$ that are contained in $\mathfrak{K}$; having $< p^{\alpha-i-x+1}$ conjugates in $\mathfrak{G}$, $\mathfrak{K}$ contains a subgroup of order p^{i+x} from every group $\mathfrak{A}$ of order p^α in $\mathfrak{G}$. If $i + x = \alpha$, $\mathfrak{G}$ is compound, by I. If $i + x < \alpha$ and $j + y < \beta$, a subgroup $\mathfrak{L}$ of order q^{j+y} of $\mathfrak{K}$ occurs in a conjugate $\mathfrak{K}'$ of $\mathfrak{K}$ different from $\mathfrak{K}$. $\mathfrak{L}$ is permutable with two groups of order p^{i+x} from the same group $\mathfrak{A}$; these with $\mathfrak{L}$ generate a group $\mathfrak{M}$ of order $p^{i+x+z}q^{j+y}$ $(z > 0)$ containing $\mathfrak{K}$ and contained in $\mathfrak{G}$ and $< \mathfrak{G}$; but this is contrary to assumption.

A simple application of Theorem II is afforded by the groups of order p^2q^β $(p > q > 2)$. A simple group of this order must contain p^2 subgroups $\mathfrak{B}$ of order q^β, p of which have a common subgroup $\mathfrak{D}$ of order $q^r(r > 0)$ invariant in a group $\mathfrak{D}'$ of order pq^{r+s} $(s > 0)$; $p^2 - 1$ is divisible by $q^{\beta-r}$, and $p - 1$ by q^s, hence $p - 1$ is divisible by $q^{\beta-r}$; $\mathfrak{D}'$ is contained in a subgroup of order pq^β of $\mathfrak{G}$. But then $\mathfrak{G}$ is compound. The theorem controls also the case $q = 2$, except in the single event that $s = 1$ and $p + 1 = 2^{\beta-r-1}$.

§ 2. *Preliminary treatment of the groups of order* p^3q^β.

A simple group of order $p^\alpha q^\beta$ $(p \lesseqgtr q)$ can occur only if $\alpha > 2\mu$, μ being the lowest index for which $p^\mu \equiv 1 \pmod{q}$.* For $\alpha = 3$, we can only take $\mu = 1$. A group $\mathfrak{G}$ of order p^3q^β can be simple only if $p \equiv 1 \pmod{q}$; also $\beta > 2\nu$, where $q^\nu \equiv 1$ (p), therefore $q^\beta > p^2$.

* BURNSIDE, *Theory of Groups*, p. 345. Cf. FROBENIUS, Acta Mathematica, vol. 26 (1902), p. 194.

A simple group $\mathfrak{G}$ of order p^3q^β must contain either p^2 or p^3 subgroups $\mathfrak{B}$ of order q^β. Since $q^\beta > p^2$, the elements of these subgroups $\mathfrak{B}$ cannot be wholly distinct in either case.

If $\mathfrak{G}$ contains only p^2 subgroups $\mathfrak{B}$, and if two of these are so chosen that the order q^r of their greatest common divisor $\mathfrak{D}$ is a maximum, then $\mathfrak{D}$ is invariant in a subgroup $\mathfrak{D}'$ of $\mathfrak{G}$ whose order is $p^x q^{r+s}$ $(x = 1, 2; s > 0)$ and which contains exactly p groups of order q^{r+s}. Here $p^2 - 1$ is divisible by $q^{\beta-r}$, therefore $p + 1 \geqq q^{\beta-r-1}$. (If $q \neq 2$, $p - 1$ is divisible by $q^{\beta-r}$, and $\mathfrak{D}'$ has less than p^2 conjugates in $\mathfrak{G}$ unless $x = 1$. For odd q, the discussion can be greatly simplified, as in the case of order p^2q^β). Each of the $p^{3-x}q^{\beta-r-s}$ conjugate groups $\mathfrak{D}$ occurs in exactly p of the groups $\mathfrak{B}$. If each of the p^2 groups $\mathfrak{B}$ contains k of the groups $\mathfrak{D}$, the total number of the groups $\mathfrak{D}$ is $p^2k/p = p^{3-x}q^{\beta-r-s}$; hence $k = p^{2-x}q^{\beta-r-s}$. If now $x = 1$, each group $\mathfrak{B}$ contains $pq^{\beta-r-s}$ groups $\mathfrak{D}$; each of the latter is contained in $p - 1$ other groups $\mathfrak{B}$ and no two of them occur together in any second group $\mathfrak{B}$. But there are only p^2 of the groups $\mathfrak{B}$ and $p^2 < pq^{\beta-r-s}(p - 1) + 1$, unless $q^{\beta-r-s} = 1$, $r + s = \beta$. Then $\mathfrak{D}'$ is of order pq^β, has exactly p^2 different conjugates in $\mathfrak{G}$, and therefore contains an element P of order p from every subgroup $\mathfrak{A}$ of order p^3 in $\mathfrak{G}$.

