ON A DEFINITION OF ABSTRACT GROUPS*

BY

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In a paper entitled *A definition of abstract groups* (Transactions, vol. 3, pp. 485–492, October, 1902), my second definition (l. c., p. 490)

$$(M'') = (1, 2, 3', 3''_l, 3''_r, 4''_l)$$

involves postulates not mutually independent. I shall prove here (as stated in October, 1904, Transactions, vol. 5, p. 549) that $(3''_r)$ is redundant, that in the new definition

$$(\overline{M}''): (1, 2, 3'', 3''_l, 4''_l)$$

the postulates are mutually independent, and that this mutual independence remains even for the system

$$(\overline{M}''_A): (1, 2, 3'', 3''_l, 4''_l, A)$$

defining an abelian group, obtained by adding to those of $(\overline{M}'')$ the postulate (A) that the multiplication or composition of two elements is commutative.

We have for consideration a set † or class (K) of elements and a multiplication-table or rule of combination $(\circ)$ whereby to every two elements a, b taken in the definite order a, b there corresponds a definite so-called product, in notation $a \circ b$, or, when without confusion, more simply, ab; this product may or may not be an element of the class. The postulates in question are then the following:

(1) If a and b are elements, then ab is an element of the class.

(A) If a and b are elements such that ab and ba are elements, then $ab = ba$.

(2) If a, b, c are elements such that $ab, bc, (ab)c, a(bc)$ are elements, then $(ab)c = a(bc)$.

$(3'')$ There exists an element a such that $aa = a$.

$(3''_l)$ If a and b are elements and $aa = a$, then $ab = b$.

$(3''_r)$ If a and b are elements and $aa = a$, then $ba = b$.

$(4''_l)$ If a and b are elements and $aa = a$, then there exists an element $b_l^{(a)}$ such that $b_l^{(a)} b = a$.

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† In the first definition (M) there was the underlying understanding (l. c., p. 485, footnote ‡) or postulate (0) that the class contain at least one element, and this carried over by implication to (M''), where however in view of $(3'')$ it was not needed; it is now omitted.

If a is an element such that $aa = a$ we designate those parts of $(3''_l)$, $(3''_r)$, $(4''_l)$ which refer to the element a by $(3''^a_l)$, $(3''^a_r)$, $(4''^a_l)$ respectively. Then we prove that $(3''_r)$ is deducible from $(1, 2, 3'', 3''_l, 4''_l)$ in that we prove that $(3''^a_r)$ is deducible from $(1, 2, 3''^a_l, 4''^a_l)$. This fact is proved by a suitable modification of the method used p. 486, 7° loc. cit. If a and b are elements and $aa = a$, then we designate by b', b'' elements, which by $(4''^a_l)$ surely exist, such that $b'b = a$, $b''b' = a$ and have in virtue of $(1, 2, 3''^a_l)$ the continued equality,

$$ba = bb'b = abb'b = b''b'bb'b = b''ab'b = b''b'b = ab = b,$$

that is, $ba = b$, which was to be proved. *

The proof that the postulates of $(\overline{M}''_A)$ are independent covers the independence of the postulates of $(\overline{M}'')$ and appears from the following proof systems $(K, \circ)$:

For (1). $K =$ all integers $0, \pm 1, \pm 2, \cdots$ except ± 1. $\circ = +$.

For (2). $K =$ an element a and any class of (at least three) elements x distinct from a. $a \circ a = a$. $a \circ x = x \circ a = x$. $x_1 \circ x_2 = a$ if $x_1 \neq x_2$; $x_1 \circ x_1 = x_3$ where, for every x_1, x_3 is any x except x_1.

For $(3'')$. $K =$ all positive integers. $\circ = +$.

For $(3''_l)$. $K =$ a class of (at least two) elements x. $x_1 \circ x_2 = a$, where a is a definite element, the same for all pairs x_1, x_2.

For $(4''_l)$. $K =$ all positive integers and 0. $\circ = +$.

For (A). $(K, \circ) =$ any non-abelian group.

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* [A second continued equality

$$ba = aba = b''b'ba = b''aa = b''a = b''b'b = ab = b$$

should be noticed.]