Let $\mathfrak{A}$ be any subgroup of order p^3 in $\mathfrak{G}$, and let $\mathfrak{B}$ occur in $\mathfrak{D}'$; having only p^2 conjugates in $\mathfrak{G}$, $\mathfrak{B}$ is invariant under an element P of order p in $\mathfrak{A}$; $\mathfrak{B}$ is also permutable with a subgroup $\mathfrak{A}_1$ of order p in $\mathfrak{A}$ not containing P. P transforms $\mathfrak{D}'$ into a conjugate of $\mathfrak{D}'$ different from $\mathfrak{D}'$ and containing the group $P^{-1}\mathfrak{A}_1P$, which is different from $\mathfrak{A}_1$ but is contained with $\mathfrak{A}_1$ in a subgroup $\mathfrak{A}_2$ of order p^2 of $\mathfrak{A}$. $\mathfrak{B}$ is permutable with both $\mathfrak{A}_1$ and $P^{-1}\mathfrak{A}_1P$ and therefore with $\mathfrak{A}_2$; $\mathfrak{A}_2$ and $\mathfrak{B}$ generate a group of order p^2q^β contained in $\mathfrak{G}$ and having only p conjugates in $\mathfrak{G}$.

Again, if $x = 2$ each group $\mathfrak{B}$ contains $q^{\beta-r-s}$ of the groups $\mathfrak{D}$. The p groups $\mathfrak{B}$ which have subgroups of order q^{r+s} in a same group $\mathfrak{D}'$ contain $p(q^{\beta-r-s} - 1) + 1$ of the groups $\mathfrak{D}$, and these are transformed among themselves by every element of order p in $\mathfrak{D}'$. All the elements of order p in $\mathfrak{D}'$ are therefore permutable with each of the remaining $p - 1$ groups $\mathfrak{D}$; they generate a group which is invariant in p groups $\mathfrak{D}'$ and therefore in a group $\mathfrak{M}$ of order $p^{2+y}q^{r+s+t}$ $(y = 0, 1)$ contained in $\mathfrak{G}$. If $y = 1$, $\mathfrak{M}$ has at most $p + 1$ conjugates in $\mathfrak{G}$. And if $y = 0$, $t > 0$ and $\mathfrak{M}$ has at most $p(p + 1)/q < p^2$ conjugates in $\mathfrak{G}$.

$\mathfrak{G}$ must therefore contain p^3 sub-groups $\mathfrak{B}$. The maximum greatest common divisor $\mathfrak{D}$, of order q^r, of two of these is again invariant in a subgroup $\mathfrak{D}'$ of order p^xq^{r+s} $(x = 1, 2)$ of $\mathfrak{G}$. Here $p^3 - 1$ is divisible by $q^{\beta-r}$, and $p - 1$, being divisible by q, is divisible by $q^{\beta-r-1}$ (in fact by $q^{\beta-r}$ if $q \neq 3$). Then, by the reasoning employed in the proof of Theorem II, if $r + s < \beta$, $\mathfrak{D}'$ is contained in a sub-group of order p^2q^β of $\mathfrak{G}$. Hence $r + s = \beta$ and $\mathfrak{D}'$ is of order

pq^β. Each group $\mathfrak{D}$ is common to p groups $\mathfrak{B}$. If each group $\mathfrak{B}$ contains k groups $\mathfrak{D}$, we have $p^3k/p = p^2$, hence $k = 1$; each group $\mathfrak{D}'$ contains precisely one group $\mathfrak{D}$. Each group $\mathfrak{B}$ occurs in only one group $\mathfrak{D}'$.

§ 3. *Final Investigation of the Groups of Order* p^3q^β.

Each of the p^3 groups $\mathfrak{B}$ of $\mathfrak{G}$ transforms among themselves the $p^3 - p$ conjugates of $\mathfrak{B}$ not contained in the group $\mathfrak{D}'$ in which $\mathfrak{B}$ occurs. Let $\mathfrak{D}'$, and $\mathfrak{B}$ in $\mathfrak{D}'$, be so chosen that a subgroup Δ common to $\mathfrak{B}$ and a conjugate of $\mathfrak{B}$ not contained in $\mathfrak{D}'$ is of the largest possible order, and let this order be q^ρ. Then $q^{\beta-\rho}$ divides $p^3 - p$, and therefore divides $p^2 - 1$; $\rho > 0$, and in general $p > q^{\beta-\rho-1}$, the only exception occurring when $q = 2$ and $p + 1 = 2^{\beta-\rho-1}$. In this exceptional case $\beta - r = 1$, $\mathfrak{D}$ is of order $2^{\beta-1}$.

The group Δ is common to two groups $\mathfrak{B}$ from different groups $\mathfrak{D}'$. Δ is invariant under subgroups $\mathfrak{R}_1$, $\mathfrak{R}_2$ of order $q^{\rho+\sigma_1}$, $q^{\rho+\sigma_2}$ (σ_1, $\sigma_2 > 0$) of these two groups $\mathfrak{B}$. $\mathfrak{R}_1$ and $\mathfrak{R}_2$ cannot be contained in any subgroup of $\mathfrak{G}$ of order $q^{\rho+\tau}$ ($\tau > 0$), for this subgroup would be common to two groups $\mathfrak{D}'$ and therefore to two groups $\mathfrak{B}$ contained one in each of these two groups $\mathfrak{D}'$. Δ is invariant in a subgroup Δ' of order $p^x q^{\rho+\sigma}$ (x, $\sigma > 0$) of $\mathfrak{G}$. Any subgroup of order p^x of Δ' transforms $\mathfrak{D}'$ containing Δ into precisely p groups $\mathfrak{D}'$ each containing Δ, for Δ cannot occur in all the p^2 groups $\mathfrak{D}'$. Δ' has one or more subgroups of order $q^{\rho+\sigma}$ common with each of these p groups $\mathfrak{D}'$, and no subgroup of order $q^{\rho+\tau}$ ($\tau > 0$) common with any other group $\mathfrak{D}'$. Hence Δ occurs in precisely p groups $\mathfrak{D}'$.

1) If $p > q^{\beta-\rho-1}$, or if $\sigma > 1$, then by Theorem II, Δ' is contained in a subgroup $\mathfrak{M}$ of order pq^β of $\mathfrak{G}$ (the order p^2q^β being inadmissible). $\mathfrak{M}$ contains p groups $\mathfrak{B}$ having Δ as their common subgroup; Δ is invariant in $\mathfrak{M}$, $\mathfrak{M} = \Delta'$, $\rho + \sigma = \beta$; and Δ has p^2 conjugates in $\mathfrak{G}$. If each group $\mathfrak{D}'$ contains k groups Δ, then since each group Δ occurs in p groups $\mathfrak{D}'$ we have $p^2k/p = p^2$, hence $k = p$.

If now two groups $\mathfrak{D}'$ have more than one group Δ in common, their greatest common divisor $\mathfrak{C}$ is of order pq^ρ and contains p groups Δ; these are all the groups Δ contained in the two groups $\mathfrak{D}'$; $\mathfrak{C}$ is the smallest group that contains them; $\mathfrak{C}$ is invariant in both groups $\mathfrak{D}'$ and has $< p^2$ conjugates is $\mathfrak{G}$.

If no two groups $\mathfrak{D}'$ have more than one group Δ in common, the p groups Δ contained in any group $\mathfrak{D}'$ are distributed among $p(p-1)+1$ groups $\mathfrak{D}'$, and none of them occur in $p - 1$ groups $\mathfrak{D}'$. All the elements of order p of a group $\mathfrak{D}'$ are therefore permutable with each of $p - 1$ other groups $\mathfrak{D}'$; these elements of order p are common to p groups $\mathfrak{D}'$ and are all the elements of order p of any of these p groups $\mathfrak{D}'$; they generate a group invariant under the p groups $\mathfrak{D}'$ and having $< p^2$ conjugates in $\mathfrak{G}$.

2) It remains to consider the special case $q = 2$, $p + 1 = 2^{\beta-\rho-1}$, $\sigma = 1$, $r = \beta - 1$. Let $\mathfrak{D}'_1$, $\mathfrak{D}'_2$ be two groups $\mathfrak{D}'$ having a group Δ in common. Δ

cannot be the greatest common divisor of $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$, since either of the latter would then transform the other into $p\,2^{\beta-\rho} > p^2$ conjugates. The greatest common divisor $\mathfrak{C}$ of $\mathfrak{D}_1'$, $\mathfrak{D}_2'$ is therefore of order $p\,2^\rho$.

Suppose first that $\mathfrak{C}$ contains only one group of order 2^ρ, that is, that Δ is invariant in $\mathfrak{C}$; then Δ' contains $\mathfrak{C}$. If Δ' were of order $p\,2^{\rho+1}$, $\mathfrak{C}$ would be invariant in Δ' and would contain every element of order p of Δ'; whereas Δ' must contain an element of order p or p^2 which transforms $\mathfrak{D}_1'$ into $\mathfrak{D}_2'$. Hence Δ' is of order $p^2 2^{\rho+1}$, and Δ has $p\,2^{\beta-\rho-1} = p(p+1)$ conjugates in $\mathfrak{G}$. If each group $\mathfrak{D}'$ contains k groups Δ, we have $p^2 k/p = p(p+1)$, $k = p+1$. Since Δ is invariant under an element p of $\mathfrak{D}'$, Δ is contained in the group $\mathfrak{D}$ occurring in $\mathfrak{D}'$; the $p+1$ groups Δ occurring in $\mathfrak{D}'$ are conjugate in $\mathfrak{D}'$ and are therefore all contained in $\mathfrak{D}$. No two groups Δ occurring in the same group $\mathfrak{D}'$ can occur together in any second group $\mathfrak{D}'$. The p groups $\mathfrak{D}'$ which have Δ in common, contain p^2+1 groups Δ, and do not contain any one of the remaining $p-1$ groups Δ. Δ' transforms among themselves the p^2+1 groups Δ occurring with Δ in groups $\mathfrak{D}'$, and transforms among themselves the remaining $p-1$ groups Δ. All the elements of order p or p^2 in Δ' are permutable with these $p-1$ groups Δ; they all occur in p groups Δ' and are all the elements of order p or p^2 of each of these groups Δ'; they generate a group invariant under p groups Δ' and having $< p^2$ conjugates in $\mathfrak{G}$.

The group $\mathfrak{C}$ common to any two groups $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$ must therefore contain p groups Δ. No group Δ is contained in a group $\mathfrak{D}$, for then Δ would be invariant in $\mathfrak{C}$. Since $\mathfrak{D}$ is of order $2^{\beta-1}$, Δ has a subgroup $\mathfrak{S}$ of order $2^{\rho-1}$ common with $\mathfrak{D}$. $\mathfrak{S}$ is invariant in $\mathfrak{C}$, since $\mathfrak{C}$ cannot have two groups of order $2^{\rho-1}$ common with $\mathfrak{D}$. $\mathfrak{S}$ is also invariant under subgroups of order $2^{\rho+\tau_1}$, $2^{\rho+\tau_2}$ $(\tau_1, \tau_2 \geqq 0)$ of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, and therefore under subgroups $\mathfrak{L}_1$, $\mathfrak{L}_2$ of order $p\,2^{\rho+\tau_1}$, $p\,2^{\rho+\tau_2}(\tau_1, \tau_2 > 0)$ of $\mathfrak{D}_1'$ and $\mathfrak{D}_2'$ respectively. $\mathfrak{L}_1$ and $\mathfrak{L}_2$ cannot both be contained in a group of order $p\,2^{\rho+\tau}$ containing $\mathfrak{C}$. The largest group $\mathfrak{S}'$ contained in $\mathfrak{G}$ and containing $\mathfrak{S}$ as invariant subgroup is therefore of order $p^2 2^{\rho+\tau}$, where we must take $\tau = 1$, by virtue of Theorem II. $\mathfrak{S}'$ transforms $\mathfrak{D}'$ containing $\mathfrak{S}$ into p groups $\mathfrak{D}'$ containing $\mathfrak{S}$ and has $p\,2^{\rho+1}$ elements in common with each of these p groups $\mathfrak{D}'$. $\mathfrak{S}$ does not occur in any other group $\mathfrak{D}_i'$. For any group of order p^2 in $\mathfrak{S}'$ could transform $\mathfrak{D}_i'$ into only p groups $\mathfrak{D}'$; $\mathfrak{S}'$ would have an element P of order p common with $\mathfrak{D}_i'$; $\mathfrak{S}$, being invariant under P, would be contained in $\mathfrak{D}_i$ and would be invariant under a group of order 2^ρ of $\mathfrak{D}_i$; this group of order 2^ρ would occur in $\mathfrak{S}'$ and therefore in one of the p groups $\mathfrak{D}'$ into which $\mathfrak{S}'$ transforms $\mathfrak{D}_1'$; and this would lead to the case already disposed of where $\mathfrak{C}$ contains only one group Δ.

The group $\mathfrak{S}$ has $p(p+1)$ conjugates in $\mathfrak{G}$ and $p+1$ conjugates in each group $\mathfrak{D}'$. The $p(p+1)$ groups $\mathfrak{S}'$ are all different. The $p+1$ conjugates of $\mathfrak{S}$ which occur in $\mathfrak{D}'$ are conjugate in $\mathfrak{D}'$ and are all contained in $\mathfrak{D}$. No two groups $\mathfrak{D}'$ can have two groups $\mathfrak{S}$ in common, since this would again lead to the rejected case where $\mathfrak{C}$ contains only one group Δ. The p groups $\mathfrak{D}'$ which have $\mathfrak{S}$ in common contains p^2+1 groups $\mathfrak{S}$.. $\mathfrak{S}'$ transforms these p^2+1 groups $\mathfrak{S}$ among themselves. All the elements of order p or p^2 in $\mathfrak{S}'$ are permutable with each of the remaining $p-1$ groups $\mathfrak{S}$; they generate a group invariant in p groups $\mathfrak{S}'$ and having $< p^2$ conjugates in $\mathfrak{G}$.

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